

Duality between Lagrangian and Legendrian invariants

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Consider a pair (X, L) , of a Weinstein manifold X with an exact Lagrangian submanifold L , with ideal contact boundary (Y, Λ) , where Y is a contact manifold and $\Lambda \subset Y$ is a Legendrian submanifold. We introduce the Chekanov-Eliashberg DG-algebra, $CE^*(\Lambda)$, with coefficients in chains of the based loop space of Λ and study its relation to the Floer cohomology $CF^*(L)$ of L . Using the augmentation induced by L , $CE^*(\Lambda)$ can be expressed as the Adams cobar construction Ω applied to a Legendrian coalgebra, $LC_*(\Lambda)$. We define a twisting cochain:

$$\mathfrak{t}: LC_*(\Lambda) \rightarrow B(CF^*(L))^\#$$

via holomorphic curve counts, where B denotes the bar construction and $\#$ the graded linear dual. We show under simply-connectedness assumptions that the corresponding Koszul complex is acyclic which then implies that $CE^*(\Lambda)$ and $CF^*(L)$ are Koszul dual. In particular, \mathfrak{t} induces a quasi-isomorphism between $CE^*(\Lambda)$ and the cobar of the Floer homology of L , $\Omega CF_*(L)$.

This generalizes the classical Koszul duality result between $C^*(L)$ and $C_{-*}(\Omega L)$ for L a simply-connected manifold, where ΩL is the based loop space of L , and provides the natural geometric ingredient explaining the computations given in [31] in the case when X is a plumbing of cotangent bundles of 2-spheres (where an additional weight grading ensured Koszulity of \mathfrak{t}).

We use the duality result to show that under certain connectivity and locally finiteness assumptions, $CE^*(\Lambda)$ is quasi-isomorphic to $C_{-*}(\Omega L)$ for any Lagrangian filling L of Λ .

Our constructions have interpretations in terms of wrapped Floer cohomology after versions of Lagrangian handle attachments. In particular, we outline a proof that $CE^*(\Lambda)$ is quasi-isomorphic to the wrapped Floer cohomology of a fiber disk C in the Weinstein domain obtained by attaching $T^*(\Lambda \times [0, \infty))$ to X along Λ (or, in the terminology of [59] the wrapped Floer cohomology of C in X with wrapping stopped by Λ). Along the way, we give a definition of wrapped Floer cohomology via holomorphic buildings that avoids the use of Hamiltonian perturbations which might be of independent interest.

1 Introduction

In this introduction we first give an overview of our results. The overview starts with a review of well known counterparts of our constructions in algebraic topology. We then introduce our Legendrian and Lagrangian invariants in Sections 1.1 and 1.2, respectively, and discuss the connection between them and applications thereof in Section 1.3. Finally, in Section 1.4 we give detailed calculations of the invariants introduced in the simple yet illustrative example of the Legendrian Hopf link filled by two Lagrangian disks intersecting transversely in one point.

The starting point for our study is a construction in classical topology. Consider the following augmented DG-algebras over a field \mathbb{K} associated to a based, connected, topological space (M, pt) :

$$C^*(M) \rightarrow \mathbb{K}, \quad C_{-*}(\Omega M) \rightarrow \mathbb{K},$$

where $C^*(M)$ is the singular cochain complex equipped with the cup-product and $C_{-*}(\Omega M)$ is the singular chain complex of the based loop space of M equipped with the Pontryagin product. (We use cohomologically graded complexes throughout the paper so that all differentials increase the grading by 1.) In the case of singular cohomology, the inclusion $i: \text{pt} \rightarrow M$ gives the augmentation $i^*: C^*(M) \rightarrow C^*(\text{pt}) = \mathbb{K}$ and in the case of the based loop space, the augmentation is given by the trivial local system $\pi_1(M, \text{pt}) \rightarrow \mathbb{K}$.

If M is of finite-type (for example, a finite CW-complex), then it is well known that one can recover the augmented DG-algebra $C^*(M)$ from the augmented DG-algebra $C_{-*}(\Omega M)$ by the Eilenberg-Moore equivalence:

$$C^*(M) \simeq \text{RHom}_{C_{-*}(\Omega M)}(\mathbb{K}, \mathbb{K}).$$

In the other direction, if M is *simply connected*, then the Adams construction gives a quasi-isomorphism

$$C_{-*}(\Omega M) \simeq \text{RHom}_{C^*(M)}(\mathbb{K}, \mathbb{K}),$$

and in this case $C^*(M)$ and $C_{-*}(\Omega M)$ are said to be *Koszul dual* DG-algebras. Koszul duality is sometimes abbreviated and simply called *duality*. For more general M , using the method of acyclic models, Brown [13] constructed a twisting cochain

$$t: C_{-*}(M) \rightarrow C_{-*}(\Omega M).$$

This is a degree 1 map that induces a DG-algebra map $\Omega C_{-*}(M) \rightarrow C_{-*}(\Omega M)$. By definition, the latter is a quasi-isomorphism when duality holds, and this can be detected by an associated *Koszul complex*, which is acyclic if and only if duality holds.

In the general case, $\Omega C_{-*}(M)$ is a certain completion of $C_{-*}(\Omega M)$ and consequently $C_{-*}(\Omega M)$ is a more refined invariant of M than $\Omega C_{-*}(M)$.

In this paper, we pursue this idea in the context of invariants associated to Lagrangian and Legendrian submanifolds. Here the role played by simple connectivity in the above discussion have two natural counterparts, one corresponds to a generalized notion of simply-connectedness for intersecting Lagrangian submanifolds and the other is the usual notion of simply-connectedness for Legendrian submanifolds.

We start with the geometric data of a Liouville domain X with convex boundary Y and an exact Lagrangian submanifold $L \subset X$ with Legendrian boundary $\Lambda \subset Y$. We assume that $c_1(X) = 0$, that the Maslov class of L vanishes (for grading purposes) and that L is relatively spin (to orient certain moduli spaces of holomorphic disks). Assume that L is subdivided into embedded components intersecting transversely $L = \bigcup_{v \in \Gamma} L_v$ and that Λ is subdivided into connected components $\Lambda = \bigsqcup_{v \in \Gamma} \Lambda_v$. To avoid notational complications, we take both parametrized by the same finite set Γ and assume that the boundary of L_v is Λ_v . We use a base field \mathbb{K} and define the semi-simple ring

$$\mathbf{k} = \bigoplus_{v \in \Gamma} \mathbb{K}e_v,$$

generated by mutually orthogonal idempotents e_v . Also, we fix a partition

$$\Gamma = \Gamma^+ \cup \Gamma^-$$

into two disjoint sets, and choose a base-point $p_v \in \Lambda_v$ for each $v \in \Gamma^+$.

For simplicity, let us restrict, in this introduction, to the following situation:

- X is a subcritical Liouville domain,
- If $v \in \Gamma^-$ then the corresponding Legendrian Λ_v is an embedded *sphere*.

From a technical point of view, these restrictions are unnecessary. We make them in order to facilitate the explanation of our constructions from the perspective of Legendrian surgery. (Note that, the topology of Λ_v is unrestricted when $v \in \Gamma^+$.)

We write X_Λ for the completion of the Liouville sector obtained from X by attaching critical Weinstein handles along Λ_v for each $v \in \Gamma^-$ and cotangent cones $T^*(\Lambda_v \times [0, \infty))$ along Λ_v for each $v \in \Gamma^+$. If $\Gamma^+ = \emptyset$, X_Λ is an ordinary Liouville manifold. In general, in the complement of a compact subset, it looks like a neighborhood of the zero section in the cotangent bundle $\bigcup_{v \in \Gamma^+} T^*(\Lambda_v \times [T, \infty))$, for some $T > 0$. Hence it is a geometrically bounded manifold, Gromov compactness holds [36], and holomorphic curve theory is well-behaved.

In X_Λ , for $v \in \Gamma^-$, there is a closed exact Lagrangian submanifold $S_v = L_v \cup D_v$, the union of the Lagrangian L_v in X and the *Lagrangian core disk* D_v of the Weinstein handle attached to Λ_v , and for $v \in \Gamma^+$, there is a non-compact Lagrangian obtained by attaching the cylindrical boundary $\Lambda_v \times [0, \infty)$ to L_v for $v \in \Gamma^+$, which we will still denote by L_v , by abuse of notation, even when we view them now in X_Λ . Dually, for each $v \in \Gamma^-$, we obtain (non-compact) exact Lagrangian disks C_v , the *Lagrangian cocore disks* of the Weinstein handles attached to Λ_v on X , and for each $v \in \Gamma^+$, we construct *dual* Lagrangian disks C_v intersecting L_v once and asymptotic to a Legendrian meridian of L_v (these can be constructed as the cotangent fiber at the point (p_v, t) , $t > 0$, in $T^*(\Lambda_v \times [0, \infty)) \subset X_\Lambda$, where p_v is the base points on Λ_v).

The invariants that we will construct are associated to the union of Lagrangian submanifolds

$$L_\Lambda := \bigcup_{v \in \Gamma^+} L_v \cup \bigcup_{v \in \Gamma^-} S_v \quad \text{and} \quad C_\Lambda := \bigcup_{v \in \Gamma} C_v.$$

The Lagrangian L_Λ will be referred to as a *Lagrangian skeleton* of X_Λ , it is a union of Lagrangian submanifolds which intersect transversely. The dual Lagrangian C_Λ is the union of Lagrangian disks which can be locally identified with cotangent fibers to irreducible components of L_Λ .

We will study two algebraic invariants associated to $(X_\Lambda, L_\Lambda, C_\Lambda)$. The first one is the *Legendrian A_∞ -algebra*, LA^* . It corresponds to the endomorphism algebra of L_Λ considered in the infinitesimal Fukaya category of X_Λ . The second one is the *Chekanov-Eliashberg DG-algebra*, CE^* . It corresponds to the endomorphism algebra of C_Λ considered in the partially wrapped Fukaya category of X_Λ . However, we will take the pre-surgery perspective as in [11] and construct all these invariants by studying Legendrian invariants of $\Lambda \subset X$ rather than Floer cohomology in X_Λ . From this perspective, the case $\Gamma^+ \neq \emptyset$ is a new construction, that generalizes the theory from [11] in a way analogous to how partially wrapped Fukaya categories [59] generalizes wrapped Fukaya categories [3].

The invariants LA^* and CE^* come equipped with canonical augmentations to the semi-simple ring \mathbf{k} and it is easy to see by construction that LA^* is determined by CE^* via the equivalence:

$$LA^* \simeq \mathrm{RHom}_{CE^*}(\mathbf{k}, \mathbf{k}).$$

The duality which would recover CE^* from LA^* holds in the ‘‘simply-connected’’ case (see Section 2.3). In the topological case discussed above, this is analogous to the simply-connectedness assumption on M , which makes the augmented algebras $C^*(M)$ and $C_{-*}(\Omega M)$ Koszul dual. In fact, the topological case is a special case of our study

for the Weinstein manifold T^*M , with the Lagrangian skeleton $L_\Lambda = M$ given by the 0-section, and the dual Lagrangian C_Λ given by a cotangent fiber T_p^*M .

We next sketch the definition of our version of the Chekanov-Eliashberg DG-algebra (without any assumption of simply-connectedness). This is the DG-algebra over \mathbf{k} called CE^* above. Its underlying \mathbf{k} -bimodule is the unital \mathbf{k} -algebra freely generated by Reeb chords between components of Λ and chains in $C_{-*}(\Omega_{p_v}\Lambda_v)$ for $v \in \Gamma^+$. (This is the crucial distinction between Γ^+ and Γ^- .)

We use the cubical chain complex (cf. [58]) $C_{-*}(\Omega_{p_v}\Lambda_v)$ for $v \in \Gamma^+$, see Section 3.1 for a discussion of other possible choices of chain models, to express CE^* as a free algebra over \mathbf{k} generated by Reeb chords c and generators of $C_{-*}(\Omega_{p_v}\Lambda_v)$ for $v \in \Gamma^+$. Since CE^* is a free algebra over \mathbf{k} , the differential is determined by its action on generators. On a generator of $C_{-*}(\Omega_{p_v}\Lambda_v)$ we simply apply the usual differential. On a generator c_0 which is a Reeb chord, the differential is determined by moduli spaces of holomorphic disks in the symplectization $\mathbb{R} \times Y$ which asymptotically converge to c_0 on the positive end and chords c_1, \dots, c_i at the negative end as follows. Consider

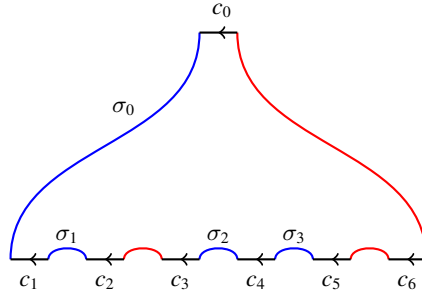


Figure 1: The differential in CE^* , the word $\sigma_0 c_1 \sigma_1 c_2 c_3 \sigma_2 c_4 \sigma_3 c_5 c_6$ appears in dc_0 .

the moduli space of J -holomorphic maps $u: D \rightarrow \mathbb{R} \times Y$, where D is a disk with $k + 1$ boundary punctures $z_j \in \partial D = S^1$ that are mutually distinct and (z_0, z_1, \dots, z_k) respects the counter-clockwise cyclic order of S^1 , and u sends the boundary component (z_{j-1}, z_j) of $S^1 \setminus \{z_0, \dots, z_k\}$ to $\mathbb{R} \times \Lambda$, and is asymptotic to c_j near the puncture at z_j for $j = 1, \dots, k$ and to c_0 near the puncture at z_0 . The moduli space, which is naturally a stratified space with manifold strata that carries a fundamental chain, comes with evaluation maps to $\Omega_{p_v}\Lambda_v$ for $v \in \Gamma^+$. The image of the fundamental chain determines a word in our chain model of $C_{-*}(\Omega_{p_v}\Lambda_v)$. Reading these together with the Reeb chords in order gives the differential of c_0 .

We remark that loop space coefficients were used in the context of Lagrangian Floer

cohomology before [7, 33]. See also [2], [18] for uses of high-dimensional moduli spaces in Floer theory.

While CE^* with loop space coefficients is a powerful invariant, it is in general hard to compute as it involves high-dimensional moduli spaces of disks. As mentioned above, duality in the Legendrian Λ will also play a role. More precisely, we define another DG-algebra CE_{\parallel}^* related to CE^* via a Morse theoretic version of Adams cobar construction whose definition involves taking parallel copies of Λ but uses only 0-dimensional moduli spaces (see Section 3.4). In fact, we prove that the two DG-algebras are quasi-isomorphic when Λ_v for $v \in \Gamma^+$ are simply-connected.

Theorem 1 *There exists a DG-algebra map*

$$CE^* \rightarrow CE_{\parallel}^*$$

which is a quasi-isomorphism when Λ_v is simply-connected for all $v \in \Gamma^+$.

Theorem 1 is restated and proved as Theorem 43 in Section 3.4.

1.1 Partially wrapped Fukaya categories by surgery

Let $\Lambda = \bigsqcup_{v \in \Gamma} \Lambda_v$ be a Legendrian submanifold and $\Gamma = \Gamma^+ \cup \Gamma^-$ as above. Furthermore, we use the notation above for co-core disks and write $CE^* = CE^*(\Lambda)$. An important result that is implicit in [11, Remark 5.9] is the following:

Theorem 2 *Suppose $\Gamma = \Gamma^-$, then there exists a surgery map defined via a holomorphic disk count that gives an A_{∞} -quasi isomorphism between the wrapped Floer cochain complex $CW^* := \bigoplus_{v,w \in \Gamma^-} CW^*(C_v, C_w)$ and the Legendrian DG-algebra CE^* .*

We prove Theorem 2 in Appendix B.2 following [11], referring to [24] for the necessary technical results omitted there. Appendix B.1, also contains a construction of wrapped Floer A_{∞} -algebras that uses only purely holomorphic disks (without Hamiltonian perturbation), and a proof that this agrees with the more standard version defined in [3] which uses Hamiltonian perturbations.

One of the main guiding principles for the results in this paper is that Theorem 2 remains true when Γ^+ is non-empty, provided the Lagrangians C_v are considered as objects of the *partially wrapped* Fukaya category of X_{Λ} , where the non-capped Legendrians Λ_v for $v \in \Gamma^+$ serve as *stops* (cf. [59]). The full proof of this result when Γ^+ is

non-empty can be reduced to the standard surgery result, Theorem 2, and will appear elsewhere. Here we give an outline of a somewhat different and more topological proof, see Appendix B.3. We will use the geometric intuition provided by this viewpoint and our constructions of Legendrian invariants provide a rigorous “working definition” of CE^* even in the case that Γ^+ is non-empty, and also a starting point for the study of “partially wrapped Fukaya categories” via Legendrian surgery (extending the scope of [11] considerably). For future reference, we state this result as a conjecture:

Conjecture 3 *There exists a surgery map defined via moduli spaces of holomorphic disks which gives an A_∞ -quasi isomorphism between the partially wrapped Floer cochain complex $CW^* := \bigoplus_{v,w \in \Gamma} CW^*(C_v, C_w)$ and the DG-algebra CE^* .*

While writing this paper, we learned that Z. Sylvan independently considered a similar conjecture [60] in relation with his theory of partially wrapped Fukaya categories [59].

1.2 Augmentations and infinitesimal Fukaya categories

We keep the notation above and now consider an exact Lagrangian filling L in X of Λ . Such a filling gives an augmentation

$$\epsilon_L: CE^* \rightarrow \mathbf{k}.$$

For chords on components Λ_v , $v \in \Gamma^-$ this is well known and given by a count of rigid disks with one positive puncture and boundary on L_v . For chords on components Λ_v , $v \in \Gamma^+$ the same count composed with the augmentation on $C_{-*}(\Omega\Lambda_v)$ corresponding to the trivial local system gives the augmentation. This allows us to write

$$CE^* = \Omega LC_*$$

for an A_∞ -coalgebra $LC_* = LC_*(\Lambda)$ that we call the *Legendrian A_∞ -coalgebra* (which depends on ϵ_L). Here Ω is the Adams cobar construction. Writing $LA^* := (LC_*)^\#$ for the A_∞ algebra which is the linear dual of LC_* , the following result recovers the Floer cochain complex of L in X_Λ .

Theorem 4 *There exists an A_∞ -quasi isomorphism between the Floer cochain complex in the infinitesimal Fukaya category of X_Λ , $CF^* := CF^*(L_\Lambda)$ and the A_∞ -module homomorphism of the ΩLC_* -module \mathbf{k} , $\mathrm{RHom}_{\Omega LC_*}(\mathbf{k}, \mathbf{k}) \cong LA^*$.*

Note that, by the general properties of bar-cobar constructions (see Section 2.2.1), the algebra $\mathrm{RHom}_{\Omega LC_*}(\mathbf{k}, \mathbf{k})$ is quasi-isomorphic to the graded \mathbf{k} -dual of the bar construction on the algebra ΩLC_* , which can be computed as

$$(1) \quad (\mathrm{B}\Omega LC_*)^\# \cong (LC_*)^\# = LA^*.$$

Remark 5 In case Γ^+ is empty, the A_∞ -algebra LA^* is obtained from the construction in [19] and [10], known as Aug_- category, by adding a copy of \mathbf{k} making it unital.

In case Γ^- is empty, the A_∞ -algebra $LA^* \approx (\text{BCE}^*)^\#$, see (1), is the endomorphism algebra of Λ with the augmentation ϵ_L in the Aug_+ category of [52]. In the setting of microlocal sheaves, a related result was obtained by Nadler [50, Theorem 1.6].

1.3 Duality between partially wrapped and infinitesimal Fukaya categories

We study duality in the setting of the two categories described above: the partially wrapped Fukaya category and the infinitesimal Fukaya category of X_Λ (after surgery), or equivalently, the augmented DG-algebra ΩLC_* and the augmented A_∞ -algebra LA^* (before surgery).

As we have seen in Theorem 4, the augmented DG-algebra ΩLC_* determines the augmented (unital) A_∞ -algebra CF^* . Now, a natural question is to what extent the quasi-isomorphism type of the A_∞ -algebra CF^* determines the quasi-isomorphism type of the augmented Legendrian DG-algebra ΩLC_* .

We emphasize here the phrase “quasi-isomorphism type”: even though it is possible to construct chain models of the A_∞ -algebra LA^* (which is A_∞ -quasi isomorphic to CF^*) and the DG-algebra ΩLC_* by counting “the same” holomorphic disks interpreted in different ways, the two algebras are considered with respect to different equivalence relations, and the resulting equivalence classes can be very different. In particular, it is *not* generally true that $f: \mathcal{C} \rightarrow \mathcal{D}$ being a quasi-isomorphism of A_∞ -coalgebras implies that $\Omega f: \Omega \mathcal{C} \rightarrow \Omega \mathcal{D}$ is a quasi-isomorphism.

We will study this question by (geometrically) constructing a *twisting cochain* (see Section 2.1.4):

$$t: LC_* \rightarrow (\text{BCF}^*)^\#,$$

where B stands for the bar construction and $\#$ is the graded \mathbf{k} -dual. This twisting cochain induces a map of DG-algebras:

$$\Omega LC_* \rightarrow \text{RHom}_{CF^*}(\mathbf{k}, \mathbf{k}),$$

which is a quasi-isomorphism if and only if t is a *Koszul* twisting cochain. For example, we will prove the following result.

Theorem 6 Suppose that LC_* is a locally finite, simply-connected \mathbf{k} -bimodule, then the natural map $\Omega LC_* \rightarrow \text{RHom}_{CF^*}(\mathbf{k}, \mathbf{k})$ is a quasi-isomorphism.

This is an instance of *Koszul duality* between the A_∞ -algebras ΩLC_* and CF^* . It has many useful implications. For example, it implies an isomorphism between Hochschild cohomologies:

$$HH^*(\Omega LC_*, \Omega LC_*) \cong HH^*(CF^*, CF^*).$$

When $\Gamma^+ = \emptyset$, an isomorphism defined via a surgery map [11] was described between *symplectic cohomology*, $SH^* = SH^*(X_\Lambda)$, and the Hochschild cohomology $HH^*(\Omega LC_*, \Omega LC_*)$. Therefore, when duality holds, we obtain a more economical way of computing SH^* .

In the case of cotangent bundles T^*M of simply-connected manifolds M , this recovers a classical result due to Jones [41], which gives:

$$H_{n-*}(\mathcal{L}M) \cong HH^*(C_{-*}(\Omega M), C_{-*}(\Omega M)) \cong HH^*(CF^*(M), CF^*(M)),$$

where M is a simply-connected manifold of dimension n and $\mathcal{L}M$ denotes the free loop space of M .

In Section 6, we give several concrete examples where the duality holds beyond the case of cotangent bundles. For example, the duality holds for plumbings of simply-connected cotangent bundles according to an arbitrary plumbing tree, see Theorem 58.

In another direction, combining duality and Floer cohomology with local coefficients, we establish the following result for relatively spin exact Lagrangian fillings $L \subset X$ with vanishing Maslov class of a Legendrian submanifold $\Lambda \subset Y$.

Theorem 7 *Let $\Gamma = \Gamma^-$ and assume that $SH^*(X) = 0$ and that Λ is simply-connected. If $CE^*(\Lambda)$ is supported in degrees < 0 , then L is simply connected. Moreover, if Λ is a sphere then $CE^*(\Lambda)$ is isomorphic to $C_{-*}(\Omega \bar{L})$, where $\bar{L} = L \cup_\Lambda D$, for a disk D with boundary $\partial D = \Lambda$.*

In general, duality between ΩLC_* and CF^* does not hold (as can be seen for example by looking at cotangent bundles of non-simply connected manifolds or, letting Λ be the standard Legendrian trefoil knot in S^3 filled by a punctured torus). However, there are cases when duality holds even if LC_* is not simply-connected, for instance because of the existence of an auxiliary weight grading, see [31], or for an example in the 1-dimensional case see [45]. It is a very interesting open question to find a geometric characterization of when duality holds.

Remark 8 *Constructions of Legendrian and Lagrangian holomorphic curve invariants require the use of perturbations to achieve transversely cut out moduli spaces. For our*

main invariant CE^* , all moduli spaces used can be shown to be transverse by classical techniques (see Theorem 63) except for the rigid holomorphic planes in X_Λ with a single positive end that are used to anchor the disks in the terminology of [11]. These are also relevant for defining the wrapped Floer cochain complex CW^* without Hamiltonian perturbations and for constructing the surgery map. In all cases, there is a distinguished boundary puncture in the main disk that determines asymptotic markers on the split off planes. Taking this marker into account removes symmetries of the planes and a specific perturbation scheme for transversality of the resulting moduli spaces was constructed in [24]. We will use that perturbation scheme here, see Section A.2 for details.

1.4 An example – the Hopf Link

In this section, we study the example of the Hopf link in order to illustrate our results in a simple and computable example. Some of the algebraic constructions used here are explained in detail only later, see Section 2.

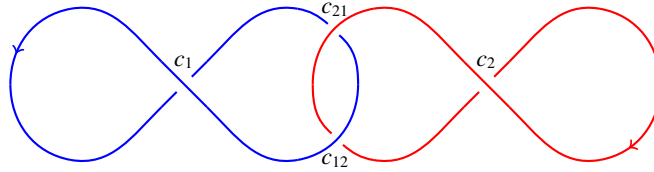


Figure 2: Hopf link when both Γ^+ and Γ^- are non-empty, the blue component lies in Γ^+ and the red in Γ^- .

Let $\Lambda \subset S^3$ be the standard Legendrian Hopf link. We work over $\mathbf{k} = \mathbb{K}e_1 \oplus \mathbb{K}e_2$ and with the Lagrangian filling L given by two disks in D^4 that intersect transversely in a single point. We choose the partition $\Lambda = \Lambda^+ \cup \Lambda^-$. This means that after attaching a Weinstein 2-handle to Λ^- and $T^*(S^1 \times [0, \infty))$ to Λ^+ , we obtain the symplectic manifold X_Λ with Lagrangian skeleton

$$L_\Lambda = S^2 \cup T_{\text{pt}}^*S^2 \subset T^*S^2,$$

or in the terminology of [59], X_Λ is T^*S^2 with wrapping stopped by a Legendrian fiber sphere. The DG-algebra $CE^* = CE^*(\Lambda)$ of Λ has coefficients in

$$C_{-*}(\Omega\Lambda^+)e_1 \oplus \mathbb{K}e_2 \cong \mathbb{K}[t, t^{-1}]e_1 \oplus \mathbb{K}e_2.$$

A free model for $\mathbb{K}[t, t^{-1}]$ is given by the tensor algebra $\mathbb{K}\langle s_1, t_1, k_1, l_1, u_1 \rangle$ with $|s_1| = |t_1| = 0$, $|k_1| = |l_1| = -1$, $|u_1| = -2$ and the differential

$$\begin{aligned} dk_1 &= e_1 - s_1 t_1 \\ dl_1 &= e_1 - t_1 s_1 \\ du_1 &= k_1 s_1 - s_1 l_1 \end{aligned}$$

The subscripts indicate that as \mathbf{k} -module generators s_1, t_1, k_1, l_1, u_1 are annihilated by the idempotent e_2 . Next, incorporating the Reeb chords, with notation as in Figure 2, we get the free algebra

$$\mathbf{k}\langle c_{12}, c_{21}, c_1, c_2, s_1, t_1, k_1, l_1, u_1 \rangle$$

with gradings

$$|u_1| = -2, |c_1| = |c_2| = |k_1| = |l_1| = -1, |c_{12}| = |c_{21}| = |s_1| = |t_1| = 0$$

and differential:

$$\begin{aligned} dc_1 &= e_1 + s_1 + c_{12}c_{21}, \\ dc_2 &= c_{21}c_{12}, \\ dk_1 &= e_1 - s_1 t_1, \\ dl_1 &= e_1 - t_1 s_1, \\ du_1 &= k_1 s_1 - s_1 l_1. \end{aligned}$$

The only augmentation to \mathbf{k} is given by $\epsilon(s_1) = \epsilon(t_1) = -e_1$, and $\epsilon(c_1) = \epsilon(c_2) = \epsilon(c_{12}) = \epsilon(c_{21}) = \epsilon(k_1) = \epsilon(l_1) = \epsilon(u_1) = 0$. After change of variables, $s_1 \rightarrow s_1 - e_1$ and $t_1 \rightarrow t_1 - e_1$, we obtain the free algebra

$$\mathbf{k}\langle c_{12}, c_{21}, c_1, c_2, s_1, t_1, k_1, l_1, u_1 \rangle$$

with non-zero differential on generators:

$$(2) \quad \begin{aligned} dc_1 &= s_1 + c_{12}c_{21}, \\ dc_2 &= c_{21}c_{12}, \\ dk_1 &= s_1 + t_1 - s_1 t_1, \\ dl_1 &= s_1 + t_1 - t_1 s_1, \\ du_1 &= l_1 - k_1 + k_1 s_1 - s_1 l_1. \end{aligned}$$

On the other hand, we can compute the Floer cochains $CF^* = CF^*(L_\Delta)$ of L_Δ as

$$CF^* = \mathbf{k} \oplus \mathbb{K}a_{12} \oplus \mathbb{K}a_{21} \oplus \mathbb{K}a_2, \quad |a_2| = 2, \quad |a_{12}| = |a_{21}| = 1$$

The cohomology level computation follows easily from the geometric picture and general properties of Floer cohomology: L_Λ is a union of a disk D^2 and a sphere S^2 that intersect transversely in one point and we have

$$\begin{aligned} HF^*(D^2, D^2) &= \mathbb{K}e_1 \\ HF^*(S^2, S^2) &= \mathbb{K}e_2 \oplus \mathbb{K}a_2 \\ HF^*(D^2, S^2) &= \mathbb{K}a_{12} \\ HF^*(S^2, D^2) &= \mathbb{K}a_{21} \end{aligned}$$

The only non-trivial product that does not involve idempotents is given by $m_2(a_{12}, a_{21}) = a_2$. For degree reasons, the only possible non-trivial higher products are:

$$m_{2k}(a_{12}, a_{21}, \dots, a_{12}, a_{21}) = ca_2 \text{ for some } k > 1 \text{ and } c \in \mathbb{K}.$$

It turns out that one can take $c = 0$ for all $k > 1$. Indeed, assuming that the A_∞ -structure is strictly unital (which can be arranged up to quasi-isomorphism), consider the A_∞ -relation that involves the term

$$m_2(m_{2k}(a_{12}, a_{21}, \dots, a_{12}, a_{21}), e_2).$$

By induction on $k > 1$, this term has to vanish, which implies $m_{2k}(a_{12}, a_{21}, \dots, a_{12}, a_{21})$ has to vanish for all $k > 1$. Let us confirm this by using the quasi-isomorphism:

$$CF^* \cong \text{RHom}_{CE^*}(\mathbf{k}, \mathbf{k}).$$

We introduce the counital A_∞ -coalgebra

$$LC_* = \mathbf{k} \oplus \mathbb{K}c_{12} \oplus \mathbb{K}c_{21} \oplus \mathbb{K}c_1 \oplus \mathbb{K}c_2 \oplus \mathbb{K}s_1 \oplus \mathbb{K}t_1 \oplus \mathbb{K}k_1 \oplus \mathbb{K}l_1 \oplus \mathbb{K}u_1$$

with $|u_1| = -3, |c_1| = |c_2| = |k_1| = |l_1| = -2, |c_{12}| = |c_{21}| = |s_1| = |t_1| = -1$ for which $\Delta_i = 0$ except for $i = 1$ or 2 where there are the following non-zero terms:

$$\begin{aligned} \Delta_1(c_1) &= s_1, \\ \Delta_1(k_1) &= s_1 + t_1, \\ \Delta_1(l_1) &= s_1 + t_1, \\ \Delta_1(u_1) &= l_1 - k_1. \end{aligned}$$

Write $\Delta_2(x) = 1 \otimes_{\mathbf{k}} x + x \otimes_{\mathbf{k}} 1 + \overline{\Delta}_2(x)$. Then

$$\begin{aligned} \overline{\Delta}_2(c_1) &= c_{12}c_{21}, \\ \overline{\Delta}_2(c_2) &= c_{21}c_{12}, \\ \overline{\Delta}_2(k_1) &= -s_1t_1, \\ \overline{\Delta}_2(l_1) &= -t_1s_1, \\ \overline{\Delta}_2(u_1) &= k_1s_1 - s_1l_1, \end{aligned}$$

where the A_∞ coalgebra operations on LC_* is defined so that ΩLC_* is isomorphic to CE^* . Thus, $\mathrm{RHom}_{CE^*}(\mathbf{k}, \mathbf{k})$ can be computed as the graded dual of LC_* which is the A_∞ -algebra:

$$LA^* = \mathbf{k} \oplus \mathbb{K}c_{12}^\vee \oplus \mathbb{K}c_{21}^\vee \oplus \mathbb{K}c_1^\vee \oplus \mathbb{K}c_2^\vee \oplus \mathbb{K}s_1^\vee \oplus \mathbb{K}t_1^\vee \oplus \mathbb{K}k_1^\vee \oplus \mathbb{K}l_1^\vee \oplus \mathbb{K}u_1^\vee,$$

with gradings

$$|u_1^\vee| = 3, |c_1^\vee| = |c_2^\vee| = |k_1^\vee| = |l_1^\vee| = 2, |c_{12}^\vee| = |c_{21}^\vee| = |s_1^\vee| = |t_1^\vee| = 1,$$

where c^\vee is the linear dual of the generator c of LC_* . The differential is

$$\mathfrak{m}_1(s_1^\vee) = c_1^\vee + k_1^\vee + l_1^\vee, \mathfrak{m}_1(t_1^\vee) = k_1^\vee + l_1^\vee, \mathfrak{m}_1(k_1^\vee) = -u_1, \mathfrak{m}_1(l_1^\vee) = u_1,$$

and the products that do not involve idempotents are

$$\begin{aligned} \mathfrak{m}_2(c_{12}^\vee, c_{21}^\vee) &= c_2^\vee, \mathfrak{m}_2(c_{21}^\vee, c_{12}^\vee) = c_1^\vee, \mathfrak{m}_2(t_1^\vee, s_1^\vee) = -k_1^\vee, \\ \mathfrak{m}_2(s_1^\vee, t_1^\vee) &= -l_1^\vee, \mathfrak{m}_2(k_1^\vee, s_1^\vee) = u_1^\vee, \mathfrak{m}_2(s_1^\vee, l_1^\vee) = -u_1^\vee. \end{aligned}$$

All the higher products vanish. We claim that that this A_∞ -algebra is quasi-isomorphic to the algebra

$$\mathbf{k} \oplus \mathbb{K}a_{12} \oplus \mathbb{K}a_{21} \oplus \mathbb{K}a_2, |a_2| = 2, |a_{12}| = |a_{21}| = 1$$

with the only non-trivial product (not involving idempotents) given by

$$\mathfrak{m}_2(a_{12}, a_{21}) = a_2.$$

Indeed, it is easy to show that the map defined by:

$$c_{12}^\vee \rightarrow a_{12}, c_{21}^\vee \rightarrow a_{21}, c_2^\vee \rightarrow a_2, \text{ and } c_1^\vee, s_1^\vee, t_1^\vee, k_1^\vee, l_1^\vee, u_1^\vee \rightarrow 0$$

is a DG-algebra (hence also A_∞ -algebra) map, which induces an isomorphism at the level of cohomology.

Dually, we can construct a DG-algebra map

$$CE^* \rightarrow \mathrm{RHom}_{CF^*}(\mathbf{k}, \mathbf{k}).$$

The Floer cochain complex CF^* has a unique augmentation given by projection to \mathbf{k} and we compute

$$\mathrm{RHom}_{CF^*}(\mathbf{k}, \mathbf{k}) \cong \Omega CF_*,$$

where CF_* is the coalgebra dual to CF^* . This is the free coalgebra

$$\mathbf{k}\langle a_{12}^\vee, a_{21}^\vee, a_2^\vee \rangle$$

with $|a_{12}^\vee| = |a_{21}^\vee| = 0$ and $|a_2^\vee| = -1$ and the only non-trivial differential not involving counits is

$$\Delta_2(a_2^\vee) = a_{21}^\vee a_{12}^\vee.$$

We have a twisting cochain

$$\mathfrak{t}: LC_* \rightarrow \Omega CF_*$$

given by

$$\begin{aligned} \mathfrak{t}(c_2) &= a_2^\vee, \quad \mathfrak{t}(c_{12}) = a_{12}^\vee, \quad \mathfrak{t}(c_{21}) = a_{21}^\vee, \\ \mathfrak{t}(c_1) &= 0, \quad \mathfrak{t}(s_1) = -a_{12}^\vee a_{21}^\vee, \quad \mathfrak{t}(t_1) = a_{12}^\vee a_{21}^\vee, \\ \mathfrak{t}(l_1) &= \mathfrak{t}(k_1) = a_{12}^\vee a_2^\vee a_{21}^\vee, \quad \mathfrak{t}(u_1) = a_{12}^\vee a_2^\vee a_2^\vee a_{21}^\vee. \end{aligned}$$

This means that \mathfrak{t} satisfies the following equations:

$$\begin{aligned} d\mathfrak{t}(c_1) &= \mathfrak{t}(s_1) + \mathfrak{t}(c_{12})\mathfrak{t}(c_{21}), \\ d\mathfrak{t}(c_2) &= \mathfrak{t}(c_{21})\mathfrak{t}(c_{12}), \\ d\mathfrak{t}(k_1) &= \mathfrak{t}(s_1) + \mathfrak{t}(t_1) - \mathfrak{t}(s_1)\mathfrak{t}(t_1), \\ d\mathfrak{t}(l_1) &= \mathfrak{t}(s_1) + \mathfrak{t}(t_1) - \mathfrak{t}(t_1)\mathfrak{t}(s_1), \\ d\mathfrak{t}(u_1) &= \mathfrak{t}(l_1) - \mathfrak{t}(k_1) + \mathfrak{t}(k_1)\mathfrak{t}(s_1) - \mathfrak{t}(s_1)\mathfrak{t}(l_1). \end{aligned}$$

Hence, it induces a DG-algebra map

$$\Omega LC_* \rightarrow \Omega CF_*.$$

We have not checked whether this is a quasi-isomorphism, or equivalently whether \mathfrak{t} is a *Koszul* twisting cochain. Note, however, the DG-algebra map $\Omega CF_* \rightarrow \Omega LC_*$ defined by

$$a_2^\vee \rightarrow c_2, a_{12}^\vee \rightarrow c_{12}, a_{21}^\vee \rightarrow c_{21},$$

shows that \mathfrak{t} is a retraction, and ΩCF_* is a retract of ΩLC_* .

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2 Algebraic preliminary

In this section, we review the homological algebra we use in our study of various invariants associated to Legendrian submanifolds and their Lagrangian fillings. Most of this material is well established, see [44] and also [55], [54], [37], [46], [47]. Note though that our sign conventions follow [57], see Remark 9.

2.1 A_∞ -algebras and A_∞ -coalgebras

In this section we will discuss the basic algebraic objects we use. These are modules over a ground ring \mathbf{k} of the following form. Fix a coefficient field \mathbb{K} (of arbitrary characteristic) and let \mathbf{k} be a semi-simple ring of the form:

$$\mathbf{k} = \bigoplus_{v \in \Gamma} \mathbb{K}e_v$$

where $e_v^2 = e_v$ and $e_v e_w = 0$ for $v \neq w$, and where the index set Γ is finite.

We will use \mathbb{Z} -graded \mathbf{k} -bimodules. If $M = \bigoplus_i M^i$ is such a module then we call M *connected* if $M^0 \cong \mathbf{k}$ and either $M^i = 0$ for all $i > 0$, or $M^i = 0$ for all $i < 0$, and we call M *simply-connected* if, in addition, in the former case $M^{-1} = 0$ and in the latter $M^1 = 0$. Further, we say that M is *locally finite* if each M^i is finitely generated as a \mathbf{k} -bimodule.

We have the usual shifting and tensor product operations on modules. If $M = \bigoplus_i M^i$ is a graded \mathbf{k} -bimodule and s is an integer then we let the corresponding shifted module $M[s] = \bigoplus_i M[s]^i$ be the module with graded components

$$M[s]^i = M^{i+s}.$$

If $N = \bigoplus_i N^i$ is another graded \mathbf{k} -bimodule then $M \otimes_{\mathbf{k}} N = \bigoplus_k (M \otimes_{\mathbf{k}} N)^k$ is naturally a graded \mathbf{k} -bimodule with

$$(M \otimes_{\mathbf{k}} N)^k = \bigoplus_{i+j=k} M^i \otimes_{\mathbf{k}} N^j.$$

For iterated tensor products we write

$$M^{\otimes_{\mathbf{k}} r} = \underbrace{M \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} M}_r.$$

Our modules will often have further structure as \mathbb{Z} -graded A_∞ -algebras and A_∞ -coalgebras over \mathbf{k} , see Sections 2.1.1 and 2.1.2. The modules are then in particular chain complexes with a differential and we will use *cohomological* grading throughout. That is, the differential *increases* the grading by 1. For example, if L is a topological space then its cohomology complex $C^*(L)$ is supported in non-negative grading, while the homology complex $C_{-*}(L)$ is supported in non-positive degrees. To be consistent with this, we denote the grading as a subscript (resp. superscript) when the underlying chain complex has a coalgebra (resp. algebra) structure.

2.1.1 A_∞ -algebras

An A_∞ -algebra over \mathbf{k} is a \mathbb{Z} -graded \mathbf{k} -module \mathcal{A} with a collection of grading preserving \mathbf{k} -linear maps

$$m_i: \mathcal{A}^{\otimes_{\mathbf{k}} i} \rightarrow \mathcal{A}[2-i],$$

for all integers $i \geq 1$, that satisfies the A_∞ -relations:

$$(3) \quad \sum_{i,j} (-1)^{|a_1|+\dots+|a_j|-j} m_{d-i+1}(a_d, \dots, a_{j+i+1}, m_i(a_{j+i}, \dots, a_{j+1}), a_j, \dots, a_1) = 0,$$

for all d .

Remark 9 We follow the sign conventions of [57]. Note that even though, m_i is written on the left of $(a_{j+i}, \dots, a_{j+1})$, the sign convention is so that m_i acts from the right. To be consistent, we will insist that all our operators act on the right independently of how they are written. This convention and the usual Koszul sign exchange rule determine the signs that appear in our formulas.

A DG-algebra over \mathbf{k} is an A_∞ -algebra \mathcal{A} such that $m_i = 0$ for $i \geq 3$. In this case, we call the first two operations the *differential* and the *product*, respectively, and use the following adjustments to obtain an (ordinary) differential graded algebra:

$$(4) \quad da = (-1)^{|a|} m_1(a) \quad \text{and} \quad a_2 a_1 = (-1)^{|a_1|} m_2(a_2, a_1).$$

In particular, the product is then associative and the graded Leibniz rule for d holds:

$$(5) \quad d(a_2 a_1) = (da_2)a_1 + (-1)^{|a_2|} a_2(da_1).$$

An A_∞ -map $\epsilon: \mathcal{A} \rightarrow \mathcal{B}$ between A_∞ -algebras \mathcal{A} and \mathcal{B} over \mathbf{k} , with operations m_i and n_i , $i \geq 1$, respectively, is a collection of \mathbf{k} -linear grading preserving maps

$$\epsilon_i: \mathcal{A}^{\otimes_{\mathbf{k}} i} \rightarrow \mathcal{B}[1-i],$$

$i \geq 1$, that satisfies the relations

$$\begin{aligned} & \sum_{i,j} (-1)^{|a_1|+\dots+|a_j|-j} \epsilon_{d-i+1}(a_d, \dots, a_{j+i+1}, m_i(a_{j+i}, \dots, a_{j+1}), a_j, \dots, a_1) = \\ & \sum_{\substack{1 \leq j \leq d, \\ 0 < i_1 < i_2 < \dots < i_j < d}} n_j(\epsilon_{d-i_j}(a_d, \dots, a_{d-i_j}), \dots, \epsilon_{i_2-i_1}(a_{i_2}, \dots, a_{i_1+1}), \epsilon_{i_1}(a_{i_1}, \dots, a_1)). \end{aligned}$$

An A_∞ -map $\epsilon: \mathcal{A} \rightarrow \mathcal{B}$ is called an A_∞ -quasi isomorphism if the map on cohomology $H^*(\mathcal{A}) \rightarrow H^*(\mathcal{B})$ induced by ϵ^1 is an isomorphism.

We say that an A_∞ -algebra \mathcal{A} is *strictly unital* if there is an element $1_{\mathcal{A}} \in \mathcal{A}$ such that $m_1(1_{\mathcal{A}}) = 0$, $m_2(1_{\mathcal{A}}, a) = m_2(a, 1_{\mathcal{A}}) = a$ for any $a \in \mathcal{A}$, and such that m_i , $i > 2$ annihilates any monomial containing $1_{\mathcal{A}}$ as a factor. Any A_∞ -algebra \mathcal{A} which has a cohomological unit, a cocycle representing the identity element in $H^*(\mathcal{A})$, is quasi-isomorphic to a strictly unital A_∞ -algebra.

An *augmentation* of a strictly unital A_∞ -algebra is an A_∞ -map $\epsilon: \mathcal{A} \rightarrow \mathbf{k}$, where \mathbf{k} is considered as a strictly unital A_∞ -algebra in degree 0 with trivial differential and higher A_∞ -products, and such that $\epsilon_1(1_{\mathcal{A}}) = 1_{\mathbf{k}}$, ϵ_i for $i > 1$ annihilates any monomial containing $1_{\mathcal{A}}$. An augmentation is called *strict* if $\epsilon_i = 0$ for $i > 1$. The category of augmented, strictly unital A_∞ -algebras is equivalent to the category of strictly augmented, strictly unital A_∞ -algebras (see [54, Section 7.2]).

2.1.2 A_∞ -coalgebras

An A_∞ -coalgebra \mathcal{C} over \mathbf{k} is a \mathbb{Z} -graded \mathbf{k} -module with a collection of \mathbf{k} -linear grading preserving maps:

$$\Delta_i: \mathcal{C} \rightarrow \mathcal{C}^{\otimes_{\mathbf{k}} i}[2 - i],$$

for all integers $i \geq 1$, with the following properties. The maps satisfy the co- A_∞ -relations:

$$(6) \quad \sum_{i=1}^d \sum_{j=0}^{d-i} (\mathbf{1}^{\otimes_{\mathbf{k}}(d-i-j)} \otimes_{\mathbf{k}} \Delta_i \otimes_{\mathbf{k}} \mathbf{1}^{\otimes_{\mathbf{k}} j}) \Delta_{d-i+1} = 0,$$

where,

$$\begin{aligned} \mathbf{1}^{\otimes_{\mathbf{k}}(d-i-j)} \otimes_{\mathbf{k}} \Delta_i \otimes_{\mathbf{k}} \mathbf{1}^{\otimes_{\mathbf{k}} j} (c_{d-i+1}, \dots, c_1) &= \\ (-1)^{|c_1| + \dots + |c_j| - j} (c_{d-i+1}, \dots, c_{j+2}) \otimes_{\mathbf{k}} \Delta_i(c_{j+1}) \otimes_{\mathbf{k}} (c_j, \dots, c_1) & \\ \in \mathcal{C}^{\otimes_{\mathbf{k}}(d-i-j)} \otimes_{\mathbf{k}} \mathcal{C}^{\otimes_{\mathbf{k}} i} \otimes_{\mathbf{k}} \mathcal{C}^{\otimes_{\mathbf{k}} j}. & \end{aligned}$$

Furthermore, the degree 1 map

$$\mathcal{C}[-1] \rightarrow \prod_{i=1}^{\infty} \mathcal{C}[-1]^{\otimes_{\mathbf{k}} i},$$

with i^{th} component equal to Δ_i , factorizes through the natural inclusion

$$\bigoplus_{i=1}^{\infty} \mathcal{C}[-1]^{\otimes_{\mathbf{k}} i} \rightarrow \prod_{i=1}^{\infty} \mathcal{C}[-1]^{\otimes_{\mathbf{k}} i},$$

of the direct sum into the direct product.

A *DG-coalgebra* over \mathbf{k} is an A_∞ -coalgebra such that $\Delta_i = 0$ for $i \geq 3$. In this case, we call the first two operations the *differential* and the *coproduct*, respectively, and use the following adjustments to obtain an (ordinary) differential graded coalgebra:

$$(7) \quad \theta c = (-1)^{|c|} \Delta_1(c) \quad \text{and} \quad \Delta(c) = \sum (-1)^{|c_{(2)}|} c_{(1)} \otimes_{\mathbf{k}} c_{(2)},$$

where we write $\Delta_2(c) = \sum c_{(1)} \otimes_{\mathbf{k}} c_{(2)}$.

In particular, the coproduct is coassociative (i.e. $(\Delta \otimes_{\mathbf{k}} \mathbf{1}) \circ \Delta = (\mathbf{1} \otimes_{\mathbf{k}} \Delta) \circ \Delta$) and the graded co-Leibniz rule holds:

$$(8) \quad \Delta \theta(c) = \sum (-1)^{|c_{(1)}|} c_{(1)} \otimes_{\mathbf{k}} \theta(c_{(2)}) + \theta(c_{(1)}) \otimes_{\mathbf{k}} c_{(2)}.$$

A DG-coalgebra \mathcal{C} is called *conilpotent* (also called *cocomplete*) if for any $c \in \mathcal{C}$, there exists an $n \geq 2$ such that c is in the kernel of the iterated comultiplication map defined recursively by $\Delta^{(2)} = \Delta$ and $\Delta^{(n)} = (\mathbf{1}^{\otimes_{\mathbf{k}}(n-2)} \otimes_{\mathbf{k}} \Delta) \circ \Delta^{(n-1)}$, for $n > 2$.

An A_∞ -comap $f: \mathcal{C} \rightarrow \mathcal{D}$ between A_∞ -coalgebras \mathcal{C} and \mathcal{D} over \mathbf{k} , with operations Δ_i and Θ_i , $i \geq 1$, respectively, is a collection of \mathbf{k} -linear grading preserving maps

$$f_i: \mathcal{C} \rightarrow \mathcal{D}^{\otimes_{\mathbf{k}} i}[1 - i],$$

$i \geq 1$, that satisfies the relations:

$$\begin{aligned} & \sum_{i=1}^d \sum_{j=0}^{d-i} (\mathbf{1}^{\otimes_{\mathbf{k}}(d-i-j)} \otimes_{\mathbf{k}} \Theta_i \otimes_{\mathbf{k}} \mathbf{1}^{\otimes_{\mathbf{k}} j}) f_{d-i+1} = \\ & \sum_{\substack{1 \leq j \leq d, \\ 0 < i_1 < i_2 < \dots < i_j < d}} (f_{d-i_j} \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} f_{i_2-i_1} \otimes_{\mathbf{k}} f_{i_1}) \Delta_j. \end{aligned}$$

where,

$$\begin{aligned} & \mathbf{1}^{\otimes_{\mathbf{k}}(d-i-j)} \otimes_{\mathbf{k}} \Theta_i \otimes_{\mathbf{k}} \mathbf{1}^{\otimes_{\mathbf{k}} j} (d_{d-i+1}, \dots, d_1) = \\ & (-1)^{|d_1| + \dots + |d_j| - j} (d_{d-i+1}, \dots, d_{j+2}) \otimes_{\mathbf{k}} \Theta_i(d_{j+1}) \otimes_{\mathbf{k}} (d_j, \dots, d_1) \\ & \in \mathcal{D}^{\otimes_{\mathbf{k}}(d-i-j)} \otimes_{\mathbf{k}} \mathcal{D}^{\otimes_{\mathbf{k}} i} \otimes_{\mathbf{k}} \mathcal{D}^{\otimes_{\mathbf{k}} j}. \end{aligned}$$

Furthermore, the degree 0 map

$$\mathcal{C}[-1] \rightarrow \prod_{i=1}^{\infty} \mathcal{D}[-1]^{\otimes_{\mathbf{k}} i}$$

with i^{th} component equal to f_i , factorizes through the natural inclusion:

$$(9) \quad \bigoplus_{i=1}^{\infty} \mathcal{D}[-1]^{\otimes_{\mathbf{k}} i} \rightarrow \prod_{i=1}^{\infty} \mathcal{D}[-1]^{\otimes_{\mathbf{k}} i},$$

of the direct sum into the direct product.

An A_∞ -comap $f: \mathcal{C} \rightarrow \mathcal{D}$ is called an A_∞ -*quasi isomorphism* if the map on cohomology $H^*(\mathcal{C}) \rightarrow H^*(\mathcal{D})$ induced by f^1 is an isomorphism.

We say that an A_∞ -coalgebra is *strictly counital* if there exists a \mathbf{k} -linear map $\epsilon: \mathcal{C} \rightarrow \mathbf{k}$ such that $(\epsilon \otimes \mathbf{1})\Delta_2 = (\mathbf{1} \otimes \epsilon)\Delta_2 = \mathbf{1}$ and $(\mathbf{1}^{\otimes_{\mathbf{k}}(i-j)} \otimes_{\mathbf{k}} \epsilon \otimes_{\mathbf{k}} \mathbf{1}^{\otimes_{\mathbf{k}}j-1})\Delta_i = 0$ for all $i \neq 2$ and j . Any A_∞ -coalgebra \mathcal{C} which has a cohomological counit, a cocycle representing the counit in $H^*(\mathcal{C})$, is quasi-isomorphic to a strictly counital A_∞ -coalgebra.

A *coaugmentation* of a strictly counital A_∞ -coalgebra \mathcal{C} is an A_∞ -comap $\eta: \mathbf{k} \rightarrow \mathcal{C}$, where \mathbf{k} is considered as a vector space in degree 0 with the trivial A_∞ -coalgebra structure, such that $\epsilon\eta_1 = \mathbf{1}_{\mathbf{k}}$ and $(\mathbf{1}^{\otimes_{\mathbf{k}}(i-j)} \otimes_{\mathbf{k}} \epsilon \otimes_{\mathbf{k}} \mathbf{1}^{\otimes_{\mathbf{k}}j-1})\eta_i = 0$ for all $i > 1$ and j . The coaugmentation is called *strict* if $\eta_i = 0$ for $i \geq 2$.

2.1.3 Graded dual

We next discuss the graded dual of a graded \mathbf{k} -module. Since we are working over a noncommutative ring \mathbf{k} , there are two \mathbf{k} -linear duals (cf. [8]).

If \mathcal{A} is a graded \mathbf{k} -module, $\mathcal{A} = \bigoplus_i \mathcal{A}_i$, then the *graded duals* $\mathcal{A}^\# = \bigoplus_i (\mathcal{A}^\#)_i$ and ${}^\#\mathcal{A} = \bigoplus_i ({}^\#\mathcal{A})_i$ are defined as follows. The graded components $(\mathcal{A}^\#)_i$ of $\mathcal{A}^\#$ are left \mathbf{k} -module maps:

$$\text{hom}_{\mathbf{k}}(\mathcal{A}_{-i}, \mathbf{k}),$$

and the \mathbf{k} -bimodule structure on $\mathcal{A}^\#$ is given as follows: if $e_v, e_w \in \mathbf{k}$, $a \in (\mathcal{A}^\#)_i$, and $c \in \mathcal{A}_{-i}$, then

$$(10) \quad (e_v \cdot a \cdot e_w)(c) = a(c e_v) e_w.$$

The graded components $({}^\#\mathcal{A})_i$ of ${}^\#\mathcal{A}$ in degree i are right \mathbf{k} -module maps, which we write as:

$$\text{hom}_{-\mathbf{k}}(\mathcal{A}_{-i}, \mathbf{k}),$$

and the \mathbf{k} -bimodule structure is given by: if $e_v, e_w \in \mathbf{k}$, $a \in ({}^\#\mathcal{A})_i$, and $c \in \mathcal{A}_{-i}$, then

$$(11) \quad (e_v \cdot a \cdot e_w)(c) = e_v a(e_w c).$$

Both canonical maps $\mathcal{A} \rightarrow ({}^\#\mathcal{A}^\#)$ and $\mathcal{A} \rightarrow ({}^\#\mathcal{A})^\#$ are \mathbf{k} -bimodule maps, which are isomorphisms if \mathcal{A} is locally finite.

If V_1, V_2, \dots, V_n are \mathbf{k} -bimodules, there is a natural map

$$V_n^\# \otimes_{\mathbf{k}} V_{n-1}^\# \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} V_1^\# \rightarrow (V_1 \otimes_{\mathbf{k}} V_2 \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} V_n)^\#$$

given by:

$$(12) \quad (a_n \otimes a_{n-1} \otimes \dots \otimes a_1)(c_1 \otimes c_2 \otimes \dots \otimes c_n) := a_1(c_1 a_2(c_2 \dots a_n(c_n) \dots)).$$

Similarly, there is a natural map

$${}^{\#}V_n \otimes_{\mathbf{k}} {}^{\#}V_{n-1} \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} {}^{\#}V_1 \rightarrow {}^{\#}(V_1 \otimes_{\mathbf{k}} V_2 \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} V_n),$$

given by:

$$(13) \quad (a_n \otimes a_{n-1} \otimes \dots \otimes a_1)(c_1 \otimes c_2 \otimes \dots \otimes c_n) := a_n(\dots a_2(a_1(c_1)c_2) \dots c_n).$$

These give the graded duals $\mathcal{C}^{\#}$ and ${}^{\#}\mathcal{C}$ of a coaugmented A_{∞} -coalgebra \mathcal{C} the structure of augmented A_{∞} -algebras with structure maps defined by:

$$(14) \quad \mathfrak{m}_i(a_i, \dots, a_1)(c) := (-1)^{|c|}(a_i \otimes \dots \otimes a_1)\Delta_i(c).$$

Note that to get a non-zero product, we must have $|\mathfrak{m}_i(a_i, \dots, a_1)| = |c|$, hence the sign $(-1)^{|c|}$ equals the sign $(-1)^{|a_1|+\dots+|a_i|-i}$.

In general, there is no natural way of equipping the graded dual of an augmented A_{∞} -algebra with an A_{∞} -coalgebra structure. However, if the grading on \mathcal{A} is locally finite (i.e. \mathcal{A}_i are finitely generated as \mathbf{k} -bimodules), then it follows that

$$\begin{aligned} \mathcal{A}^{\#} \otimes_{\mathbf{k}} \mathcal{A}^{\#} \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} \mathcal{A}^{\#} &\cong (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} \mathcal{A})^{\#}, \\ {}^{\#}\mathcal{A} \otimes_{\mathbf{k}} {}^{\#}\mathcal{A} \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} {}^{\#}\mathcal{A} &\cong {}^{\#}(\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} \mathcal{A}). \end{aligned}$$

Therefore, the graded duals $\mathcal{A}^{\#}$ and ${}^{\#}\mathcal{A}$ of an augmented A_{∞} -algebra \mathcal{A} with locally finite grading are naturally coaugmented A_{∞} -coalgebras.

2.1.4 Twisting cochains

Let $(\mathcal{C}, \Delta_{\bullet})$ be an A_{∞} -coalgebra and let $(\mathcal{A}, \mathfrak{m}_1, \mathfrak{m}_2)$ be a DG-algebra. A *twisting cochain* is a \mathbf{k} -linear map $\mathfrak{t}: \mathcal{C} \rightarrow \mathcal{A}$ of degree 1 that satisfies:

$$(15) \quad \mathfrak{m}_1 \circ \mathfrak{t} - \mathfrak{t} \circ \Delta_1 + \sum_{d \geq 2} (-1)^d \mathfrak{m}_2^{(d)} \circ \mathfrak{t}^{\otimes_{\mathbf{k}} d} \circ \Delta_d = 0,$$

where $\mathfrak{m}_2^{(2)} := \mathfrak{m}_2$, and $\mathfrak{m}_2^{(d)} := \mathfrak{m}_2 \circ (\text{Id}_{\mathcal{A}} \otimes_{\mathbf{k}} \mathfrak{m}_2^{(d-1)})$. Note that for $c \in \mathcal{C}$, $\Delta_i(c) \neq 0$ for only finitely many i , and hence the potentially infinite sum in (15) is actually finite when it acts on c .

If the coalgebra \mathcal{C} is coaugmented by $\eta: \mathbf{k} \rightarrow \mathcal{C}$ and the algebra \mathcal{A} is augmented $\epsilon: \mathcal{A} \rightarrow \mathbf{k}$, we require in addition that its twisting cochains \mathfrak{t} are compatible in the sense that

$$(16) \quad \mathfrak{t} \circ \eta = \epsilon \circ \mathfrak{t} = 0.$$

We denote the set of twisting cochains from \mathcal{C} to \mathcal{A} by $\text{Tw}(\mathcal{C}, \mathcal{A})$.

Let $\mathfrak{t} \in \text{Tw}(\mathcal{C}, \mathcal{A})$ be a twisting cochain. Consider the twisted tensor product $\mathcal{A} \otimes_{\mathbf{k}}^{\mathfrak{t}} \mathcal{C}$ as a chain complex with differential $d^{\mathfrak{t}}: \mathcal{A} \otimes_{\mathbf{k}}^{\mathfrak{t}} \mathcal{C} \rightarrow \mathcal{A} \otimes_{\mathbf{k}}^{\mathfrak{t}} \mathcal{C}$ defined as follows,

$$(17) \quad d^{\mathfrak{t}} = \mathfrak{m}_1 \otimes_{\mathbf{k}} \text{Id}_{\mathcal{C}} + \text{Id}_{\mathcal{A}} \otimes_{\mathbf{k}} \Delta_1 + \sum_{d \geq 2} (\mathfrak{m}_2^{(d)} \otimes \text{Id}_{\mathcal{C}}) \circ (\text{Id}_{\mathcal{A}} \otimes_{\mathbf{k}} \mathfrak{t}^{\otimes_{\mathbf{k}} d-1} \otimes_{\mathbf{k}} \text{Id}_{\mathcal{C}}) \circ (\text{Id}_{\mathcal{A}} \otimes_{\mathbf{k}} \Delta_d).$$

Here the differential squares to zero, $d^{\mathfrak{t}} \circ d^{\mathfrak{t}} = 0$, since \mathfrak{t} satisfies (15). This complex is the *Koszul complex* associated with \mathfrak{t} . It is called acyclic if the projection to \mathbf{k} is a quasi-isomorphism.

Note that one also has an analogous complex of the form $\mathcal{C} \otimes_{\mathbf{k}}^{\mathfrak{t}} \mathcal{A}$.

The \mathbb{K} -vector space of \mathbf{k} -bimodule morphisms $\text{hom}_{\mathbf{k}-\mathbf{k}}(\mathcal{C}, \mathcal{A})$ carries an A_{∞} -algebra structure with operations \mathfrak{n}_d , $d \geq 1$ given by

$$\mathfrak{n}_1(t) = \mathfrak{m}_1 \circ t + (-1)^{|t|} t \circ \Delta_1$$

and

$$\mathfrak{n}_d(t_d, t_{d-1}, \dots, t_1) = (-1)^{d(|t_d| + \dots + |t_1|)} \mathfrak{m}_2^{(d)} \circ (t_d \otimes t_{d-1} \otimes \dots \otimes t_1) \circ \Delta_d, \quad d \geq 2.$$

where the composition $(t_d \otimes t_{d-1} \otimes \dots \otimes t_1) \circ \Delta_d$ is defined component-wise. Thus, if $\Delta_d(c) = c_d \otimes \dots \otimes c_1$, then

$$(t_d \otimes_{\mathbf{k}} t_{d-1} \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} t_1) \Delta_d(c) = (-1)^{\dagger} \mathfrak{t}_d(c_d) \otimes_{\mathbf{k}} \mathfrak{t}_{d-1}(c_{d-1}) \dots \otimes_{\mathbf{k}} \mathfrak{t}_1(c_1),$$

where $\dagger = \sum_{j=2}^d \sum_{i=1}^{j-1} |c_i| |t_j|$. In this setting, a twisting cochain $\mathfrak{t}: \mathcal{C} \rightarrow \mathcal{A}$ corresponds to a solution of the Maurer-Cartan equation:

$$(18) \quad \sum_{i \geq 1} \mathfrak{n}_i(\mathfrak{t}, \mathfrak{t}, \dots, \mathfrak{t}) = 0.$$

(As before, this sum is effectively finite since for any $c \in \mathcal{C}$, $\Delta_i(c) \neq 0$ only for finitely many i .)

In particular, a twisting cochain $\mathfrak{t}: \mathcal{C} \rightarrow \mathcal{A}$ defines a twisted A_{∞} -structure on $\text{hom}_{\mathbf{k}}(\mathcal{C}, \mathcal{A})$, with operations $\mathfrak{n}_d^{\mathfrak{t}}$ given by

$$\mathfrak{n}_d^{\mathfrak{t}}(t_d, t_{d-1}, \dots, t_1) = \sum_{l_i \geq 0} \mathfrak{n}_{d+l_0+l_1+\dots+l_d}(\overbrace{\mathfrak{t}, \dots, \mathfrak{t}}^{l_d}, t_d, \overbrace{\mathfrak{t}, \dots, \mathfrak{t}}^{l_{d-1}}, t_{d-1}, \dots, t_1, \overbrace{\mathfrak{t}, \dots, \mathfrak{t}}^{l_0}).$$

We will denote this twisted A_{∞} -structure $\text{hom}_{\mathbf{k}}^{\mathfrak{t}}(\mathcal{C}, \mathcal{A})$.

There are direct analogs of the above construction if we instead consider a DG-coalgebra $(\mathcal{C}, \Delta_1, \Delta_2)$ and an A_∞ -algebra \mathcal{A} with operations m_j . The module $\text{hom}_{\mathbf{k}-\mathbf{k}}(\mathcal{C}, \mathcal{A})$ has the structure of an A_∞ -algebra with operations n_d given by:

$$n_1(t) = m_1 \circ t + (-1)^{|t|} t \circ \Delta_1$$

and

$$n_d(t_d, t_{d-1}, \dots, t_1) = m_d \circ (t_d \otimes t_{d-1} \otimes \dots \otimes t_1) \circ \Delta_2^{(d)}, \quad d \geq 2.$$

To make sense of the twisting cochain equation (18), one needs to make additional assumptions to ensure the convergence of the infinite sum. This holds, for example, if \mathcal{C} is conilpotent.

We remark that if both \mathcal{C} and \mathcal{A} are A_∞ -(co)algebras, then defining a twisting cochain is a more complicated matter (cf. [55, Introduction]). We will not need this here.

2.2 Bar-cobar duality for A_∞ -(co)algebras

In this section we first introduce the bar and cobar constructions and then discuss basic relations between them.

2.2.1 Bar and cobar constructions

Let $(\mathcal{A}, \{m_j\}_{j \geq 1})$ be a strictly unital A_∞ -algebra with a strict augmentation $\epsilon: \mathcal{A} \rightarrow \mathbf{k}$. Define the *augmentation ideal* $\overline{\mathcal{A}} = \ker(\epsilon)$. Note that if we are given a non-unital A_∞ -algebra $\overline{\mathcal{A}}$, then we can turn it into a strictly unital A_∞ -algebra $\mathcal{A} := \mathbf{k} \oplus \overline{\mathcal{A}}$ with an augmentation given by projection to \mathbf{k} .

We next recall the construction of the (reduced) *bar construction* $B\mathcal{A}$. For any augmented A_∞ -algebra \mathcal{A} , $B\mathcal{A}$ is a coaugmented conilpotent DG-coalgebra. As a coaugmented coalgebra $B\mathcal{A}$ is defined as

$$B\mathcal{A} = \mathbf{k} \oplus \overline{\mathcal{A}}[1] \oplus \overline{\mathcal{A}}[1]^{\otimes_{\mathbf{k}} 2} \oplus \dots,$$

where $[1]$ denotes the downwards shift by 1. We write a typical monomial using Eilenberg and MacLane's notation:

$$[a_d | a_{d-1} | \dots | a_1] = sa_d \otimes_{\mathbf{k}} sa_{d-1} \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} sa_1,$$

where for $a \in \overline{\mathcal{A}}$, $sa \in \overline{\mathcal{A}}[1]$ denotes the corresponding element in $\overline{\mathcal{A}}[1]$ with degree shifted down by 1.

The differential $b: \mathbf{B}\mathcal{A} \rightarrow \mathbf{B}\mathcal{A}$ is defined to vanish on $\mathbf{k} \subset \mathcal{A}$, $b|_{\mathbf{k}} = 0$, and as follows on monomials:

$$b([a_d|a_{d-1}|\dots|a_1]) = \sum_{i,j} (-1)^{|a_1|+\dots+|a_j|-j} [a_d|\dots|a_{j+i+1}|\mathbf{m}_i(a_{j+i},\dots,a_{j+1})|a_j|\dots|a_1],$$

where strictness of the augmentation ensures that $\mathbf{m}_i(a_{j+i},\dots,a_{j+1}) \in \overline{\mathcal{A}}[1]$ for $(a_{j+i},\dots,a_{j+1}) \in \overline{\mathcal{A}}[1]^i$. (If the augmentation is not strict, one applies a quasi-isomorphism of \mathcal{A} to make it strict.)

The coproduct $\Delta_2: \mathbf{B}\mathcal{A} \rightarrow \mathbf{B}\mathcal{A} \otimes_{\mathbf{k}} \mathbf{B}\mathcal{A}$ is defined by

$$\Delta_2([a_d|a_{d-1}|\dots|a_1]) = \sum_{i=0}^d (-1)^{|a_i|+\dots+|a_1|-i} [a_d|a_{d-1}|\dots|a_{i+1}] \otimes_{\mathbf{k}} [a_i|a_{i-1}|\dots|a_1].$$

The slightly unusual sign $(-1)^{|a_i|+\dots+|a_1|-i}$ appears as a consequence of the following two facts:

- (i) The equation $b^2 = 0$ is equivalent to the A_∞ -relations (3) for $(\mathbf{m}_i)_{i \geq 1}$,
- (ii) The pair (b, Δ_2) satisfies the co- A_∞ -relations (6).

Redefining (b, Δ_2) to (θ, Δ) using (7) removes the sign in Δ_2 , and (θ, Δ) becomes a (usual) coassociative DG-coalgebra, where the co-Leibniz rule (8) holds. The coaugmentation $\eta: \mathbf{k} \rightarrow \mathbf{B}\mathcal{A}$ is defined by letting η_1 be the inclusion of \mathbf{k} and $\eta_i = 0$ for $i > 0$.

There is an increasing, exhaustive, and bounded below (hence, complete Hausdorff) filtration on the complex $\mathbf{B}\mathcal{A}$:

$$\mathbf{k} = \mathcal{F}^0 \mathbf{B}\mathcal{A} \subset \mathcal{F}^1 \mathbf{B}\mathcal{A} \subset \dots \subset \mathbf{B}\mathcal{A},$$

where

$$\mathcal{F}^p \mathbf{B}\mathcal{A} := \mathbf{k} \oplus \overline{\mathcal{A}}[1] \oplus \dots \oplus \overline{\mathcal{A}}[1]^{\otimes_{\mathbf{k}} p}.$$

This induces the *word-length spectral sequence* with

$$E_1^{p,q} = H^{p+q}(\mathcal{F}^p \mathbf{B}\mathcal{A} / \mathcal{F}^{p-1} \mathbf{B}\mathcal{A})$$

converging strongly to

$$E_\infty^{p,q} = \mathcal{F}^p H^{p+q}(\mathbf{B}\mathcal{A}) / \mathcal{F}^{p-1} H^{p+q}(\mathbf{B}\mathcal{A})$$

by the classical convergence theorem ([62, Theorem 5.5.1]). It can be proved using this spectral sequence that an A_∞ -map $\epsilon: \mathcal{A} \rightarrow \mathcal{B}$ is a quasi-isomorphism if and only if the naturally induced DG-coalgebra map $\mathbf{B}\epsilon: \mathbf{B}\mathcal{A} \rightarrow \mathbf{B}\mathcal{B}$ is a quasi-isomorphism, see [46, Proposition 2.2.3].

There is a universal twisting cochain $t_{\mathcal{A}}: B_{\mathcal{A}} \rightarrow \mathcal{A}$ which is non-zero only on $\overline{\mathcal{A}}[1] \subset B_{\mathcal{A}}$ and is given by the inclusion map $\overline{\mathcal{A}}[1] \rightarrow \mathcal{A}$. The twisting cochain $t_{\mathcal{A}}$ gives rise to a free \mathcal{A} -bimodule resolution of \mathcal{A} obtained as a twisted tensor product

$$\mathcal{A} \otimes_{\mathbf{k}}^{t_{\mathcal{A}}} B_{\mathcal{A}} \otimes_{\mathbf{k}}^{t_{\mathcal{A}}} \mathcal{A},$$

with the differential d given by the following formula:

(19)

$$\begin{aligned} d &= m_1 \otimes_{\mathbf{k}} \text{Id}_{B_{\mathcal{A}}} \otimes_{\mathbf{k}} \text{Id}_{\mathcal{A}} + \text{Id}_{\mathcal{A}} \otimes_{\mathbf{k}} b \otimes_{\mathbf{k}} \text{Id}_{\mathcal{A}} + \text{Id}_{\mathcal{A}} \otimes_{\mathbf{k}} \text{Id}_{B_{\mathcal{A}}} \otimes_{\mathbf{k}} m_1 \\ &+ \left(\sum_{d \geq 2} (m_d \otimes_{\mathbf{k}} \text{Id}_{B_{\mathcal{A}}}) \circ (\text{Id}_{\mathcal{A}} \otimes_{\mathbf{k}} t^{\otimes_{\mathbf{k}} d-1} \otimes_{\mathbf{k}} \text{Id}_{B_{\mathcal{A}}}) \circ (\text{Id}_{\mathcal{A}} \otimes_{\mathbf{k}} \Delta_2^{(d)}) \right) \otimes_{\mathbf{k}} \text{Id}_{\mathcal{A}} \\ &+ \text{Id}_{\mathcal{A}} \otimes_{\mathbf{k}} \left(\sum_{d \geq 2} (\text{Id}_{B_{\mathcal{A}}} \otimes_{\mathbf{k}} m_d) \circ (\text{Id}_{B_{\mathcal{A}}} \otimes_{\mathbf{k}} t^{\otimes_{\mathbf{k}} d-1} \otimes_{\mathbf{k}} \text{Id}_{\mathcal{A}}) \circ (\Delta_2^{(d)} \otimes_{\mathbf{k}} \text{Id}_{\mathcal{A}}) \right). \end{aligned}$$

This can be used to compute Hochschild homology and cohomology of \mathcal{A} with coefficients in an \mathcal{A} -bimodule \mathcal{M} .

Consider instead a strictly counital A_{∞} -coalgebra \mathcal{C} with operations Δ_i and with a strict coaugmentation $\eta: \mathbf{k} \rightarrow \mathcal{C}$. Let $\overline{\mathcal{C}} = \text{coker}(\eta)$ be the coaugmentation ideal. We next recall the *cobar construction* which associates a DG-algebra $\Omega\mathcal{C}$ to \mathcal{C} . As an augmented algebra $\Omega\mathcal{C}$ is:

$$\Omega\mathcal{C} = \mathbf{k} \oplus \overline{\mathcal{C}}[-1] \oplus \overline{\mathcal{C}}[-1]^{\otimes_{\mathbf{k}} 2} \oplus \dots$$

As before, we write a typical monomial as

$$[c_d | c_{d-1} | \dots | c_1] = s^{-1} c_d \otimes_{\mathbf{k}} s^{-1} c_{d-1} \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} s^{-1} c_1,$$

where for $c \in \overline{\mathcal{C}}$, $s^{-1}c \in \overline{\mathcal{C}}[-1]$ denotes the corresponding element in $\overline{\mathcal{C}}[-1]$ with degree shifted up by 1. The differential m_1 on $\Omega\mathcal{C}$ vanishes on \mathbf{k} , $m_1|_{\mathbf{k}} = 0$, and acts as follows on monomials:

$$m_1([c_m | \dots | c_1]) = \sum_{i,j} (-1)^{|c_1| + \dots + |c_i| - i} [c_m | \dots | c_{i+2} | \Delta_j(c_{i+1}) | c_i | \dots | c_1].$$

Here, by abuse of notation, we write Δ_j for the induced coproduct $\overline{\mathcal{C}}[-1] \rightarrow \overline{\mathcal{C}}[-1]^{\otimes_j}$.

The product $m_2: \Omega\mathcal{C} \otimes \Omega\mathcal{C} \rightarrow \Omega\mathcal{C}$ is given by:

$$m_2([c_m | \dots | c_{i+1}], [c_i | \dots | c_1]) = (-1)^{|c_1| + \dots + |c_i| - i} [c_m | \dots | c_{i+1} | c_i | \dots | c_1]$$

The slightly unusual sign $(-1)^{|c_1| + \dots + |c_i| - i}$ appears as a consequence of the following two facts:

- (i) The equation $\mathfrak{m}_1^2 = 0$ is equivalent to co- A_∞ -relations (6) for $(\Delta_j)_{j \geq 1}$.
- (ii) The pair $(\mathfrak{m}_1, \mathfrak{m}_2)$ satisfies the A_∞ -relations (3).

Redefining $(\mathfrak{m}_1, \mathfrak{m}_2)$ to (d, \cdot) using (4) removes the sign in \mathfrak{m}_2 , and (d, \cdot) becomes a (usual) associative DG-algebra, where Leibniz rule (5) holds. The augmentation $\epsilon: \Omega\mathcal{C} \rightarrow \mathbf{k}$ is given by letting ϵ_1 to be the projection to \mathbf{k} and $\epsilon_i = 0$ for $i > 0$.

There is a decreasing, exhaustive, bounded above filtration on the complex $\Omega\mathcal{C}$:

$$\Omega\mathcal{C} = \mathcal{F}^0\Omega\mathcal{C} \supset \mathcal{F}^1\Omega\mathcal{C} \supset \dots,$$

given by

$$\mathcal{F}^p\Omega\mathcal{C} := \overline{\mathcal{C}}[-1]^{\otimes_{\mathbf{k}} p} \oplus \overline{\mathcal{C}}[-1]^{\otimes_{\mathbf{k}} (p+1)} \oplus \dots$$

This gives the *word length spectral sequence* with

$$E_1^{p,q} = H^{p+q}(\mathcal{F}^p\Omega\mathcal{C} / \mathcal{F}^{p+1}\Omega\mathcal{C}).$$

Unlike the case of the word length filtration on the bar construction, for the cobar construction, in general, convergence may fail. Thus, we introduce completions. We define the *completed cobar construction* to be:

$$\widehat{\Omega}\mathcal{C} = \varprojlim_s (\Omega\mathcal{C}) / (\mathcal{F}^s\Omega\mathcal{C})$$

The length filtration on $\Omega\mathcal{C}$ induces a filtration $\widehat{\mathcal{F}}$ on $\widehat{\Omega}\mathcal{C}$ defined by:

$$\widehat{\mathcal{F}}^p\widehat{\Omega}\mathcal{C} = \varprojlim_s (\mathcal{F}^p\Omega\mathcal{C}) / (\mathcal{F}^s\Omega\mathcal{C})$$

which is decreasing, exhaustive, bounded above and complete Hausdorff. The spectral sequence associated to the filtration $\widehat{\mathcal{F}}$ on $\widehat{\Omega}\mathcal{C}$ is isomorphic to the length spectral sequence associated with the filtration $\widehat{\mathcal{F}}$ on $\Omega\mathcal{C}$ and converges conditionally to $H^*(\widehat{\Omega}\mathcal{C})$ (see [9, Theorem 9.2]). It converges strongly to $H^*(\widehat{\Omega}\mathcal{C})$ if the spectral sequence is regular, i.e., only finitely many of the differentials $d_r^{p,q}$ are nonzero for each p and q (see [9, Theorem 7.1]). This holds, for example, if $\Omega\mathcal{C}$ is locally finite.

We say that $\Omega\mathcal{C}$ is complete if the natural map $\Omega\mathcal{C} \rightarrow \widehat{\Omega}\mathcal{C}$ is a quasi-isomorphism. For example, it is easy to see that this is the case if \mathcal{C} is locally finite and simply-connected.

If $\mathfrak{f}: \mathcal{C} \rightarrow \mathcal{D}$ is an A_∞ -comap which is a quasi-isomorphism of A_∞ -coalgebras and if $\Omega\mathcal{C}$ is complete then $\Omega\mathfrak{f}$ is a quasi-isomorphism. (This follows from [21, Theorem 7.4], see also [62, Theorem 5.5.11]). The condition that $\Omega\mathcal{C}$ is complete is necessary and is related to the completeness of the word length filtration. A counter-example when the completeness assumption is dropped can be found in [46, Section 2.4.1].

There is a universal twisting cochain $\mathfrak{t}^\mathcal{C}: \mathcal{C} \rightarrow \Omega\mathcal{C}$ given by the composition of canonical projection $\mathcal{C} \rightarrow \overline{\mathcal{C}}[-1]$ and the canonical inclusion $\overline{\mathcal{C}}[-1] \rightarrow \Omega\mathcal{C}$.

2.2.2 Bar-cobar adjunction

Suppose that \mathcal{C} is a coaugmented A_∞ -coalgebra and \mathcal{A} is an augmented DG-algebra, then we have a canonical bijection

$$(20) \quad \mathrm{hom}_{\mathrm{DG}}(\Omega\mathcal{C}, \mathcal{A}) \rightarrow \mathrm{Tw}(\mathcal{C}, \mathcal{A}),$$

given by $\phi \mapsto \phi \circ \mathfrak{t}^{\mathcal{C}}$. Similarly, if \mathcal{C} is a coaugmented conilpotent DG-coalgebra and \mathcal{A} is an augmented A_∞ -algebra, then we have a canonical bijection

$$(21) \quad \mathrm{hom}_{\mathrm{coDG}}(\mathcal{C}, \mathbf{B}\mathcal{A}) \rightarrow \mathrm{Tw}(\mathcal{C}, \mathcal{A}),$$

given by $\phi \mapsto \mathfrak{t}_{\mathcal{A}} \circ \phi$ (see [55, Lemme 3.17]).

Therefore, when \mathcal{C} is a coaugmented conilpotent DG-coalgebra, and \mathcal{A} is an augmented DG-algebra, we have the bar-cobar adjunction:

$$\mathrm{hom}_{\mathrm{DG}}(\Omega\mathcal{C}, \mathcal{A}) \cong \mathrm{hom}_{\mathrm{coDG}}(\mathcal{C}, \mathbf{B}\mathcal{A}).$$

Notice, in particular, that the natural DG-maps

$$(22) \quad \Omega\mathbf{B}\mathcal{A} \rightarrow \mathcal{A}, \quad \mathcal{C} \rightarrow \mathbf{B}\Omega\mathcal{C}$$

are quasi-isomorphisms for any DG-algebra \mathcal{A} and conilpotent DG-coalgebra \mathcal{C} (see [54, Section 6.10]). It is also true that for any A_∞ -algebra \mathcal{A} , the A_∞ -algebra map

$$\mathcal{A} \rightarrow \Omega\mathbf{B}\mathcal{A}$$

given by the adjunction map $\mathbf{B}\mathcal{A} \rightarrow \mathbf{B}\Omega\mathbf{B}\mathcal{A}$ is an A_∞ -quasi-isomorphism (see [44, Lemme 2.3.4.3]). Note that any A_∞ -quasi-isomorphism is invertible up to homotopy ([57, Corollary 1.4]).

Similarly, for any A_∞ -coalgebra \mathcal{C} , the A_∞ -comap

$$\mathbf{B}\Omega\mathcal{C} \rightarrow \mathcal{C}$$

given by the adjunction map $\Omega\mathbf{B}\Omega\mathcal{C} \rightarrow \mathcal{C}$ is an A_∞ -quasi isomorphism.

However, an A_∞ -quasi isomorphism for a general A_∞ -coalgebra is not usually a convenient notion, since as we remarked above a quasi-isomorphism of A_∞ -coalgebras between \mathcal{C} and \mathcal{C}' does not necessarily induce a quasi-isomorphism of DG-algebras $\Omega\mathcal{C}$ and $\Omega\mathcal{C}'$.

For this reason, one considers the category of *conilpotent* A_∞ -coalgebras. Let \mathcal{C} be a coaugmented A_∞ -coalgebra generated over \mathbf{k} by variables $(c_i)_{i \in I}$ with I some countable index set, such that there exists a total ordering

$$c_{\sigma(1)} < c_{\sigma(2)} < \dots$$

where $\sigma: I \rightarrow I$ is a bijection. This produces an increasing filtration

$$\mathcal{F}^0 = \mathbf{k} \subset \mathcal{F}^1 \subset \dots \subset \Omega\mathcal{C}$$

by setting $\mathcal{F}^p = \mathbf{k}\langle c_{\sigma(1)}, \dots, c_{\sigma(p)} \rangle$. Suppose that the structure maps $(\Delta_i)_{i \geq 1}$ are compatible with this filtration, in the sense that, $\Delta_i(c_{\sigma(p)}) \subset \mathcal{F}^{p-1}$ for all i and p . Then we call \mathcal{C} a conilpotent A_∞ -coalgebra. (More generally, a homotopy retract of such A_∞ -coalgebras are called conilpotent, cf. [54, Sections 6.10 and 9]. This notion is called *finite type* in [43].). Given two such A_∞ -coalgebras \mathcal{C} and \mathcal{C}' , one considers filtered A_∞ -comaps between them. In the case of a conilpotent DG-coalgebra \mathcal{C} there exists an increasing filtration on $\Omega\mathcal{C}$ given by the subalgebras $\text{Ker}(\Delta^{(n)})$ that plays the same role (see [44, Lemme 1.3.2.3]).

We next state the following elementary lemma for later convenience.

Lemma 10 *Let \mathcal{A} be an augmented A_∞ -algebra, such that the \mathbf{k} -bimodule structure on \mathcal{A} and $\text{B}\mathcal{A}$ are locally finite, then there are quasi-isomorphisms of augmented DG-algebras:*

$$\Omega(\mathcal{A}^\#) \rightarrow (\text{B}\mathcal{A})^\# \quad \text{and} \quad \Omega(\# \mathcal{A}) \rightarrow \#(\text{B}\mathcal{A}).$$

Note that the assumption is satisfied when \mathcal{A} is simply-connected. We shall also consider the case when \mathcal{A} is assumed to be connected, then we have:

Lemma 11 *Let $\mathcal{A} = \bigoplus_i \mathcal{A}^i$ be a connected, locally finite \mathbf{k} -bimodule equipped with an augmented A_∞ -algebra structure, then there are maps of DG-algebras*

$$\Omega(\mathcal{A}^\#) \rightarrow (\text{B}\mathcal{A})^\# \quad \text{and} \quad \Omega(\# \mathcal{A}) \rightarrow \#(\text{B}\mathcal{A}),$$

which become quasi-isomorphisms after completion:

$$\widehat{\Omega}(\mathcal{A}^\#) \rightarrow (\text{B}\mathcal{A})^\# \quad \text{and} \quad \widehat{\Omega}(\# \mathcal{A}) \rightarrow \#(\text{B}\mathcal{A}).$$

2.3 Koszul duality

Suppose \mathcal{C} is a coaugmented conilpotent A_∞ -coalgebra and \mathcal{A} is an augmented DG-algebra, via the bijection (20), any twisting cochain $\mathfrak{t} \in \text{Tw}(\mathcal{C}, \mathcal{A})$ is of the form $\mathfrak{t} = \phi \circ \mathfrak{t}^\mathcal{C}$ for some unique $\phi \in \text{hom}_{\text{DG}}(\Omega\mathcal{C}, \mathcal{A})$. Similarly, if \mathcal{C} is a coaugmented DG-algebra and \mathcal{A} is an augmented A_∞ -algebra, any twisting cochain $\mathfrak{t} \in \text{Tw}(\mathcal{C}, \mathcal{A})$ is of the form $\mathfrak{t} = \mathfrak{t}_{\mathcal{A}} \circ \phi$ for some $\phi \in \text{hom}_{\text{coDG}}(\mathcal{C}, \text{B}\mathcal{A})$.

Definition 12 *In either case above we call \mathfrak{t} a Koszul twisting cochain if ϕ is a quasi-isomorphism, and we denote the set of Koszul twisting cochains by $\text{Kos}(\mathcal{C}, \mathcal{A})$.*

The terminology of Koszul twisting cochains is taken from [46]. They are also called *acyclic twisting cochains* in other sources (cf. [44], [54]). This terminology is due to the well known fact that, under various locally finiteness assumptions, a twisting cochain \mathfrak{t} is Koszul if and only if the Koszul complex (17) associated \mathfrak{t} is acyclic (cf. [54, Appendix A]).

Informally, if $\mathfrak{t} \in \text{Kos}(\mathcal{C}, \mathcal{A})$ then, depending on whether we write $\mathfrak{t} = \phi \circ \mathfrak{t}^{\mathcal{C}}$ or $\mathfrak{t} = \mathfrak{t}_{\mathcal{A}} \circ \phi$, either \mathcal{A} can be used in place of $\Omega\mathcal{C}$ or \mathcal{C} can be used in place of $\mathbf{B}\mathcal{A}$ in various resolutions. This, in turn, may lead to smaller complexes to compute with. For example, one can compute Hochschild homology and cohomology of \mathcal{A} and $\Omega\mathcal{C}$ using the \mathcal{A} -bimodule resolution of \mathcal{A} given by the complex:

$$\mathcal{A} \otimes_{\mathbf{k}}^{\mathfrak{t}} \mathcal{C} \otimes_{\mathbf{k}}^{\mathfrak{t}} \mathcal{A}$$

with the differential as in (19), see [37].

Suppose that \mathcal{A} is an A_{∞} -algebra with an augmentation $\epsilon: \mathcal{A} \rightarrow \mathbf{k}$. The augmentation ϵ makes \mathbf{k} into a left \mathcal{A} -module or equivalently a right \mathcal{A}^{op} -module.

Definition 13 *The Koszul dual of an augmented A_{∞} -algebra \mathcal{A} is the DG-algebra of left \mathcal{A} -module maps from \mathbf{k} to itself:*

$$E(\mathcal{A}) := \text{RHom}_{\mathcal{A}}(\mathbf{k}, \mathbf{k}).$$

(Here, the \mathbf{k} -bimodule structure on $E(\mathcal{A})$ is given by (10) since \mathbf{k} is viewed as a left \mathbf{k} -module induced from its structure as a left \mathcal{A} -module. If instead, we have an augmentation of \mathcal{A}^{op} , then we view \mathbf{k} as a right \mathcal{A} -module, and the \mathbf{k} -bimodule structure on $\text{RHom}_{\mathcal{A}}(\mathbf{k}, \mathbf{k})$ would be given by (11)).

Recall that for a unital A_{∞} -algebra \mathcal{A} over a field \mathbb{K} (or a semisimple ring such as \mathbf{k}) any A_{∞} -module is both h -projective and h -injective, that is, if M is an A_{∞} -module over \mathcal{A} and N is an acyclic A_{∞} -module over \mathcal{A} , then the complexes $\text{RHom}_{\mathcal{A}}(M, N)$ and $\text{RHom}_{\mathcal{A}}(N, M)$ are acyclic [57, Lemma 1.16]. Hence, the DG-algebra $\text{RHom}_{\mathcal{A}}(\mathbf{k}, \mathbf{k})$ can be computed as the A_{∞} -module homomorphisms from \mathbf{k} to itself. (More generally, this holds if \mathcal{A} is h -projective as a complex of \mathbf{k} -modules, which implies that \mathbf{k} is h -projective as an A_{∞} -module over \mathcal{A} .) Therefore, we have the following.

Proposition 14 *If \mathcal{A} (resp. \mathcal{A}^{op}) is an augmented unital A_{∞} -algebra then*

$$\text{RHom}_{\mathcal{A}}(\mathbf{k}, \mathbf{k}) \cong (\mathbf{B}\mathcal{A})^{\#} \quad (\text{resp. } \#(\mathbf{B}\mathcal{A})).$$

Proof Recall that $\mathcal{A} \otimes_{\mathbf{k}} \mathbf{B}\mathcal{A}$ is quasi-isomorphic to \mathbf{k} as an \mathcal{A} -module. Hence, by the hom-tensor adjunction, we have $\mathrm{RHom}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathbf{k}} \mathbf{B}\mathcal{A}, \mathbf{k}) \cong \mathrm{RHom}_{\mathbf{k}}(\mathbf{B}\mathcal{A}, \mathbf{k})$. Since \mathcal{A} is h -projective as a complex of \mathbf{k} -modules, so is $\mathbf{B}\mathcal{A}$, hence the latter is computed by $(\mathbf{B}\mathcal{A})^{\#}$. \square

The cohomology is a graded algebra:

$$\mathrm{Ext}_{\mathcal{A}}(\mathbf{k}, \mathbf{k}) := H^*(\mathrm{RHom}_{\mathcal{A}}(\mathbf{k}, \mathbf{k})) \cong H^*((\mathbf{B}\mathcal{A})^{\#}).$$

Dually, we also have the derived tensor product $\mathbf{k} \widehat{\otimes}_{\mathcal{A}} \mathbf{k}$, which can be computed by the complex $\mathbf{B}\mathcal{A}$. The cohomology is a graded coalgebra:

$$\mathrm{Tor}_{\mathcal{A}}(\mathbf{k}, \mathbf{k}) := H^*(\mathbf{k} \widehat{\otimes}_{\mathcal{A}} \mathbf{k}) \cong H^*(\mathbf{B}\mathcal{A}).$$

In particular, note that if \mathbf{k} is a field, we have that $\mathrm{Ext}_{\mathcal{A}}(\mathbf{k}, \mathbf{k}) \cong (\mathrm{Tor}_{\mathcal{A}}(\mathbf{k}, \mathbf{k}))^{\#}$ by the universal coefficient theorem.

Remark 15 If \mathcal{A} is a **commutative algebra** (or more generally an E_2 -algebra), then $\mathrm{Tor}_{\mathcal{A}}(\mathbf{k}, \mathbf{k})$ also has a graded algebra structure defined via:

$$\mathrm{Tor}_{\mathcal{A}}(\mathbf{k}, \mathbf{k}) \otimes \mathrm{Tor}_{\mathcal{A}}(\mathbf{k}, \mathbf{k}) \rightarrow \mathrm{Tor}_{\mathcal{A} \otimes \mathcal{A}}(\mathbf{k} \otimes \mathbf{k}, \mathbf{k} \otimes \mathbf{k}) \rightarrow \mathrm{Tor}_{\mathcal{A}}(\mathbf{k}, \mathbf{k})$$

induced by the algebra map $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ (which exists since \mathcal{A} is commutative). This should not be confused with the natural coalgebra structure above.

Note that \mathcal{A} itself can be viewed as a left \mathcal{A} -module and the map $\epsilon: \mathcal{A} \rightarrow \mathbf{k}$ is a map of left \mathcal{A} -modules, hence it induces a map of DG-algebras:

$$\tilde{\epsilon}: \mathrm{RHom}_{\mathcal{A}}(\mathbf{k}, \mathbf{k})^{op} \rightarrow \mathrm{RHom}_{\mathcal{A}}(\mathcal{A}, \mathbf{k}),$$

which can in turn be viewed as an augmentation of $E(\mathcal{A})^{op} = \mathrm{RHom}_{\mathcal{A}}(\mathbf{k}, \mathbf{k})^{op}$ since $\mathrm{RHom}_{\mathcal{A}}(\mathcal{A}, \mathbf{k})$ can again be identified with \mathbf{k} as it is the Yoneda image of \mathbf{k} as an \mathcal{A} -module. Hence, \mathbf{k} can be viewed as a right $E(\mathcal{A})$ -module.

Definition 16 The double-dual of \mathcal{A} is defined to be $E(E(\mathcal{A})) := \mathrm{RHom}_{E(\mathcal{A})}(\mathbf{k}, \mathbf{k})$.

There is a natural map from \mathcal{A} to its double-dual:

$$\Phi: \mathcal{A} \rightarrow \mathrm{RHom}_{E(\mathcal{A})}(\mathbf{k}, \mathbf{k})$$

defined via viewing the right $E(\mathcal{A})$ -module \mathbf{k} as $\mathrm{RHom}_{\mathcal{A}}(\mathcal{A}, \mathbf{k})$ and acting on the left by $\mathcal{A} \cong \mathrm{RHom}_{\mathcal{A}}(\mathcal{A}, \mathcal{A})$.

Definition 17 We say that \mathcal{A} and $E(\mathcal{A})$ are Koszul dual if $\Phi: \mathcal{A} \rightarrow \mathrm{RHom}_{E(\mathcal{A})}(\mathbf{k}, \mathbf{k})$ is a quasi-isomorphism.

One standard situation when Koszul duality holds is the following:

Theorem 18 *Suppose $\mathcal{C} = \bigoplus_{i \leq 0} \mathcal{C}^i$ is a locally finite, simply-connected \mathbf{k} -bimodule equipped with an A_∞ -coalgebra structure and the coaugmentation $\mathbf{k} \cong \mathcal{C}^0 \rightarrow \mathcal{C}$. Let $\mathcal{A} = \Omega\mathcal{C}$, which is an augmented connected DG-algebra. Then $E(\mathcal{A}) \cong \mathcal{C}^\#$ and \mathcal{A} and $\mathcal{C}^\#$ are Koszul dual. In other words, the natural morphism*

$$\Omega\mathcal{C} \rightarrow \mathrm{RHom}_{E(\mathcal{A})}(\mathbf{k}, \mathbf{k})$$

is a quasi-isomorphism.

Proof First, observe that indeed $E(\mathcal{A}) \cong (\mathbf{B}\mathcal{A})^\# \cong (\mathbf{B}\Omega\mathcal{C})^\# \cong \mathcal{C}^\#$ by (22). Next, we have that

$$\mathrm{RHom}_{E(\mathcal{A})}(\mathbf{k}, \mathbf{k}) \cong {}^\#(\mathbf{B}(\mathcal{C}^\#)) \cong \Omega\mathcal{C}$$

where we applied Lemma 10 to $\mathcal{C}^\#$ and used the fact that ${}^\#(\mathcal{C}^\#) \cong \mathcal{C}$ since \mathcal{C} is locally finite. \square

In the case that $\mathcal{C} = \bigoplus_{i \leq 0} \mathcal{C}^i$ is a locally finite, connected (but not simply-connected) \mathbf{k} -bimodule, Lemma 10 no longer applies. We instead use Lemma 11 to deduce the following weaker duality result.

Proposition 19 *Let $\mathcal{C} = \bigoplus_{i \leq 0} \mathcal{C}^i$ be a connected, locally finite \mathbf{k} -bimodule, equipped with an A_∞ -coalgebra structure and coaugmentation $\mathbf{k} \cong \mathcal{C}^0 \rightarrow \mathcal{C}$, and $\mathcal{A} = \Omega\mathcal{C} = \mathbf{k} \oplus \bigoplus_{j \geq 1} (\overline{\mathcal{C}}[-1])^{\otimes j}$, which is an augmented DG-algebra where augmentation is given by projection to \mathbf{k} . Then $E(\mathcal{A}) \cong \mathcal{C}^\#$ and there is a quasi-isomorphism*

$$\widehat{\Omega}\mathcal{C} \rightarrow \mathrm{RHom}_{E(\mathcal{A})}(\mathbf{k}, \mathbf{k}).$$

Note that in Proposition 19, $\mathcal{A} = \Omega\mathcal{C}$ is not connected, and may admit other augmentations $\epsilon: \mathcal{A} \rightarrow \mathbf{k}$ than that induced by the cobar construction. Such augmentations will be considered below. For example, suppose that $\overline{\mathcal{C}} \cong \mathbf{k} \oplus \overline{\mathcal{C}}$ is a coaugmented A_∞ -coalgebra such that $\overline{\mathcal{C}} = \mathbb{K}\langle c \mid c \in \mathcal{R} \rangle$ is generated by elements c from an indexing set \mathcal{R} and that $\epsilon: \Omega\mathcal{C} \rightarrow \mathbf{k}$ is an augmentation, which is induced by a map $\overline{\mathcal{C}} \rightarrow \mathbf{k}$ since $\Omega\mathcal{C}$ is free. Now we can consider the coaugmented A_∞ -coalgebra $\overline{\mathcal{C}}^\epsilon = \mathbf{k} \oplus \overline{\mathcal{C}}^\epsilon$ such that

$$\overline{\mathcal{C}}^\epsilon = \mathbb{K}\langle c - \epsilon(c)1_{\mathbf{k}} \mid c \in \mathcal{R} \rangle$$

Then $\Omega\mathcal{C}$ and $\Omega\overline{\mathcal{C}}^\epsilon$ are quasi-isomorphic as non-augmented DG-algebras, and the augmentation on $\Omega\overline{\mathcal{C}}^\epsilon$ induced by the cobar construction coincides with the given augmentation ϵ on Ω .

Remark 20 When \mathcal{C} is not simply-connected, the proof of duality fails precisely because $B\mathcal{C}^\#$ is not locally finite. Nevertheless, the duality result can still be proved in certain cases where an extra weight grading (internal degree, or Adams degree) is available (see [47], [54, Appendix A.2], [37]). We will not study this situation systematically in this paper, but it is important as it extends the range of applicability of Koszul duality theory. In the setting of Chekanov-Eliashberg DG-algebras, such a situation was considered in [31].

3 Legendrian (co)algebra

In this section we introduce our Legendrian invariants. We start with discussing a model for loop space coefficients in Section 3.1. In Section 3.2 we define the Chekanov-Eliashberg algebra with loop space coefficients using moduli spaces of disks of all dimensions, and in Section 3.4 we give a more computable version which uses only rigid disks and which carries the same information if the Legendrian submanifold is simply connected.

3.1 Coefficients

Before defining our Legendrian invariants, we describe chain models for their coefficients $C_{-*}(\Omega_{p_v}\Lambda_v)$ for $v \in \Gamma^+$. (Notation is as above, Λ_v is a $+$ decorated connected component of the Legendrian Λ .) We work over a field \mathbb{K} .

Let $\Omega_{p_v}\Lambda_v$ denote the topological monoid of Moore loops based at p_v , where the monoid structure comes from concatenation of loops (see [5]). Write $C_{-*}(\Omega_{p_v}\Lambda_v)$ for the cubical chain complex (graded cohomologically). Since $\Omega_{p_v}\Lambda_v$ is a topological monoid, the complex $C_{-*}(\Omega_{p_v}\Lambda_v)$ becomes a DG-algebra using the natural product map \times on cubical chains, where the DG-algebra product is given as follows:

$$C_{-*}(\Omega_{p_v}\Lambda_v) \otimes C_{-*}(\Omega_{p_v}\Lambda_v) \xrightarrow{\times} C_{-*}(\Omega_{p_v}\Lambda_v \times \Omega_{p_v}\Lambda_v) \xrightarrow{\circ} C_{-*}(\Omega_{p_v}\Lambda_v).$$

We point out that the \times -map

$$\times : C_{-*}(\Omega_{p_v}\Lambda_v) \otimes C_{-*}(\Omega_{p_v}\Lambda_v) \rightarrow C_{-*}(\Omega_{p_v}\Lambda_v \times \Omega_{p_v}\Lambda_v),$$

when both sides are equipped with the Pontryagin product, is a DG-algebra map.

In what follows, we shall also make use of an inverse to the \times , known as the Serre diagonal [58] and the cubical analogue of the Alexander-Whitney map,

$$\eta : C_{-*}(\Omega_{p_v}\Lambda_v \times \Omega_{p_w}\Lambda_w) \rightarrow C_{-*}(\Omega_{p_v}\Lambda_v) \otimes C_{-*}(\Omega_{p_w}\Lambda_w).$$

To define this map consider the n -cube I^n with coordinates (x_1, \dots, x_n) . For an ordered j -element subset $J \subset \{1, 2, \dots, n\}$, $J = (i_1, \dots, i_j)$, $i_1 < \dots < i_j$ and for $\epsilon \in \{0, 1\}$ let $\iota_J^\epsilon: I^j \rightarrow I^n$ be the map given in coordinates $y = (y_1, \dots, y_j)$ by

$$x_{i_r}(\iota_J(y)) = y_r, \quad x_m(\iota_J(y)) = \epsilon \text{ if } m \notin J.$$

Consider a cubical chain $(\sigma, \tau): I^m \rightarrow \Omega_{p_v} \Lambda_v \times \Omega_{p_w} \Lambda_w$. If J is an ordered subset of $\{1, \dots, m\}$ let J' denote its complement ordered in the natural way. Define η by

$$\eta(\sigma, \tau) = \sum_J (-1)^{JJ'} (\sigma \circ \iota_J^0) \otimes (\tau \circ \iota_{J'}^1),$$

where the sum ranges over all ordered subsets J and $(-1)^{JJ'}$ is the sign of the permutation JJ' . This is a strictly associative chain map inducing a quasi-isomorphism. Note also that there are obvious extensions of η to several products of loop spaces.

As the cubical chain complex $C_{-*}(\Omega_{p_v} \Lambda_v)$ is very large it is not the most effective complex for computation. We next discuss smaller models. Starting with a 0-reduced simplicial set X with geometric realization $|X| = \Lambda_v$, an explicit economical model for $C_{-*}(\Omega_{p_v} \Lambda_v)$ is obtained by taking normalized chains on the Kan loop group GX [42]. We will not say much about this, but point out that GX is a free simplicial group such that its geometric realization $|GX|$ is homotopy equivalent to $\Omega|X|$ (cf. [35, Corollary 5.11]). Hence, by the monoidal Dold-Kan correspondence (cf. [56]) the normalized chains on GX gives a (weakly) equivalent model of $C_{-*}(\Omega_{p_v} \Lambda_v)$. (Another similar construction is sketched in [43] and leads to a free model.)

Alternatively, one can work with CW complexes. We start with the simply connected case: for a 1-reduced (unique 0-cell and no 1-cells) CW-structure on Λ_v the Adams-Hilton construction [5] gives a free DG-algebra model for $C_{-*}(\Omega_{p_v} \Lambda_v)$ as follows. Denote the k -cells of Λ_v by e_k^i , $k \geq 2$ and $i = 1, \dots, m_k$. The Adams-Hilton construction gives a CW monoid with a single 0-cell and generating cells \bar{e}_k^i in dimension $k - 1$ that is quasi-isomorphic to $\Omega_{p_v}(\Lambda_v)$ as a monoid (cf. [14]). This gives a DG-algebra structure on the free algebra:

$$A(\Lambda_v) := \mathbb{K}\langle \bar{e}_2^1, \dots, \bar{e}_2^{m_2}, \bar{e}_3^1, \dots, \bar{e}_3^{m_3}, \dots, \dots \rangle, \quad |\bar{e}_k^i| = 1 - k.$$

and a DG-algebra map

$$A(\Lambda_v) \xrightarrow{\Psi} C_{-*}(\Omega_{p_v} \Lambda_v)$$

which is a quasi-isomorphism. The differential d on $A(\Lambda_v)$ is generally not explicit. It is defined recursively as follows. For every 2-cell e_2^i , we have $d(\bar{e}_2^i) = 0$. In general, assume that d_{k-1} and Ψ_{k-1} have been defined on the k -skeleton $\Lambda_v^{(k)}$ of Λ_v , then for each $(k + 1)$ -cell e , with attaching map $f: S^k \rightarrow \Lambda_v^{(k)}$, define $d_k \bar{e} = c$ so

that $(\Psi_{k-1})(c) = (\Omega f)_*(\xi)$ where ξ a generator of $H_{k-1}(\Omega S^k)$, and $\Psi_k(\bar{e})$ to be the k -chain of loops in e (which then depends on earlier choices along the boundary of \bar{e}). We remark that $A(\Lambda_\nu)$ can be identified isomorphically with $\Omega C_*^{\text{CW}}(\Lambda_\nu)$ for a suitable A_∞ -coalgebra structure on the cellular chain complex $C_*^{\text{CW}}(\Lambda)$.

This construction can be generalized to the non-simply-connected case as follows. (See [39, 40], a generalization was given earlier in [32], however that paper contains an error.) Begin with a 0-reduced CW-structure on Λ_ν . Denote the k -cells by e_k^i for $i = 1, \dots, m_k$. For each k -cell e_k^i with $k \geq 2$, we have a free variable in degree $1 - k$ which we again denote by \bar{e}_k^i . For each 1-cell e_1^j , $j = 1, \dots, m_1$ we have two variables t_j and t_j^{-1} in degree 0 such that $t_j t_j^{-1} = 1 = t_j^{-1} t_j$. Thus, the underlying algebra is the “almost free” algebra of the form

$$A(\Lambda_\nu) := \mathbb{K}\langle t_1^{\pm 1}, \dots, t_{m_1}^{\pm 1}, \bar{e}_2^1, \dots, \bar{e}_2^{m_2}, \bar{e}_3^1, \dots, \bar{e}_3^{m_3}, \dots \rangle.$$

This presentation is often more efficient than the presentation one gets from the Kan loop group construction using a simplicial set presentation of Λ_ν . However, the differential in the Adams-Hilton model is not easy to describe explicitly. Note that we have

$$d(t_j) = d(t_j^{-1}) = 0,$$

for degree reasons. For every 2-cell e_2^i , we have

$$d(\bar{e}_2^i) = 1 - c_i$$

where $c_i \in \mathbb{K}\langle t_j^{\pm 1} \mid j = 1, \dots, m_1 \rangle$ represents the class of the attaching map of e_2^i . The differential on higher dimensional cells is generally harder to compute and is exactly as in the simply-connected case discussed above.

Augmentations $\epsilon: A(\Lambda_\nu) \rightarrow \mathbb{K}$ correspond to solutions of the equations

$$\begin{cases} \epsilon(t_j)\epsilon(t_j^{-1}) = 1, & j = 1, \dots, m_1, \\ \epsilon(d\bar{e}_2^i) = 0, & i = 1, \dots, m_2. \end{cases}$$

Since $\mathbb{K}\langle t_j^{\pm 1}, j = 1, \dots, m_1 \mid d\bar{e}_2^i, i = 1, \dots, m_2 \rangle$ is a presentation of the fundamental group algebra $\mathbb{K}[\pi_1(\Lambda_\nu, p_\nu)]$, augmentations correspond exactly to local systems $\pi_1(\Lambda_\nu, p_\nu) \rightarrow \mathbb{K}$.

We will use the cubical chain complex $C_{-*}(\Omega_{p_\nu} \Lambda_\nu)$ to define Legendrian invariants below. Cubical chains work uniformly for all spaces Λ_ν and are convenient for showing that the fundamental classes of moduli spaces of pseudoholomorphic disks \mathcal{M}^{sy} , via evaluation maps, take values in the chain complex. The Legendrian invariants can also be studied using any of the smaller models discussed above. It is however important to

note that in the non-simply connected case, we only have either weak equivalence in the homotopy category of DG-algebras or Morita equivalence [39, 40] of these models and the cubical chain complex $C_{-*}(\Omega_{p_v}\Lambda_v)$.

In the case that Λ_v is simply-connected we can use a DG-algebra map

$$\Phi: C_{-*}(\Omega_{p_v}\Lambda_v) \rightarrow A(\Lambda_v)$$

that goes in the opposite direction to the Adams-Hilton map to pass to a more economical quasi-isomorphic model. Such a homotopy equivalence Φ is constructed in two steps: first, in [51] using Eilenberg-Moore methods, a DG-algebra quasi-isomorphism:

$$C_{-*}(\Omega_{p_v}\Lambda_v) \rightarrow \Omega C_*(\Lambda_v)$$

is constructed where in both instances C_* refers to the normalised singular chains. Second, using the standard A_∞ -coalgebra quasi-isomorphism between the DG-coalgebra of singular chains $C_*(\Lambda_v)$ and the A_∞ -coalgebra $C_*^{\text{CW}}(\Lambda_v)$ of normalised cellular chains, one obtains a DG-algebra quasi-isomorphism

$$\Omega C_*(\Lambda_v) \rightarrow \Omega C_*^{\text{CW}}(\Lambda_v) = A(\Lambda_v),$$

since we assumed that the complexes C_* and C_*^{CW} are simply-connected. (In Section 3.5, we also give a more geometric construction of a DG-algebra quasi-isomorphism Φ using Morse flow trees.)

Similarly, if Λ_v is homotopy equivalent to an Eilenberg-MacLane space $\mathbb{K}(\pi_1, 1)$, then the singular chains can be replaced with the group algebra $\mathbb{K}[\pi_1]$: there exists a quasi-isomorphism of DG-algebras

$$C_{-*}(\Omega_{p_v}\Lambda_v) \rightarrow \mathbb{K}[\pi_1]$$

given by sending a 0-chain to its homology class, and sending all higher dimensional chains to 0. Note that this DG-algebra map exists for any space Λ_v , but is a quasi-isomorphism only in the case that Λ_v is homotopy equivalent to $\mathbb{K}(\pi_1, 1)$.

It is often convenient to use a cofibrant (or free) replacement for $\mathbb{K}[\pi_1]$. For example, if $\Lambda_v = S^1$, then $\mathbb{K}[\pi_1] \cong \mathbb{K}[t, t^{-1}]$ and a cofibrant replacement is given by the free graded algebra

$$\mathbb{K}\langle s_1, t_1, k_1, l_1, u_1 \rangle, \quad |s_1| = |t_1| = 0, |k_1| = |l_1| = -1, |u_1| = -2$$

with the differential

$$\begin{aligned} dk_1 &= 1 - s_1 t_1 \\ dl_1 &= 1 - t_1 s_1 \\ du_1 &= k_1 s_1 - s_1 l_1 \end{aligned}$$

A DG-algebra defined over $\mathbb{K}[t, t^{-1}]$ can be pulled back to a weakly equivalent DG-algebra over this cofibrant replacement. (See [61] for background in model categories on DG-algebras that we are using in a very simple case here.)

3.2 Construction of Legendrian invariants

As above, let X be a Liouville domain with $c_1(X) = 0$ (for \mathbb{Z} -grading) and $\partial X = Y$ its contact boundary. Let $\Lambda = \bigsqcup_{v \in \pi_0(\Lambda)} \Lambda_v$ be a Legendrian submanifold in Y where Λ_v is a connected component of Λ . Assume that Λ is relatively spin and that its Maslov class vanishes. Let each connected component Λ_v be decorated with a sign and write Λ^+ and Λ^- for the union of the components decorated accordingly. (Our different treatments of Λ^+ and Λ^- is natural from the point of view of handle attachments; recall from the introduction that when Λ^- is a union of spheres we attach usual Lagrangian disk-handles to Λ^- and handles with cotangent ends to Λ^+ .) When we have an exact Lagrangian filling L of Λ (relatively spin and with vanishing Maslov class), L can also be decomposed into embedded components $L = \bigcup_{v \in \Gamma} L_v$. These embedded components are not disjoint, they are allowed to intersect transversely at finitely many points. There is a bijection between Γ and the embedded components of L .

We require that if two components Λ_{w_1} and Λ_{w_2} are boundary components of the same embedded component L_v then either both belong to Λ^- or both to Λ^+ . Using this property, we get a decomposition $\Gamma = \Gamma^+ \sqcup \Gamma^-$, corresponding to the decomposition $\Lambda = \Lambda^+ \sqcup \Lambda^-$.

Let \mathbf{k} be the semi-simple ring generated by mutually orthogonal idempotents $\{e_v\}_{v \in \Gamma}$. If we are not given a filling of Λ , then the index set Γ is taken to be the connected components, $\pi_0(\Lambda)$, instead. If we need to distinguish between the two choices, we will denote them as \mathbf{k}_Λ and \mathbf{k}_L . Note that there is an injective ring map $\mathbf{k}_L \rightarrow \mathbf{k}_\Lambda$ which takes the idempotent e_v corresponding to an embedded component L_v to the sum $e_{w_1} + \cdots + e_{w_r}$ of idempotents of its boundary components Λ_{w_j} . In particular, this map turns any \mathbf{k}_Λ -bimodule into a \mathbf{k}_L -bimodule.

Let \mathcal{R} denote the set of non-empty Reeb chords of Λ . This is a graded set: the grading of a chord $c \in \mathcal{R}$ is given by $|c| = -\text{CZ}(c)$ where $\text{CZ}(c)$ is the Conley-Zehnder grading, see Section A. (With this convention, the unique chord c of the standard Legendrian unknot in \mathbf{R}^3 has $|c| = -2$ and for the corresponding Legendrian unknot in \mathbf{R}^{2n-1} with one Reeb chord c , $|c| = -n$. See also Remark 30.)

Note that \mathcal{R} is a \mathbf{k} -bimodule, where $e_v \mathcal{R} e_w$ corresponds to the set of Reeb chords from Λ_v to Λ_w . The underlying algebra of the standard Chekanov-Eliashberg DG-algebra

is generated freely by \mathcal{R} over \mathbf{k} . We need to modify this in the case Λ^+ is non-empty to incorporate chains in the based loop space of Λ_ν for $\nu \in \Gamma^+$. Let us first do this using cubical chains.

For each $\nu \in \Gamma^+$, consider the cubical chains $C_{-*}(\Omega_{p_\nu}\Lambda_\nu)$ as a \mathbf{k} -algebra by requiring that the left or right action of e_w is trivial except if $w = \nu$ when it acts as identity. Let CE^* be the algebra over \mathbf{k} given by freely adjoining elements of \mathcal{R} to the union of $C_{-*}(\Omega_{p_\nu}\Lambda_\nu)$ for $\nu \in \Gamma^+$. Thus an element of CE^* is a word in Reeb chords and chains on based loops of Λ_ν for $\nu \in \Gamma^+$ that respects the \mathbf{k} -bimodule structure.

Now the differential on CE^* is defined by extending the differential on the cubical complexes $C_{-*}(\Omega_{p_\nu}\Lambda_\nu)$ for $\nu \in \Gamma_+$. Since the elements of \mathcal{R} are added freely, it suffices to describe the differential on a single Reeb chord and extend it by the graded Leibniz rule. The differential d on a Reeb chord decomposes to a sum

$$d = \sum_{i \geq 0} \Delta_i,$$

where for any Reeb chord c_0 only finitely many $\Delta_i(c_0)$ are non-zero. The operations $\Delta_i(c_0)$ are defined as follows.

Consider moduli spaces of holomorphic disks with positive puncture at c_0 , see Section A for definitions and notation. More precisely, consider Reeb chords c_i, \dots, c_1 such that $c_0 c_i \dots c_1$ is a composable word and let $\mathbf{c} = c_0^+ c_i^- \dots c_1^-$. Consider the disk D_{i+1} with one distinguished positive puncture and i negative punctures (across which the boundary numbering is constant in the terminology of Section A). Consider the moduli space $\mathcal{M}^{\text{sy}}(\mathbf{c})$. Theorems 63 and 64 imply that this is a smooth orientable manifold with a natural compactification as a stratified space that carries a fundamental chain. It follows that, via the evaluation map at a point in the boundary arcs of D_{i+1} , $\mathcal{M}^{\text{sy}}(\mathbf{c})$ parameterizes a chain of paths in the $(i+1)$ -fold product $\Lambda^{\times(i+1)}$, which after adding reference paths to the base point becomes chain of based loops in $(\Omega_p \Lambda)^{\times(i+1)}$. We treat two cases separately. First, if all boundary components of D_{i+1} map to components in Λ^- then we let

$$(23) \quad [\mathcal{M}^{\text{sy}}(\mathbf{c})] = \begin{cases} n c_i \dots c_1 & \text{if } \dim(\mathcal{M}^{\text{sy}}(\mathbf{c})) = 1, \\ 0 & \text{if } \dim(\mathcal{M}^{\text{sy}}(\mathbf{c})) \neq 1, \end{cases}$$

where n is the algebraic number of \mathbf{R} -components in the moduli space. Second, if some boundary component maps to a component in Λ^+ then we write $[\mathcal{M}^{\text{sy}}(\mathbf{c})]$ for the chain of paths in $(\Omega_p \Lambda)^{i+1}$, where we separate the components in the product by the Reeb chords $\mathbf{c}' = c_i \dots c_1$:

$$[\mathcal{M}^{\text{sy}}(\mathbf{c})] = \sigma_{i+1} c_i \sigma_i \dots \sigma_2 c_1 \sigma_1,$$

where σ_j are the components of the fundamental chain $\sigma: \mathcal{M}^{\text{sy}}(\mathbf{c}) \rightarrow (\Omega\Lambda)^{\times(i+1)}$. Further, we write e_ν for each boundary component that maps to a component in Λ^- in between the Reeb chords $c_i \dots c_1$ as above.

A subtle point here is that the moduli space $\mathcal{M}^{\text{sy}}(\mathbf{c})$ naturally gives rise to a chain σ in $C_{-*}(\Omega\Lambda^{\times(i+1)})$ rather than in $C_{-*}(\Omega\Lambda)^{\otimes(i+1)}$. Note that σ_i are simply components of σ , they are not considered as chains. To separate these out we apply the cubical Alexander-Whitney map, recalled in Section 3.1:

$$\eta: C_{-*}((\Omega\Lambda)^{\times(i+1)}) \rightarrow C_{-*}(\Omega\Lambda)^{\otimes(i+1)}$$

With these conventions we then define for $i \geq 0$,

$$\Delta_i(c_0) := \sum_{\mathbf{c}=c_0^+ c_i^- \dots c_1^-} \eta[\mathcal{M}^{\text{sy}}(\mathbf{c})],$$

where we separate the components of the tensor product by the Reeb chords in \mathbf{c}' in analogy with the notation for the product chain and where η is the Serre diagonal. The output of $\Delta_i(c_0)$ is thus a sum of alternating words of chains of loops in $C_{-*}(\Omega\Lambda)$ and Reeb chords, and Δ_i is an operation of degree $2 - i$ on $LC_*(\Lambda)$. We point out that if there are Λ^+ components then higher dimensional moduli spaces contribute to the differential (unlike the case when $\Lambda = \Lambda^-$). Note also that it is possible to have holomorphic disks contributing to Δ_0 , which means that the chord c_0 is the positive puncture of a disk without negative punctures.

Proposition 21 *Let $d: CE^* \rightarrow CE^*$ be the map extended to CE^* by the graded Leibniz rule. Then d is a differential, $d^2 = 0$. We call CE^* with the differential d the Chekanov-Eliashberg DG-algebra.*

Remark 22 *When $\Lambda = \Lambda^-$, CE^* was called LCA^* in [31] - this is cohomologically graded version of the usual Legendrian homology algebra LHA_* in [11]. By definition, we have $LHA_* = CE^{-*}$.*

Proof In case there are only components in Λ^- involved the result follows from standard arguments involving the boundary of 1-dimensional moduli spaces. Consider therefore the case when there are chains in the loop space involved.

Let $\mathbf{c} = c_0^+ c_m^- \dots c_1^-$. The d -dimensional moduli space $\mathcal{M}^{\text{sy}}(\mathbf{c})$ contributes to dc_0 . Its boundary consists of broken curves with one level of dimension $d - k$ and one of dimension k for $0 < k < d$, we find, with ∂ denoting the natural tensor extension of

the boundary operator in singular homology over boundary components involved in Λ^+ , that

$$\partial[\mathcal{M}^{\text{sy}}(\mathbf{c})] = [\mathcal{M}^{\text{sy}}(\mathbf{c}') \star \mathcal{M}^{\text{sy}}(\mathbf{c}'')]]$$

where \star denotes the Pontryagin product, where $\mathbf{c}' = c_0^+ c_m^- \dots c_j^- b^- c_{j-k}^- \dots c_1^-$, and $\mathbf{c}'' = b^+ c_{j-1}^- \dots c_{j-k+1}^-$. We next apply the cubical Alexander-Whitney map η to this formula to deduce

$$\begin{aligned} \partial \circ \eta[\mathcal{M}^{\text{sy}}(\mathbf{c})] &= \eta \circ \partial[\mathcal{M}^{\text{sy}}(\mathbf{c})] = \eta[\mathcal{M}^{\text{sy}}(\mathbf{c}') \star \mathcal{M}^{\text{sy}}(\mathbf{c}'')]] \\ &= \eta[\mathcal{M}^{\text{sy}}(\mathbf{c}')] \star \eta[\mathcal{M}^{\text{sy}}(\mathbf{c}'')]], \end{aligned}$$

where we used that the η is a chain map and is compatible with the product. The fact that η is a chain map is well known. We verify that it is compatible with the product below. It follows that the terms contributing to d^2 which arises from the differential acting on chains and acting on Reeb chords cancel.

It remains to check that η is compatible with the products, to this end we provide the simplest non-trivial case, the general case is done similarly. Thus, we need to check that the following compositions agree:

$$\begin{aligned} &C_{-*}(\Omega_{p_u}\Lambda_u \times \Omega_{p_v}\Lambda_v) \otimes C_{-*}(\Omega_{p_v}\Lambda_v \times \Omega_{p_w}\Lambda_w) \\ &\quad \xrightarrow{\times} C_{-*}(\Omega_{p_u}\Lambda_u \times \Omega_{p_v}\Lambda_v \times \Omega_{p_v}\Lambda_v \times \Omega_{p_w}\Lambda_w) \\ &\quad \xrightarrow{\mathbf{1} \times \circ \times \mathbf{1}} C_{-*}(\Omega_{p_u}\Lambda_u \times \Omega_{p_v}\Lambda_v \times \Omega_{p_w}\Lambda_w) \\ &\quad \xrightarrow{\eta} C_{-*}(\Omega_{p_u}\Lambda_u) \otimes C_{-*}(\Omega_{p_v}\Lambda_v \times \Omega_{p_w}\Lambda_w) \\ &\quad \xrightarrow{\mathbf{1} \otimes \eta} C_{-*}(\Omega_{p_u}\Lambda_u) \otimes C_{-*}(\Omega_{p_v}\Lambda_v) \otimes C_{-*}(\Omega_{p_w}\Lambda_w) \end{aligned}$$

and

$$\begin{aligned} &C_{-*}(\Omega_{p_u}\Lambda_u \times \Omega_{p_v}\Lambda_v) \otimes C_{-*}(\Omega_{p_v}\Lambda_v \times \Omega_{p_w}\Lambda_w) \\ &\quad \xrightarrow{\eta \otimes \eta} C_{-*}(\Omega_{p_u}\Lambda_u) \otimes C_{-*}(\Omega_{p_v}\Lambda_v) \otimes C_{-*}(\Omega_{p_v}\Lambda_v) \otimes C_{-*}(\Omega_{p_w}\Lambda_w) \\ &\quad \xrightarrow{\mathbf{1} \otimes \times \otimes \mathbf{1}} C_{-*}(\Omega_{p_u}\Lambda_u) \otimes C_{-*}(\Omega_{p_v}\Lambda_v \times \Omega_{p_v}\Lambda_v) \otimes C_{-*}(\Omega_{p_w}\Lambda_w) \\ &\quad \xrightarrow{\mathbf{1} \otimes \circ \otimes \mathbf{1}} C_{-*}(\Omega_{p_u}\Lambda_u) \otimes C_{-*}(\Omega_{p_v}\Lambda_v) \otimes C_{-*}(\Omega_{p_w}\Lambda_w) \end{aligned}$$

This is easily checked by evaluating it on a test chain (σ, τ) . Note that this uses the fact that the cubical chain complex is a quotient. Namely, degenerate cubical chains are divided out. \square

Remark 23 As discussed in Section A, the moduli spaces in the definition of the differential on CE^* above are defined in terms of anchored moduli spaces \mathcal{M}^{sy} , i.e.,

moduli spaces of disks with additional interior punctures where holomorphic planes in the filling with asymptotic markers are attached. We point out that in order to calculate the differential one need only take into account rigid such holomorphic planes of dimension zero. For higher dimensional moduli spaces of planes of dimension $d_0 > 0$ the dimension of the curves in the symplectization is $d + 1 - d_0$ and does not contribute to the d -dimensional chain $[\mathcal{M}^{\text{sy}}]$.

Remark 24 As mentioned in the introduction we relate CE^* as defined above to a parallel copies version of the same algebra which is defined solely in terms of rigid moduli spaces. In order to do so it is convenient to use a topologically simpler but algebraically more complicated model of CE^* , defined as follows. The generating set of our algebra is extended to chains in the product $(\Omega\Lambda)^{\times(i+1)}$, where we separate the coordinate functions by Reeb chords. We define the product of two such chains by taking the Pontryagin product of the chains at adjacent factors giving an operation

$$C_{-*}((\Omega\Lambda)^{\times(i+1)}) \otimes C_{-*}((\Omega\Lambda)^{\times(j+1)}) \rightarrow C_{-*}((\Omega\Lambda)^{\times(i+j+1)}).$$

The differential on this version of CE^* is then defined by the singular differential on the chain and as

$$\Delta_i(c_0) := \sum_{\mathbf{c}=c_0^+c_i^- \dots c_1^-} [\mathcal{M}^{\text{sy}}(\mathbf{c})],$$

on Reeb chord generators. (In other words we define it as above but disregard the diagonal approximation.) It follows from the Künneth formula that the two versions of CE^* are quasi-isomorphic.

Although, the definitions of CE^* given above works generally, from a computational perspective, it is hard to get our hands on, as the cubical chain complexes $C_{-*}(\Omega_{p_v}\Lambda_v)$ have uncountably many elements.

Next, we provide a modification of the definition that gives a quasi-isomorphic DG-algebra, under the assumption that for each $v \in \Gamma^+$, there exists a DG-algebra quasi-isomorphism

$$\Phi: C_{-*}(\Omega_{p_v}\Lambda_v) \rightarrow \mathbb{K}\langle \mathcal{E}_v \rangle$$

where $\mathbb{K}\langle \mathcal{E}_v \rangle$ is a DG-algebra structure on a free algebra generated by a graded finite set \mathcal{E}_v . For example, as discussed in Section 3.1, such a DG-algebra map exists when Λ_v is simply-connected (or if Λ_v is a $K(\pi_1, 1)$ space, one can first work with the group ring $\mathbb{K}[\pi_1]$ and base-change to a cofibrant replacement of it).

We define a graded quiver \mathcal{Q}_Λ with vertex set $\mathcal{Q}_0 = \Gamma$ and arrows in correspondence with

$$\mathcal{Q} := \mathcal{R} \cup \bigcup_{v \in \Gamma^+} \mathcal{E}_v.$$

More precisely, there are arrows from vertex v to w corresponding to the set of Reeb chords from Λ_v to Λ_w . In addition, for each $v \in \Gamma^+$, there are arrows from v to v corresponding to the elements in \mathcal{E}_v . Let $LC_*(\Lambda)$ be the graded \mathbf{k} -bimodule generated by \mathcal{Q} . Thus, there is one generator for each arrow in \mathcal{Q} and an idempotent e_v for each vertex $v \in \mathcal{Q}_0$ and write $\overline{LC}_*(\Lambda)$ for the submodule without the idempotents.

Let $CE^*(\Lambda)$ be \mathbf{k} -algebra given by the tensor algebra

$$CE^*(\Lambda) = \mathbf{k} \oplus \bigoplus_{i=1}^{\infty} \overline{LC}_*(\Lambda)[-1]^{\otimes i}$$

Recall that the path algebra of a quiver is defined as a vector space having all paths in the quiver as basis (including, for each vertex v an idempotent e_v), and multiplication given by concatenation of paths. Thus, the \mathbf{k} -bimodule $CE^*(\Lambda)$ is the path algebra of the quiver \mathcal{Q}_Λ where the grading of each arrow is shifted up by 1. Just like in the cobar construction we write elements in $CE^*(\Lambda)$ as

$$[x_m | \dots | x_1] = s^{-1}x_m \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} s^{-1}x_1 \in CE^*(\Lambda),$$

where $x_j \in \overline{LC}_*(\Lambda)$.

Next, we equip the \mathbf{k} -algebra $CE^*(\Lambda)$ with a differential using the moduli spaces of holomorphic disks defined in Section A. This differential is induced by operations

$$\Delta_i: \overline{LC}_*(\Lambda) \rightarrow \overline{LC}_*(\Lambda)^{\otimes i}, \quad i = 0, 1, \dots,$$

where $\overline{LC}_*(\Lambda)^{\otimes 0} = \mathbf{k}$, that give $LC_*(\Lambda)$ the structure of an A_∞ -coalgebra if $\Delta_0 = 0$.

Consider a generator of $\overline{LC}_*(\Lambda)$. If it is a generator $\sigma \in \mathcal{E}_v$ of the free model of $C_{-*}(\Omega_{p_v}, \Lambda_v)$ for some component $\Lambda_v \subset \Lambda^+$ then we define

$$(24) \quad \Delta_i \sigma = d_i \sigma,$$

where d_i is the co-product that corresponds to i^{th} homogeneous piece of the differential in the free model $\mathbb{K}\langle \mathcal{E}_v \rangle$ of $C_{-*}(\Omega_{p_v}, \Lambda_v)$. If it is a Reeb chord c_0 the define $\Delta_i(c_0)$ as before using moduli spaces $\mathcal{M}^{\text{sy}}(\mathbf{c})$ but now take the image of all the singular chains in $C_{-*}(\Omega_{p_v}, \Lambda_v)$ under the map $\Phi: C_{-*}(\Omega_{p_v}, \Lambda_v) \rightarrow \mathbb{K}\langle \mathcal{E}_v \rangle$. Since the map Φ is a DG-algebra map, the proof that d is differential on CE^* is the same. Furthermore, since Φ is a quasi-isomorphism, we get a quasi-isomorphic chain complex CE^* if we use $\mathbb{K}\langle \mathcal{E}_v \rangle$ coefficients instead of $C_{-*}(\Omega_{p_v}, \Lambda_v)$.

From now on, unless otherwise specified, we will always assume that we work with a free (over \mathbf{k}) model of CE^* .

If there exists an augmentation $\epsilon: CE^*(\Lambda) \rightarrow \mathbf{k}$, then there is a change of coordinates which turn $LC_*(\Lambda)$ into a A_∞ -coalgebra. More precisely, consider the restriction of ϵ , $\epsilon_1: LC_*(\Lambda) \rightarrow \mathbf{k}$, where we think of $LC_*(\Lambda)$ as the degree 1 polynomials in $CE^*(\Lambda)$. Define

$$LC_*^\epsilon = \mathbf{k} \oplus \ker(\epsilon_1).$$

Note that $\ker(\epsilon_1)$ is generated idempotents e_ν and by $c - \epsilon(c)$, where c ranges over the generators of $\overline{LC}_*(\Lambda)$. Let

$$\phi_\epsilon: \bigoplus_{i \geq 0} LC_*^{\otimes \mathbf{k}^i} \rightarrow \bigoplus_{i \geq 0} LC_*^{\otimes \mathbf{k}^i},$$

be the \mathbf{k} -algebra automorphism defined on generators as $\phi_\epsilon(c) = c + \epsilon(c)$. Define the operations $\Delta_i^\epsilon: LC_*^\epsilon(\Lambda) \rightarrow LC_*^\epsilon(\Lambda)^{\otimes \mathbf{k}^i}$ by

$$\Delta_i^\epsilon = \phi_\epsilon \circ \Delta_i \circ \phi_\epsilon^{-1}.$$

Theorem 25 *The operations $(\Delta_i)_{i \geq 1}$ satisfy the A_∞ -coalgebra relations and with these operations $LC_*^\epsilon(\Lambda)$ is a coaugmented conilpotent coalgebra.*

Proof Let d denote the differential on $CE^*(\Lambda)$ and let c be a generator of $\overline{LC}_*(\Lambda)$. Since ϵ is an augmentation, $\epsilon(dc) = 0$ and it follows that $\Delta_0^\epsilon = 0$. The A_∞ -coalgebra relations then follows by combining the equation $d^2 = 0$ from Theorem 25 with the automorphism ϕ_ϵ .

The coaugmentation is simply the inclusion of \mathbf{k} . The fact that LC_*^ϵ is conilpotent follows from Stokes' theorem: the sum of the actions of the Reeb chords at the negative end of a disk contributing to the differential is bounded above by the action of the Reeb chord at the positive end. This gives the desired finiteness. \square

Remark 26 *Note that if the original operation Δ_0 on $LC_*(\Lambda)$ equals 0 then the map ϵ which takes all generators of $\overline{LC}_*(\Lambda)$ to 0 is an augmentation and in this case $LC_*(\Lambda)^\epsilon = LC_*(\Lambda)$ by construction.*

Remark 27 *If there is an augmentation $\epsilon: CE^*(\Lambda) \rightarrow \mathbf{k}$ then $CE^*(\Lambda)$ can be expressed as the cobar construction of a coalgebra: by construction*

$$CE^*(\Lambda) = \Omega(LC_*^\epsilon(\Lambda)).$$

We next consider the \mathbf{k} -linear dual $LA_\epsilon^*(\Lambda) := (LC_\epsilon^*(\Lambda))^\#$ of $LC_\epsilon^*(\Lambda)$. It follows from Section 2.1.3 that this is an augmented A_∞ -algebra. We call it the *Legendrian A_∞ -algebra*.

Remark 28 *In case $\Lambda = \Lambda^-$, it can be shown that this A_∞ -algebra can be obtained from the endomorphism algebra of the augmentation ϵ in Aug_- category of [10] by adjoining a unit to it, but is, in general, different from the endomorphism algebra in the Aug_+ category of [52].*

Definition 29 *Given an augmentation $\epsilon: CE^*(\Lambda) \rightarrow \mathbf{k}$, we define the completed Chekanov-Eliashberg DG-algebra to be $\widehat{CE}_\epsilon^* := B(LA_\epsilon^*)^\#$. The underlying \mathbf{k} -algebra is the completed tensor algebra:*

$$\widehat{CE}^*(\Lambda) = \varprojlim_i CE^*(\Lambda)/I^i = \mathbf{k}\langle \overline{LC}_\epsilon^*(\Lambda)[-1] \rangle$$

where $\overline{LC}_\epsilon^*(\Lambda)$ is the ideal determined by the natural augmentation.

Note that there is a natural chain map:

$$\Phi: CE^*(\Lambda) \rightarrow \widehat{CE}_\epsilon^*(\Lambda).$$

Remark 30 *To illustrate the various gradings, the unique Reeb chord for the standard Legendrian unknot Λ in \mathbf{R}^{2n-1} has degree is $-n$ in LC_* , n in LA^* and $-(n-1)$ in CE^* (while it is $(n-1)$ in LHA_*). Therefore, we have the graded isomorphisms: $H_*(LC) \cong H_{-*}(S^n)$, $H^*(LA) \cong H^*(S^n)$ and $H^*(CE) \cong H_{-*}(\Omega S^n)$.*

3.3 Parallel copies

In this section we will describe the perturbation scheme we will use to define various version of Lagrangian Floer cohomology. For exact Lagrangian submanifolds with Legendrian boundary we get an induced perturbation scheme for the Legendrian boundary that will allow us to define a simpler version of the Chekanov-Eliashberg algebra which is isomorphic to it in case the Legendrian is simply connected.

Let X be a Weinstein manifold and let $L \subset X$ be an exact Lagrangian submanifold with Legendrian boundary Λ . We assume that Λ is embedded but allow L to be a several component Lagrangian with components that intersect transversely. Assume that the components of L are decorated with signs and write L^+ and L^- for the union of the components decorated $+$ and $-$, respectively. We will use specific families of Morse

functions to shift Lagrangian and Legendrian submanifolds off of themselves in order to relate holomorphic curve theory to Morse theory, and to perform Floer cohomology calculations without Hamiltonian perturbations.

Before we give the details on how to choose the collection of Morse functions we explain the main strategy. Let L be a Lagrangian with Legendrian boundary Λ , then a neighborhood of L in X looks like T^*L and in the cylindrical end $[0, \infty) \times \Lambda$, the vector tangent to T^*L in the direction of the dual of the $[0, \infty)$ -direction corresponds to the Reeb direction. We will define a collection of parallel copies $L_j, j = 0, 1, 2, \dots, L = L_0$. The copies are graphs in T^*L of the differentials dF_j of Morse functions $F_j: L \rightarrow \mathbf{R}$. These Morse functions F_j have critical points in the compact part of L . In the cylindrical ends they look like Morsifications of the Reeb push-off, i.e. we have $F_j(t, q) = f_j(q)t + b_j$, where f_j is a Morse function on Λ , t a coordinate in $[0, \infty)$, and b_j is constant.

The first Morse function $H_1 = F_1$ gives the first parallel copy at small distance $\epsilon > 0$ from $L_0 = L$. We define L_1 as the graph of the differential of ϵF_1 . We want all other copies to be good approximations of L_1 as seen from L_0 , so that flow lines between L_j and L_0 and between L_1 and L_0 are approximately the same. Let $L_j^1 = L_1$ for $j > 1$. We next construct $L_2 = L_2^2$ as the graph of the differential of $\epsilon^2 H_2$ over L_1 . Here H_2 should be a good approximation of H_1 in order for flow lines between L_2 and L_1 to be approximately the same as those between L_0 and L_1 . Thus we take

$$H_2 = F_1 + \epsilon H_2 = H_1 + \epsilon H_2.$$

Again we need L_j to be close to L_2 for all $j > 2$ and we define $L_j^2 = L_2$. We next define $L_3 = L_3^3$ as the graph of the function $\epsilon^3 H_3$ over L_2 . In order for L_3 to look like L_1 from the point of view of L_2 and like L_2 from the point of view of L_1 we take

$$H_3 = F_2 + \epsilon^2 H_3 = H_1 + \epsilon H_2 + \epsilon^2 H_3.$$

Continuing in the obvious way we get

$$H_k = F_{k-1} + \epsilon^{k-1} H_k = \sum_{k=1}^k \epsilon^{k-1} H_k.$$

This collection of parallel copies then has the following features: flow trees with boundary on any increasing collection L_{i_1}, \dots, L_{i_k} are arbitrarily close to flow trees of L_0, \dots, L_{k-1} . To see how this works consider for example trivalent trees of L_j, L_{j+k}, L_{j+k+s} . By construction these trees are arbitrarily well approximated by trees of L_j, L_{j+1}, L_{j+2} and after scaling the corresponding shifts are well approximated by H_1 and H_2 which gives the trees of L_0, L_1, L_2 .

We next turn to a more detailed description of the above outline of how to choose functions for parallel copies. Consider a collection of pairs of Morse functions (F_j, f_j) , where $F_j: L \rightarrow \mathbf{R}$ and $f_j: \Lambda \rightarrow \mathbf{R}$, $j = 1, 2, \dots$ which have the following properties.

Write $[0, \infty) \times Y$ and $[0, \infty) \times \Lambda$ for the ends of X and L , respectively. In the end, with coordinates $(t, q) \in [0, \infty) \times \Lambda$, we require that all pairs (F_j, f_j) satisfy,

$$F_j(t, q) = \epsilon f_j(q) e^t + b, \quad \text{for } 1 \gg \epsilon > 0, \text{ and } b > 0,$$

and that F_j does not have any local maxima. We next discuss further restrictions related to critical points.

Let (F_1, f_1) be any pair of positive Morse functions as above. Let z_1^1, \dots, z_1^m be the critical points of F_1 and let x_1^1, \dots, x_1^l be the critical points of f_1 . Fix disjoint coordinate balls $B_j^1 \subset L$ around z_j^1 and $D_j^1 \subset \Lambda$ of x_j^1 such that F_1 and f_1 are given by a quadratic polynomial in these coordinates. Fix small $\sigma > 0$ and $\tau > 0$ such that

$$\begin{aligned} |dF_1| > \sigma_1 = \sigma \quad \text{on} \quad L - \bigcup_{j=1}^m B_1^j, \\ |df_1| > \tau \sigma_1 = \tau \sigma \quad \text{on} \quad \Lambda - \bigcup_{j=1}^l D_1^j, \end{aligned}$$

Let (F_2, f_2) be another pair of positive Morse functions with m respectively l critical points z_2^1, \dots, z_2^m and x_2^1, \dots, x_2^l , where

$$\begin{aligned} z_2^j \in B_1^j \quad \text{and} \quad \text{index}(z_2^j) = \text{index}(z_1^j), \\ x_2^j \in D_1^j \quad \text{and} \quad \text{index}(x_2^j) = \text{index}(x_1^j) \end{aligned}$$

Let $\sigma_2 < \sigma \sigma_1$ and fix coordinate balls $B_2^j \subset B_1^j$ and $D_2^j \subset D_1^j$ such that

$$\begin{aligned} |dF_2| > \sigma_2 \quad \text{on} \quad L - \bigcup_{j=1}^m B_2^j, \\ |df_2| > \tau \sigma_2 \quad \text{on} \quad \Lambda - \bigcup_{j=1}^l D_2^j. \end{aligned}$$

Finally, we make sure that $F_2 < \sigma F_1$, and $f_2 < \sigma f_1$ which we obtain by over all scaling. Note that we might have to shrink σ_2 after scaling.

We continue inductively and construct a family of pairs (F_k, f_k) , $k = 1, 2, \dots$ of positive Morse functions with the following properties. Each F_k has m critical points

z_k^1, \dots, z_k^m , each f_k has l critical points x_k^1, \dots, x_k^l . There is $0 < \sigma_k$ and disjoint coordinate balls B_k^j around z_k^j and D_k^j around x_k^j such that

$$\begin{aligned} |dF_k| > \sigma_k & \quad \text{on} \quad L - \bigcup_{j=1}^m B_k^j, \\ |df_k| > \tau\sigma_k & \quad \text{on} \quad L - \bigcup_{j=1}^l D_k^j. \end{aligned}$$

Furthermore, the following hold

- $B_k^j \subset B_{k-1}^j$ and $D_k^j \subset D_{k-1}^j$,
- $\sigma_k \leq \sigma_{k-1}$,
- $F_k < \sigma F_{k-1}$ and $f_k \leq \sigma f_{k-1}$.

We next take into account the sign decoration. Assume that L^- is non-empty. In this case we first construct functions $\{(G_j, g_j)\}_{j=1}^\infty$ exactly as above on all components of L . The actual functions $\{(F_j, f_j)\}$ on (L, Λ) are then $(F_j, f_j) = (G_j, g_j)$ on L^+ and $(F_j, f_j) = (-G_j, -g_j)$ on L^- .

We will use the pair of Morse functions

$$\left(H_k = \sum_{j=1}^k F_j, h_k = \sum_{j=1}^k f_j \right).$$

(Compared (F_j, f_j) to the function $\epsilon^k F_k$ in the outline.)

Consider a cotangent neighborhood of L which agrees with the symplectization of the 1-jet space of Λ at infinity. Using this neighborhood we extend the Morse function trivially in the auxiliary directions and let L_k be the time 1 flow of the Hamiltonian vector field X_{H_k} of the function H_k . Let $k_1 > k_0$. Then L_k has Legendrian boundary Λ_k which is the cylinder over the 1-jet graph of the function h_k on Λ .

Intersection points $L_{k_0} \cap L_{k_1}$ corresponds to intersection between the components of L and critical points of $H_{k_1} - H_{k_0}$. Since

$$H_{k_1} - H_{k_0} = F_{k_0+1} + \mathcal{O}(\sigma F_{k_0+1}),$$

we find that from the point of view of L_{k_0} , L_{k_1} can be viewed as a small perturbation of L_{k_0+1} . In particular, if $k_0 \neq k_1$ then $L_{k_0} \cap L_{k_1}$ is transverse and there is a unique intersection point near each intersection point in $L \cap L_1$ which corresponds to critical points of F_1 and self-intersections of L . Similarly, at infinity Reeb chords in a small neighborhood of Λ connect Λ_{k_0} to Λ_{k_1} , $k_0 < k_1$ and there is a unique transverse such

chord near each critical point f_1 , and there are no local Reeb chords connecting Λ_{k_1} to Λ_{k_0} if $k_1 > k_0$.

Remark 31 *In Sections B.2 and B.3 we will also apply this construction to Lagrangian submanifolds $C \subset W$, where W is a Weinstein cobordism with both positive and negative ends. Our Lagrangian submanifolds C that will be equipped with systems of parallel copies will however have only positive ends in that case and the above discussion applies without change. See Remark 55 for a version when both positive and negative ends are perturbed.*

3.4 Chekanov-Eliashberg algebra with parallel copies

We will relate the Legendrian invariants LA^* and LC_* to Floer (co)homology of exact Lagrangian submanifolds. When studying Lagrangian Floer (co)homology we employ the technique of parallel copies to achieve transversality and it will be convenient to express the Legendrian invariants in the same language. As it turns out, in the case that the Legendrians are simply connected this technique leads to a simpler formulation of the theory which incorporates a model of chains on the based loop space automatically. Recall that Lagrangian fillings of Legendrian submanifolds induce augmentations, which after change of variables lead to non-curved Legendrian A_∞ -coalgebras. Geometrically, this means that one uses anchored holomorphic disks. We will assume that an augmentation of Λ has been fixed in this section and all disks considered will be anchored with respect to this augmentation. We now turn to the description of this theory in the Legendrian setting.

Let Λ be as above with decomposition $\Lambda = \Lambda^+ \sqcup \Lambda^-$. Fix a Morse function $f: \Lambda \rightarrow \mathbf{R}$ which is positive on Λ^+ and negative on Λ^- . Use it as described in Section 3.3 to construct a system $\bar{\Lambda} = \{\Lambda_j\}_{j=1}^\infty$ of parallel copies of $\Lambda = \Lambda_0$.

Let \mathcal{Q}_0 , \mathbf{k} , and \mathcal{R} be as above. Let \mathcal{R}^+ denote the Reeb chords connecting Λ_0 to Λ_1 that lie in a small neighborhood of Λ_0 . By construction there is then a natural 1-1 correspondence between \mathcal{R}^+ and the set of critical points of f on Λ^+ . (Since f shifts Λ^- in the negative Reeb direction there are no such short chords near Λ^- .) Write

$$\mathcal{R}_\parallel = \mathcal{R} \cup \mathcal{R}^+,$$

and think of chords in \mathcal{R}^+ as connecting a component Λ_ν to itself. Again the set \mathcal{R}_\parallel is a graded set: Reeb chords $c \in \mathcal{R}$ are graded as above $|c| = -CZ(c)$ and the grading of a short chord $c \in \mathcal{R}^+$ equals the negative of the Morse index of the critical point of f corresponding to c .

We define a graded quiver $\mathcal{Q}_{\parallel, \Lambda}$ with vertex set $\mathcal{Q}_0 = \Gamma$ and arrows in correspondence with

$$\mathcal{Q}_{\parallel} := \mathcal{R}_{\parallel}.$$

More precisely, there are arrows from vertex v to w corresponding to the set of Reeb chords from Λ_v to Λ_w if $v \neq w$ and corresponding to short Reeb chords from $\Lambda_v = \Lambda_{v_0}$ to Λ_{v_1} if $v = w$.

Let $LC_{*}^{\parallel}(\Lambda)$ be the graded \mathbf{k} -bimodule generated by \mathcal{Q}_{\parallel} . We define an A_{∞} -coalgebra structure on $LC_{*}^{\parallel}(\Lambda)$ given by operations Δ_i as follows. Given a chord c_0 (input) and a chords c_i, \dots, c_1 (outputs), we consider the disk D_{i+1} with distinguished puncture at c_0 and an *strictly increasing* boundary decoration κ . Let $\mathbf{c} = c_0^+ c_i^- \dots c_1^-$. Consider the moduli space $\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)$. We write $|\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)|$ for the algebraic number of \mathbf{R} -components in this moduli space provided $\dim(\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)) = 1$ and $|\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)| = 0$ otherwise. Define for $i > 0$,

$$\Delta_i(c_0) := \sum_{\mathbf{c} = c_0^+ c_i^- \dots c_1^-} |\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)| \mathbf{c}',$$

where $\mathbf{c}' = c_i \dots c_1$. This gives an operation of degree $2 - i$ on $LC_{*}^{\parallel}(\Lambda)$. Note that $\Delta_0 = 0$ trivially, since the decoration κ is strictly increasing.

It is not a priori clear that the maps Δ_i are well-defined as the sum may involve infinitely many terms. This can be avoided easily if Λ_v are simply-connected for $v \in \Gamma^+$. Namely, the simply-connectedness of Λ_v guarantees that the short chords cost index, whereas the long chords cost energy. In the non-simply connected case, we will only consider a completed version of LC_{*}^{\parallel} where infinite expressions are allowed.

Lemma 32 *Suppose Λ is simply-connected or more generally, suppose that Δ_i factorizes through the natural inclusion*

$$\bigoplus_{i=1}^{\infty} LC_{*}^{\parallel}[-1]^{\otimes k_i} \rightarrow \prod_{i=1}^{\infty} LC_{*}^{\parallel}[-1]^{\otimes k}$$

for all $i \geq 1$, then LC_{*}^{\parallel} equipped with the operations $(\Delta_i)_{i \geq 1}$ is an A_{∞} -coalgebra.

Proof Recall that the moduli spaces involved in the definition of the operations Δ_i are moduli spaces of anchored disks, where we use the augmentation induced by the filling L . Consider a 1-dimensional moduli space $\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)$ and note that its boundary consists of two level rigid disks by Theorems 64 and 63. There are two cases, either the levels are joined at a chord connecting the same copy of Λ to itself or the chord connects distinct copies. Since we count anchored disks and since the augmentation satisfies

the chain map equation, the first type of breaking cancels algebraically. Theorem 65 then shows that there are canonical identifications between moduli spaces for different increasing numberings and the proof follows since the second type of breakings are exactly what contributes to the coalgebra relations and are then in algebraic one to one correspondence with the endpoints of an oriented compact 1-manifold. \square

We next introduce the parallel copies version of the Chekanov-Eliashberg algebra. Consider a generator of LC_*^{\parallel} which is a small Reeb chord z that corresponds to a critical point of the Morse function f . The coalgebra operations $\Delta_i(z)$ then counts holomorphic disks with positive puncture at z which, by an action argument, must lie in a small neighborhood of $\mathbf{R} \times \Lambda$. In particular, rigid such holomorphic disks are in natural correspondence with rigid Morse flow trees. In the simply connected case, i.e., for Morse functions without critical points of index 1 and $n - 1$ we have the following result.

Lemma 33 *Suppose Λ is simply connected. If the Morse functions for parallel copies described above are sufficiently close to the first function (i.e., if $\epsilon > 0$ in the construction of shifting copies is sufficiently small) then if x_v is the minimum of the Morse function on the component Λ_v , the following holds: $\Delta_1(x_v) = 0$, $\Delta_2(x_v) = x_v \otimes x_v$, and $\Delta_i(x_v) = 0$ for $i > 2$. Furthermore, if c is any other generator corresponding to an arrow from v to w then $\Delta_2(c) = c \otimes x_v + (-1)^{|c|} x_w \otimes c$ and $\Delta_i(c)$ does not contain any factor x_v for $i \neq 2$.*

Proof The equations involving only Δ_1 and Δ_2 follows easily from considerations of one and two Morse functions. To see that the other equations hold we show that the corresponding spaces of flow trees are empty for perturbation sufficiently close to multiples of one Morse function. Consider a flow tree on a strip with slits with positive puncture at a and negative punctures at b_1, \dots, b_m , see Section 3.5 for notation and background on flow trees. Note that as we take the limit to multiples of the same Morse function the flow tree limits to a broken flow line starting at a connecting to b_{i_1} , then from b_{i_1} to b_{i_2} , continuing in this way until all negative punctures have been met.

Consider now a tree with positive puncture at x_v . In the limit this converges to a flow line emanating from x_v , which must be constant. This shows that all negative punctures must be x_v as well and hence $\Delta_i(x_v) = 0$ if $i > 2$ for dimensional reasons.

Consider next a tree with positive puncture at some critical point c and assume that one of the negative punctures is x_v . Consider first the case when x_v is at the last puncture corresponding to the smallest function difference. Assume that the negative punctures

are x_v and a word of punctures b then $\|c\| - \|b\| - \|x_v\| - 1 = \|c\| - \|b\| = 0$, where $\|y\| = \text{index}(y) - 1$. Consider the limit when this function difference goes to zero then the flow tree goes to a flow tree with a flow line to x_v attached. The remaining flow tree has dimension $\|c\| - \|b\| - 1 = -1$ and hence does not exist by genericity for function differences before the last one.

Consider next the case that x_v is at some other function difference. Then we have negative punctures b before x_v and a after x_v and thus a are at smaller function differences. Consider the limit when all smaller function differences shrink. In the limit we find a flow tree with negative punctures at (b, x_v) with a partial flow tree with negative punctures at a attached. The evaluation dimension of the latter tree (i.e. the dimension of the partial tree with a free positive puncture) is

$$(n - 1) - \|a\| < n - 1,$$

where we use the simple connectivity to get strict inequality. Applying the degeneration above to the remaining tree we get a tree with a flow line to x attached and its evaluation dimension is

$$\|c\| - \|b\|.$$

Now, $\|c\| - \|b\| - \|a\| - 1 = -1$ so these two does not meet.

The remaining possibility is that the small tree intersects the flow line towards x . Note however that such a tree can be viewed as the original partial tree merging with a flow line from the minimum and then continuing. At the scale of the tree with punctures c and b the flow line from the minimum and the flow at the positive puncture of the partial tree are very close to parallel, therefore the evaluation map at the positive puncture of the partial tree with negative punctures (x_v, a) is arbitrarily close to the evaluation map of the original partial tree and taking the limit, the above dimension count: $\|c\| - \|b\| - \|a\| - 1 = -1$ shows that these does not intersect for generic data and sufficiently small perturbation. \square

Remark 34 *The simple connectivity is used in the above proof to ensure that cutting with a small tree really reduces dimension. Here cutting means, intersect and start a flow from the intersection locus. In the case that there are index 1 critical points one could have $|a| = 0$ in the above and indeed there are trees with arbitrarily many punctures at index 1 critical points and then a puncture at x_v .*

Lemma 33 shows that there is a strict coaugmentation

$$(25) \quad \eta: \mathbf{k} \rightarrow LC_*^{\parallel}, \quad \eta(e_v) = x_v.$$

Definition 35 If Λ is simply-connected, the parallel copies Chekanov-Eliashberg DG-algebra is

$$CE_{\parallel}^* = \Omega LC_{*}^{\parallel}.$$

If Λ is not simply-connected, then we define the completed DG-algebra

$$\widehat{CE}_{\parallel}^*(\Lambda) = \widehat{\Omega}LC_{*}^{\parallel}$$

with the differential induced from the operations $(\Delta_i)_{i \geq 1}$.

As in the case with loop space coefficients we also define the Legendrian A_{∞} -algebra of Λ as the linear dual of the Legendrian co-algebra:

$$LA_{\parallel}^* = (LC_{*}^{\parallel})^{\#}.$$

Note that since there is no finiteness condition necessary for LA^* , this is well-defined for all Λ without any need to complete.

3.5 Isomorphism between Chekanov-Eliashberg algebras in the simply connected case

We next show that if Λ is simply connected then $CE_{\parallel}^*(\Lambda)$ is in fact isomorphic to $CE^*(\Lambda)$. To this end we first establish a Morse theoretic version of Adams result mentioned in the introduction, which here corresponds to the purely local situation of the zero-section in a 1-jet space.

Let Q be a *simply-connected* smooth manifold with a base point $q \in Q$. Fix a system of positive Morse functions $\tilde{f} = \{f_j\}_{j=1}^{\infty}$ with properties as in Section 3.3 and assume that the functions have only one minimum and no index one critical points. (This can always be arranged by handle cancellation if $\dim(Q) \geq 5$, see Remarks 41 and 44 for the lower dimensional case.) We will first discuss a Morse flow tree model for chains on Q that we denote $CM_{-*}(Q)$. Our treatment of Morse flow trees follows [22].

Consider a strip $\mathbf{R} \times [0, m]$ or half-strip $[T, \infty) \times [0, m]$ with coordinates $s + i\tau$ and with $m - 1$ slits along $[a_j, \infty) \times j$, $j = 1, \dots, m - 1$, and $T \leq a_j$ in the half-strip case. In the half strip case the vertical segment $T \times [0, m]$ is a finite end that will be used as an input, and we do not consider it as a part of the boundary of the strip with slits. In the strip case, the input is at the puncture $-\infty \times [0, m]$, and in both cases we call punctures at $+\infty$ output. Order the boundary components according to the positive boundary orientation of the disk with punctures starting from the input and decorate its boundary components by a strictly increasing sequence of positive integers $\kappa_1 < \kappa_2 < \dots < \kappa_m$.

Let $\kappa = \{\kappa_i\}_{i=1}^m$ denote this decoration. Cutting the strip by line segments $a_j \times [0, m]$, $j = 1, \dots, m-1$, subdivides it into *strip regions* of the form $[s_0, s_1] \times [\tau_0, \tau_1]$, where $s_0 \in \{-\infty, a_1, \dots, a_m\}$, $s_1 \in \{a_1, \dots, a_m, \infty\}$, and $\tau_0, \tau_1 \in \{0, 1, \dots, m\}$ and with a numbering κ_j on each boundary component $[s_0, s_1] \times \{\tau_0\}$ and $[s_0, s_1] \times \{\tau_1\}$.

Definition 36 A flow tree is a continuous map from a strip with slits into Q which in each strip region $[s_0, s_1] \times [\tau_0, \tau_1]$ depends only on the first coordinate $s \in [s_0, s_1]$ and there satisfies the gradient equation

$$\dot{x}(s) = -\nabla(f_{\kappa_i} - f_{\kappa_j})(x(s)),$$

where κ_i is the numbering of the upper horizontal boundary of the strip region and κ_j that of the lower.

A partial flow tree is defined analogously except that the domain is a half-strip with slits $[T, \infty) \times [0, m]$.

If y is a critical point of f_1 then we let $|y| = -\text{index}(y)$ denote the negative Morse index of y . If $\mathbf{y} = y_0 y_1 \dots y_m$ is a word of critical points of f_1 then the space of flow trees $\mathcal{T}(\mathbf{y})$ with input puncture at y_0 and output punctures $\mathbf{y}' = y_1 \dots y_m$, in the order induced by the boundary orientation, has dimension

$$\dim(\mathcal{T}(\mathbf{y})) = |\mathbf{y}'| - |y_0| + (m - 2).$$

For sufficiently small perturbing system of Morse functions \tilde{f} , the space of flow trees is independent of the increasing boundary decoration κ . Let $CM_{-*}(Q)$ denote the \mathbb{K} -module generated by critical points of f_1 and equip $CM_{-*}(Q)$ with the structure of a coalgebra with operations Δ_i given by

$$\Delta_i(y_0) = \sum_{|\mathbf{y}'| = |y_0| - (i-2)} |\mathcal{T}(\mathbf{y})| \mathbf{y}',$$

where the sum ranges over all \mathbf{y}' of word length i . It is not hard to see that the boundary of a 1-dimensional space of flow trees consists of broken rigid flow trees from which it follows that the operations Δ_i satisfy the coalgebra relations. Furthermore, the coalgebra has a natural co-unit, the critical point which is the minimum of f_1 . We will call this critical point the *co-unit critical point*. We say that a flow tree with no puncture mapping to the co-unit critical point is *co-unit free*.

The coalgebra $CM_{-*}(Q)$ is in fact known to agree with the Floer coalgebra $CF_*(Q)$, where we view Q as the zero section in its own cotangent bundle T^*Q as follows. Let $\bar{Q} = \{Q_j\}_{j=0}^\infty$ be the system of parallel copies of the 0-section $Q \subset T^*Q$ corresponding to the system of functions \tilde{f} . Then by a special case of [22, Theorem 1.1] (the case

when fronts have no singularities at all), there is, for all sufficiently small shifts, a natural one to one correspondence between rigid holomorphic disks with boundary on \bar{Q} and Morse flow trees in Q determined by \tilde{f} . This gives a chain isomorphism $CM_{-*}(Q) \rightarrow CF_*(Q)$.

We now return to the Morse theoretic approach to Adams' result. We will define a map

$$\phi: C_{-*}(\Omega Q) \rightarrow \Omega CM_{-*}(Q),$$

in terms of the operation of attaching co-unit free partial flow trees to Moore loops $\sigma_v: [0, r_v] \rightarrow Q$, $r_v \geq 0$, based at $q \in Q$, parameterized by a simplex, $v \in \Delta$. To define this operation we will use the following notion: we say that a partial flow tree parameterized by a half-strip $\gamma: [T, \infty) \times [0, m] \rightarrow Q$ starts at a point $p \in Q$ if its input puncture maps to p , $\gamma(T \times [0, m]) = p$.

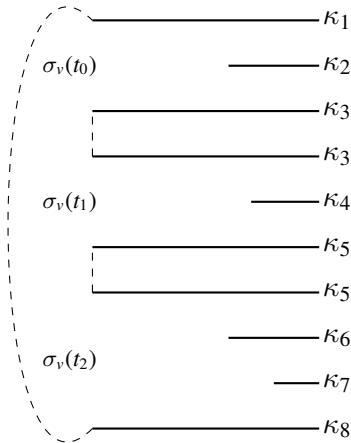


Figure 3: A configuration with 3 partial flow trees attached to σ_v at the points $\sigma_v(t_0)$, $\sigma_v(t_1)$, $\sigma_v(t_2)$. The numbers on the right determines the gradient equation at that end. The dashed part represents the loop σ_v .

Attaching partial flow trees to $\sigma_v: [0, r_v] \rightarrow Q$ then means fixing points $0 \leq t_0 \leq t_1 \cdots \leq t_m \leq r_v$ and partial flow trees Γ_j , $j = 1, \dots, m$ that start at $\sigma_v(t_j)$. Our map ϕ takes values in $\Omega CM_{-*}(Q)$ and will accordingly be defined by attaching flow trees which have no output at the co-unit. Note that the set of positive punctures of

minimum free partial flow trees for a fixed decreasing boundary numbering constitutes a codimension two subset in Q and that the corresponding subset for any numbering lies in an arbitrarily small neighborhood of it. Choosing the Morse data generically means in particular that the base point q does not lie on any minimum free partial flow tree.

If $\sigma_v: [0, r_v] \rightarrow Q$ is a loop with flow trees attached at $0 \leq t_0 \leq t_1 \cdots \leq t_m \leq r_v$, we also introduce a numbering of the components of $[0, r_v] - \{t_1, \dots, t_m\}$ induced by the flow trees attached as follows. The right most interval $(t_m, r_v]$ is numbered by κ_0 . The right boundary segment of the strip with slits attached at $\sigma_v(t_m)$ is numbered by κ_0 as well, whereas the left boundary segment of its domain is numbered by κ_{k_m} . We number the boundary segment (t_{m-1}, t_m) and the right boundary segment of the domain of the flow tree attached at $\sigma_v(t_{m-1})$ by κ_{k_m} as well. The left boundary segment of the flow tree attached at $\sigma_v(t_{m-1})$ is then numbered by $\kappa_{k_{m-1}}$, which determines the numbering of the segment (t_{m-2}, t_{m-1}) as well as the right boundary segment in the flow tree at $\sigma_v(t_{m-2})$, etc. We view the end result of this process as the domain for a loop with flow trees attached with numbering κ that decreases, see Figure 3.

Note next that if $\sigma: I^d \rightarrow \Omega Q$ is a d -dimensional cube in general position with respect to \tilde{f} , the set of σ_v , $v \in I^d$ for which a single partial flow tree can be attached is at most $(d-1)$ -dimensional. Attaching more partial flow trees, the dimension decreases further, by at least one for each flow tree. We say that the loops in σ with flow trees attached which form 0-dimensional family are the rigid loops with flow trees in σ . If σ is a cubical simplex in ΩQ and if \mathbf{y} is a word of critical points then we let

$$\mathcal{T}(\sigma; \mathbf{y})$$

denote the space of loops with flow trees in σ , where the critical points at punctures of the flow trees read in order give the word \mathbf{y} . The dimension of $\mathcal{T}(\sigma; \mathbf{y})$ is then

$$\dim(\mathcal{T}(\sigma; \mathbf{y})) = |\mathbf{y}| + \dim(\sigma) + (\ell(\mathbf{y}) - 1),$$

where ℓ is the word length.

Note that if the set of loops in σ with flow trees is transversely cut out then by definition of the system of perturbing Morse functions, the loops with flow trees corresponding to a different decreasing numbering κ' is canonically diffeomorphic to that corresponding κ . We define the map ϕ by counting rigid loops with flow trees in cubical simplices σ :

$$(26) \quad \phi(\sigma) = \sum_{\dim(\mathcal{T}(\sigma; \mathbf{y}))=0} |\mathcal{T}(\sigma; \mathbf{y})| \mathbf{y}.$$

Remark 37 *The map ϕ is defined only for chains that are suitably transverse to the Morse flow trees defined by the system of shifting functions \bar{f} . Since the Morse functions do not have any index one critical points a partial flow tree has at most $\dim(M)$ punctures. There is a natural evaluation map on partial flow trees that takes a flow tree to the location of its positive puncture. The image of this map for partial flow trees not involving the minimum is a stratified space of codimension one and by definition of the perturbation scheme the corresponding set for partial flow trees defined by distinct boundary numberings lie arbitrarily close to each other. The map ϕ above is defined for chains of loops with evaluation maps that are smooth and transverse to this stratified subset. It is straightforward to see that any chain of loops is homotopic to such a transverse chain and that any two transverse chains that are homologous are homologous through a transverse chain of loops. This means that we can replace all chains by chains that represent these transversality conditions.*

In order to connect this to the path loop fibration we consider a similar map

$$\hat{\phi}: C_{-*}(\mathbf{PQ}) \rightarrow \Omega CM_{-*}(Q) \otimes^{\mathfrak{t}} CM_{-*}(Q),$$

where \mathfrak{t} denotes the canonical twisting cochain of the cobar construction and \mathbf{PQ} the based path space. The map $\hat{\phi}$ again counts rigid paths with flow trees, where a path with flow trees is defined as a loop with flow trees except that we also attach a flow tree at the endpoint of the path. Furthermore, the last puncture in the flow tree attached at the endpoint is the distinguished last factor and is allowed to be the co-unit critical point. (Other punctures are still not the co-unit critical point, which means that flow trees before the last are still co-unit free and that in the last flow tree only the last puncture is allowed to be the co-unit). We point out that the differential on the twisted complex acting on the last factor in the right hand side counts trees where the last puncture is distinguished.

Lemma 38 *The maps ϕ and $\hat{\phi}$ are chain maps.*

Proof To see this note that the contributions to the chain map equation in both cases correspond to the boundary of an oriented compact 1-dimensional manifold. \square

Remark 39 *The codimension one boundary of $\mathcal{T}(\sigma; \mathbf{y})$ corresponds either to the loop or path moving to the boundary of σ or to the breaking of a flow tree at a critical point. Instances when two trees are attached at the same point are naturally interior points of the moduli space where the disks with slits joins to a new disk with a slit of width equal to the sum of the widths. See Figure 4.*

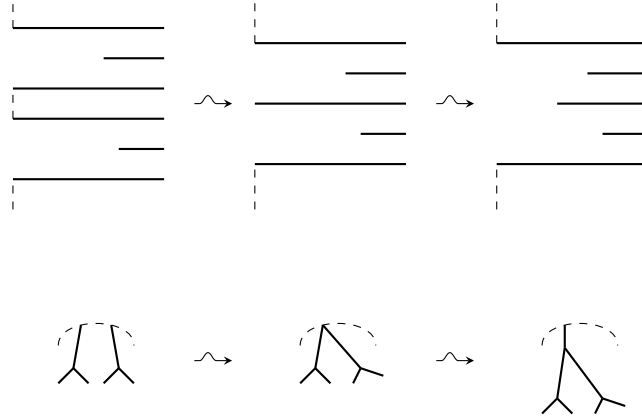


Figure 4: Flow trees attached at the same point are interior points: in the source (top), in the target (bottom).

With this established we can now prove Adams' result:

Theorem 40 *The flow tree map $\phi: C_{-*}(\Omega Q) \rightarrow \Omega CM_{-*}(Q)$ is a quasi-isomorphism.*

Proof The first observation is that the chain complex $\Omega CM_{-*}(Q) \otimes^t CM_{-*}(Q)$ is acyclic. To see that, note that for each critical point y there is a unique flow tree with positive puncture y and two negative punctures, one at the co-unit x and one at y . Filtering by degree we find that this is the only part of the differential which contributes to the differential on the first page of the corresponding spectral sequence. Since it gives an isomorphism from words not ending with the co-unit x to those which does end with x , the desired acyclicity follows. Clearly, $C_{-*}(PQ)$ is also acyclic.

Note next that the map $\hat{\phi}$ respects the filtration by degree at the end point. The corresponding E_1 -terms with induced maps are

$$C_{-*}(Q; H_{-*}(\Omega Q)) \rightarrow H_*(\Omega CM_{-*}(Q) \otimes CM_{-*}(Q)),$$

and E_2 -terms

$$H_{-*}(Q; H_{-*}(\Omega Q)) \rightarrow H_{-*}(Q; H_*(\Omega CM_{-*}(Q))).$$

Zeeman's comparison theorem [48, Sec. 3.3] then establishes the result. \square

Remark 41 *If $\dim(Q) = 4$ then the Morse function may have critical points of index one. In this case we use stabilization as follows. Multiply Q by \mathbf{R}^N for any $N \geq 2$, and consider the function $F(q, x) = f(q) + x^2$, then F has the same critical points as f and $-\nabla F$ is inward point at infinity. In $Q \times \mathbf{R}^N$ there is room to cancel 1-handles and*

the above applies. In this case we define $CM_{-*}(Q)$ to be $CM_{-*}(Q \times \mathbf{R}^N)$ (which is a 1-reduced version of the original complex). Noting that $C_{-*}(Q \times \mathbf{R}^N)$ and $C_{-*}(Q)$ are canonically isomorphic the result follows also in this case.

We next show that $CE_{\parallel}^*(\Lambda)$ and $CE^*(\Lambda)$ are isomorphic if Λ is simply connected. This is a more or less direct consequence of the description of rigid disks on a Legendrian with parallel copies and the isomorphism in Theorem 40. Since components in Λ_- are not affected by this choice of CE_{\parallel}^* versus CE^* , we disregard them and assume that $\Lambda = \Lambda_+$ in what follows.

Recall the definition of $CE^*(\Lambda)$ given in Remark 24, generated by chains $C_{-*}((\Omega_p \Lambda)^{\times(i+1)})$ in the product of the based loop space of Λ with factors separated by Reeb chords. Here the differential of a chain is just the usual differential of the chain whereas the differential of a Reeb chord is a sum over all moduli spaces of disks with one positive puncture at the chord and any number of negative punctures. Such a moduli space carries a fundamental chain and the contribution to the differential is alternating word of chains in the based loop space corresponding to the boundary arcs of the disk carried by the fundamental chain and Reeb chords at the negative puncture:

$$dc_0 = \sum_{\mathbf{c}'} [\mathcal{M}^{\text{sy}}(\mathbf{c})],$$

where $\mathbf{c} = c_0 \mathbf{c}'$ and \mathbf{c}' is a word of Reeb chords.

We next consider a system of parallel copies $\bar{\Lambda} = \{\Lambda_j\}_{j=0}^{\infty}$ defined by a system of positive Morse functions $\bar{f} = \{f_j\}_{j=1}^{\infty}$, where $\Lambda_0 = \Lambda$. Recall that the generators of the algebra $CE_{\parallel}^*(\Lambda)$ are the Reeb chords connecting Λ_0 to Λ_1 , and that these are long, corresponding to Reeb chords of Λ and short, corresponding to critical points of f_1 , except for the minimum. The differential counts rigid disks with one positive puncture in $\mathcal{M}^{\text{sy}}(\mathbf{b}; \kappa)$ where κ is a decreasing boundary numbering, $\mathbf{b} = b_0 \mathbf{b}'$.

If \mathbf{b}' is a word of Reeb chords let \mathbf{b}'_{lo} denote the word obtained by removing all short Reeb chords and \mathbf{b}'_{sh} that obtained by removing all long Reeb chords. Let $\bar{\Lambda}_{\delta}$ denote the system of parallel copies defined by the system $\delta \bar{f}$.

Theorem 42 *For all sufficiently small $\delta > 0$ there is a natural one to one correspondence between rigid holomorphic disks with one positive puncture in $\mathcal{M}^{\text{sy}}(\mathbf{b}; \kappa)$ and rigid configurations of a holomorphic disk with boundary on Λ in $\mathcal{M}^{\text{sy}}(b_0 \mathbf{b}'_{\text{lo}})$ with partial flow trees attached along the boundary, where the partial flow trees have negative punctures corresponding to \mathbf{b}'_{sh} and such that the total word of negative asymptotics is \mathbf{b}' .*

Proof This was proved in the setting of knot contact homology in [22, Corollary 5.14, 5.19, and 5.20] and in the case when there is only one flow line attached in [25, Theorem 3.6]. The proof has three parts, convergence, gluing, and surjectivity of gluing. All relevant technicalities in this more general case can be handled as there. \square

We next consider the map

$$\phi: CE^*(\Lambda) \rightarrow CE_{\parallel}^*(\Lambda),$$

which takes every Reeb chord to itself and takes a chain σ in the based loop space to $\phi(\sigma)$, where ϕ is as in (26) and where we identify the critical points of f_1 with the corresponding Reeb chords connecting Λ_0 to Λ_1 .

Theorem 43 *The map ϕ is a DG-algebra map and if Λ^+ is simply connected then ϕ is a quasi-isomorphism.*

Proof The fact that ϕ is a chain map follows from Theorem 42. Filter the algebras by action of Reeb chords in the left hand side and actions of long Reeb chords in the right hand side. The map respects this filtration and gives an isomorphism on the E_2 -pages by Theorem 40. Since the sum of actions of the Reeb chords at the negative end is bounded by that at the positive the spectral sequences converges. The theorem follows. \square

Remark 44 *Note that the isomorphism in Theorem 43 is compatible with the stabilization of Remark 41. To see this we multiply the ambient contact manifold Y with contact form α by $T^*\mathbf{R}^N$ and consider $\Lambda \times \mathbf{R}^N \subset Y \times T^*\mathbf{R}^N$ with contact form $\theta = (\alpha - y \cdot dx)$. The Reeb chord of $\Lambda \times \mathbf{R}^N$ then come in \mathbf{R}^N -families, one for each Reeb chord of Λ . Consider the contact form $e^{x^2}\theta$ and note that with respect to this contact form the Reeb chords of $\Lambda \times \mathbf{R}^N$ are in natural one to one correspondence with those of Λ and there is a canonical isomorphism between $CE^*(\Lambda)$ and $CE^*(\Lambda \times \mathbf{R}^N)$. In fact the disks in the differential are canonically identified. It follows in particular, that Theorem 43 holds also if $\dim(Q) \leq 4$.*

4 Lagrangian (co)algebra

As before, X is a Liouville manifold with $c_1(X) = 0$ and L is an exact relatively spin Lagrangian in X with vanishing Maslov class and ideal boundary given by the Legendrian Λ .

We will associate several chain level structures to L . To begin with, let us first assume that L is an embedded Lagrangian. Since L has boundary, in classical topology, one can consider either $C^*(L)$ or $C^*(L, \partial L)$. In our case, these two choices are reflected in the choices of $+$ or $-$ decorations on L , respectively. More generally, let L^v , $v \in \Gamma$ be the (irreducible) components of L . As with the Legendrian submanifolds in Section 3.2, we assume these components of L are decorated by signs and we write $L = L^+ \cup L^-$ for the corresponding decomposition. Let $F: L \rightarrow \mathbf{R}$ be a Morse function with prescribed behavior at infinity (depending on the $+$ or $-$ decoration) as explained in Section 3.3. We use this to construct a system of parallel copies $\bar{L} = \{L_j\}_{j=1}^\infty$, as in Section 3.3, shifted at infinity along the Reeb flow either in the positive or negative direction on L^+ and L^- , respectively.

Now, using the parallel copies, $\{L_j\}_{j=1}^\infty$, we define a graded quiver \mathcal{Q}_L as follows. The parallel copies $\{L_j\}_{j=1}^\infty$ give rise to following sets, fix $i_1 < i_2$ positive integers, and $v, w \in \Gamma$ with $v \neq w$:

- Intersection points $L_{i_1}^v \cap L_{i_2}^v$ in bijection with the union of critical points of $F|_{L^v}$. (These critical points may depend on the $+$ or $-$ decoration on L^v , for example, one can turn a $-$ decorated component L^v into $+$ decorated, by introducing critical points corresponding to the topology of ∂L^v , see Figure 5).
- Intersection points $L_{i_1}^v \cap L_{i_2}^w$ in bijection with $L^v \cap L^w$.

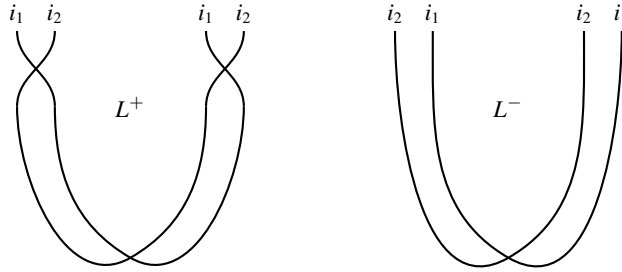


Figure 5: Difference between $+$ and $-$ generators. Both the left and the right hand side depicts shifts corresponding to Morse functions with a maximum. One of the intersection points in L_+ is the minimum and corresponds to the unit for the Floer cohomology product.

Furthermore, by the construction in Section 3.3, there are canonical bijections between the above sets associated to any pairs (i_1, i_2) and (i'_1, i'_2) with $i_1 < i_2$ and $i'_1 < i'_2$. So, fix a pair (i_1, i_2) such that $i_1 < i_2$ and define a graded quiver \mathcal{Q}_L with vertex set Γ and with an arrow connecting v to w (possibly equal to v) for each element of the above sets. Let \mathcal{I} denote the set of arrows.

Alternatively, one can describe the generators as the set of intersection points in $L_0 \cap L_1$, between the original L and the first shifted copy.

Let $CF^*(L)$ be the graded \mathbf{k} -bimodule generated by \mathcal{I} . Thus, there is one generator x_{vw} in degree $|x_{vw}|$ for each arrow in \mathcal{Q}_L from v to w . We endow $CF^*(L)$ with the structure of an A_∞ -algebra. Let x_0 be an intersection point generator and let $\mathbf{x}' = x_i \dots x_1$ be a word of intersection points. Consider the disk D_{i+1} with $i+1$ boundary punctures and with a decreasing numbering of its boundary arcs κ . Let $\mathbf{x} = x_0 x_i \dots x_1$. Consider the moduli space $\mathcal{M}^{\text{fi}}(\mathbf{x}; \kappa)$, see Appendix A for notation. Define the operations m_i by

$$m_i(\mathbf{x}') = \sum_{\ell(\mathbf{x}')=i, |x_0|=|\mathbf{x}'|+(2-i)} |\mathcal{M}^{\text{fi}}(\mathbf{x}; \kappa)| x_0$$

where $\ell(\mathbf{y})$ denotes the word length of \mathbf{y} and where $|\mathcal{M}^{\text{fi}}(\mathbf{x}; \kappa)|$ denotes the algebraic number of points in the oriented 0-dimensional manifold.

Lemma 45 *The operations m_i satisfy the A_∞ -algebra relations.*

Proof This follows by the usual argument: after noting that the decreasing boundary numbering ensures that there is no boundary bubbling, one observes that the terms in the A_∞ -algebra relations count the ends of a 1-dimensional oriented compact manifold by Theorems 63 and 64. Note also that the operations compose because of Theorem 65. \square

We call $CF^*(L)$ the Lagrangian Floer cohomology algebra of L . Let u_v denote the generator corresponding to the minimum on the component $L_v \subset L^+$. If L_v is simply connected then by Lemma 33 u_v is a strict idempotent. We write $\mathbf{k}_- \oplus CF^*(L)$ for the augmented algebra where we adjoined an idempotent e_w for each component $L_w \subset L^-$. (On these components the shifting function is decreasing at infinity and has a maximum rather than a minimum in the compact part.) Note that this is a connected algebra over \mathbf{k} .

Remark 46 *The two different choices of perturbations at infinity corresponding to $+$ and $-$ are the two extremal constructions where one pushes the copies either always in positive direction or always in the negative direction. One can also choose perturbations at infinity to depend on the topology of the manifold at infinity (see, for example, Section 4 of [1]). All our constructions should extend meaningfully to this more general setting but we have not pursued this direction in this paper.*

We next consider various linear duals of $CF^*(L)$ and associated algebraic objects. The simplest case occurs when $CF^*(L)$ is simply connected. In this case the linear dual $CF_*(L)$ is a coalgebra with operations Δ_i dual to m_i , and as before we can adjoin \mathbf{k}_- so that $\mathbf{k}_- \oplus CF_*(L)$ is co-augmented over \mathbf{k} . Then, we define the *Adams-Floer DG-algebra*

$$\Omega(\mathbf{k}_- \oplus CF_*(L)),$$

by applying the cobar construction.

In the non-simply connected case, we replace this object by the *completed Adams-Floer DG-algebra*:

$$(\mathbf{B}(\mathbf{k}_- \oplus CF^*(L)))^\#.$$

Example 47 *Let L be the standard Lagrangian D^n filling the standard Legendrian unknot $\Lambda \subset S^{2n-1}$. The Floer cohomology can be computed as follows:*

$$CF^*(L) = \begin{cases} \mathbb{K}x, |x| = 0, & \text{if } L \text{ is decorated with } +, \\ \mathbb{K}c, |c| = n. & \text{if } L \text{ is decorated with } -. \end{cases}$$

Alternatively, if we want compatibility with the inclusion $C^(D^n, \partial D^n) \rightarrow C^*(D^n)$ as*

$$CF^*(L) = \begin{cases} \mathbb{K}c \oplus \mathbb{K}y \oplus \mathbb{K}x, \\ |c| = n, |y| = n - 1, |x| = 0, dy = c & \text{if } L \text{ is decorated with } +, \\ \mathbb{K}c, |c| = n. & \text{if } L \text{ is decorated with } -. \end{cases}$$

In Section B.1, we introduce a model for wrapped Floer cohomology without Hamiltonian term and prove it is quasi-isomorphic to the usual wrapped Floer cohomology. We refer to there for details and give only a short description here. The chain complex underlying $CW^*(L)$ is the following. Let $L = L_0$ and shift L off itself to L_1 by a Morse function that is positive at infinity (as in the definition of parallel copies when L is decorated $+$). The generators of $CW^*(L)$ are then Reeb chords connecting L_1 to L_0 and intersection points in $L_0 \cap L_1$.

There is an A_∞ -functor, often called the *acceleration functor*,

$$CF^*(L) \rightarrow CW^*(L).$$

If L is decorated $+$, it can be shown that this functor is unital.

5 Maps relating Legendrian and Lagrangian (co)algebras

We continue with our usual set-up, where X is a Liouville manifold with $c_1(X) = 0$ and L is an exact Lagrangian in X with vanishing Maslov class and ideal boundary given by the Legendrian Λ . Let Γ be the set of embedded components of L subdivided into $\Gamma^+ \cup \Gamma^-$. Let Θ be the set of components of Λ with induced subdivision $\Theta^+ \cup \Theta^-$.

In this section we will define twisting cochains relating the parallel copies version $CE_{\parallel}^*(\Lambda)$ and the Floer cohomology $CF^*(L)$. Since L is an exact filling, we have an augmentation $\epsilon_L: CE^*(\Lambda) \rightarrow \mathbf{k}$. As in Section 3.2 we use this augmentation throughout to change coordinates in such a way that $\Delta_0 = 0$. As usual let $LC_*^{\parallel}(\Lambda)$ denote the corresponding coalgebra. When Λ^+ has simply connected components this is a co-unital coalgebra with co-unit

$$\sum_{v \in \Theta^+} x_v + \sum_{w \in \Theta^-} e_w,$$

and let $\eta: \mathbf{k} \rightarrow LC_*^{\parallel}(\Lambda)$ denote the co-augmentation

$$\eta(e_v) = \begin{cases} x_v & \text{if } v \in \Theta^+, \\ e_v & \text{if } v \in \Theta^-, \end{cases}$$

see (25), so that $CE_{\parallel}^* = \Omega LC_*^{\parallel}$. Note that this means that x_v are traded for e_v for $v \in \Theta^+$. Specifically, write $\overline{LC}_*^{\parallel}$ for $\text{coker}(\eta)$ and $\mathbf{k} \oplus \overline{LC}_*^{\parallel}$ for the corresponding strictly counital co-algebra. Then we have

$$CE_{\parallel}^* = \Omega LC_*^{\parallel} := \Omega(\mathbf{k} \oplus \overline{LC}_*^{\parallel}(\Lambda))$$

Consider the Floer cohomology A_{∞} -algebra $CF^*(L)$. In case all components of L^+ are simply connected there exists a strict idempotent $u_v \in CF^*(L)$ for each $v \in \Gamma^+$ corresponding to the minimum of the shifting Morse function and that we make $CF^*(L)$ unital by adding an idempotent e_w for each $w \in \Gamma^-$. We write the strictly unital algebra $\mathbf{k}_- \oplus CF^*(L)$. Let $\epsilon: \mathbf{k}_- \oplus CF^*(L) \rightarrow \mathbf{k}$ be the augmentation that maps u_v to e_v for $v \in \Gamma^+$ and e_w to e_w for $w \in \Gamma^-$. Consider the dual of the bar construction:

$$(27) \quad \mathcal{A} = (B(\mathbf{k}_- \oplus CF^*(L)))^{\#},$$

or in other words the completed Adams-Floer DG-algebra. In what follows we will represent \mathcal{A} as a quotient in way that generalizes to the non-simply connected case.

Consider adding extra idempotents e_v , $v \in \Gamma^+$ to $\mathbf{k}_- \oplus CF^*(L)$. This gives $\mathbf{k} \oplus CF^*(L)$ and we equip it with the trivial augmentation ϵ' which is the projection to \mathbf{k} . Let

$$\mathcal{A}' = (B(\mathbf{k} \oplus CF^*(L)))^{\#},$$

and let \mathcal{I} denote the subalgebra of \mathcal{A}' generated by all monomials that contains a dual of u_v for some $v \in \Gamma^+$. Let $\rho: \mathcal{A}' \rightarrow \mathcal{A}$ denote the restriction map.

Lemma 48 *The subalgebra \mathcal{I} is closed under the differential. In the simply connected case, the map ρ is a chain map with kernel \mathcal{I} and consequently, \mathcal{A} is quasi-isomorphic to \mathcal{A}'/\mathcal{I} .*

Proof To see that \mathcal{I} is closed under the differential we note that there is no negative gradient flowline that starts at the minimum, therefore any negative gradient tree that starts at u_v must also have a negative puncture at u_v . In the simply connected case, it is clear that \mathcal{I} is the kernel of ρ . The fact that ρ is a chain map follows from the u_v being strict idempotents. \square

In the general case we define $\mathcal{A} = \mathcal{A}'/\mathcal{I}$. Lemma 48 shows that in the simply connected case this definition agrees with the alternate definition of \mathcal{A} given in (27).

We next define a map

$$\mathfrak{t}': LC_*^{\parallel}(\Lambda) \rightarrow \mathcal{A}',$$

which will induce a twisting co-chain

$$\mathfrak{t}: \mathbf{k} \oplus \overline{LC}_*^{\parallel}(\Lambda) \rightarrow \mathcal{A}.$$

The map \mathfrak{t}' is defined by the following curve count for generators of $LC_*^{\parallel}(\Lambda)$. Fix systems of parallel copies \bar{L} of L . Let c be a Reeb chord of $\bar{\Lambda}$ and let $\mathbf{x}_0 = x_{0;1} \dots x_{0;j}$ be a word of intersection points of L . Let

$$\mathbf{c} = c\mathbf{x}_0$$

and define

$$(28) \quad \mathfrak{t}'(c) = \sum_{|\mathbf{x}_0|=|c|+(1-i)} |\mathcal{M}^{\text{fi}}(\mathbf{c})|_{\mathbf{x}_0}.$$

We then have the following.

Proposition 49 *If $v \in \Theta^+$ is such that Λ_v is a boundary component of L_w for $w \in \Gamma^+$ then*

$$\mathfrak{t}'(x_v) = u'_w,$$

where u'_w is the dual of the minimum u_w . Furthermore, \mathfrak{t}' satisfies the equation of a twisting cochain.

Proof For the first property we need to understand rigid holomorphic disks with positive puncture at the Reeb chord x_v . By small action such a holomorphic disk must lie in a neighborhood of L_w and is hence given by Morse flow trees. There is only one flow line emanating from the minimum in Λ_v and the flow line generically ends at the minimum of the shifting of L_w , see Figure 6. The first equation follows.

To see the twisting co-chain equation, we need to check that

$$m_1 \circ t' - t' \circ \Delta_1 + \sum_{d \geq 2} (-1)^d m_2^{(d)} \circ t'^{\otimes_{\mathbf{k}} d} \circ \Delta_d = 0$$

where $m_2^{(2)} := m_2$, and $m_2^{(i)} := m_2 \circ (\text{Id}_{\mathcal{A}} \otimes_{\mathbf{k}} m_2^{(d-1)})$. To this end, we consider the boundary of the 1-dimensional moduli space $\mathcal{M}^{\text{co}}(c_0 \mathbf{x})$. By Theorem 64 this corresponds to two level curves which by Theorems 63 and 65 form the boundary of an oriented compact 1-manifold. \square

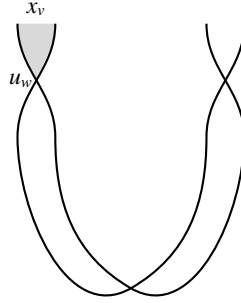


Figure 6: Minimum x_v is sent to the minimum u_w

Proposition 49 shows that t' maps the ideal generated by x_v , $v \in \Theta^+$ into $\mathcal{I} \subset \mathcal{A}'$. Hence t' induces a map

$$t: \mathbf{k} \oplus \overline{LC}_*^{\parallel}(\Lambda) \rightarrow \mathcal{A} / \mathcal{I} = \mathcal{A}.$$

Note that if $\eta': \mathbf{k} \oplus \overline{LC}_*^{\parallel}(\Lambda)$ and $\epsilon: \mathcal{A} \rightarrow \mathbf{k}$ is the trivial augmentation then $\epsilon \circ t = t \circ \eta = 0$.

Corollary 50 *The map t is a twisting co-chain.*

Proof Since t' satisfies the twisting co-chain equation so does t . \square

This twisting cochain is always defined, and determines a map:

$$(29) \quad CE_{\parallel}^*(\Lambda) \rightarrow \mathcal{A}.$$

We next consider the question whether $\mathfrak{t} \in \text{Kos}(\mathcal{C}, \mathcal{A})$. The following theorem gives a sufficient condition for \mathfrak{t} to be Koszul:

Theorem 51 *Suppose that $LC_*(\Lambda)$ is locally finite, simply-connected \mathbf{k} -bimodule. Suppose, in addition, that $HW^*(L) = 0$, then \mathcal{A} is quasi-isomorphic to $\Omega(\mathbf{k}_- \oplus CF_*(L))$ and $\mathfrak{t}: \mathcal{C} \rightarrow \mathcal{A}$ is a Koszul twisted cochain. In other words, the induced DG-algebra map:*

$$CE^*(\Lambda) \rightarrow \mathcal{A} \approx \Omega(\mathbf{k}_- \oplus CF_*(L))$$

is a quasi-isomorphism.

Corollary 52 *In the situation of Theorem 51, suppose that L is connected and decorated by $-$ and that $\partial L = \Lambda$ is diffeomorphic to a sphere S^{n-1} . Writing $\bar{L} = L \cup_{\partial} D^n$, there exists a quasi-isomorphism of DG-algebras*

$$CE^*(\Lambda) \rightarrow C_{-*}(\Omega\bar{L}),$$

where $\Omega\bar{L}$ is the based loop space of \bar{L} .

Proof We first note that there exists a quasi-isomorphism $\mathbf{k} \oplus CF^*(L) \rightarrow C^*(\bar{L})$ since L is an exact Lagrangian. We next use Theorem 51 and Adams' cobar equivalence ([4]):

$$\Omega(C_*(\bar{L})) \simeq C_{-*}(\Omega\bar{L})$$

which holds for the simply-connected topological space \bar{L} . □

Theorem 51 is obtained as a corollary of Theorem 18 and the following result:

Theorem 53 *Suppose that $HW^*(L) = 0$. Then, there exists a quasi-isomorphism of augmented A_{∞} -algebras:*

$$\Psi: \mathbf{k}_- \oplus CF^*(L) \rightarrow LA_{\parallel}^*(\Lambda),$$

such that

$$\Psi(e_v) = \begin{cases} \sum_{\Lambda_w \subset \partial L_v} e_w & \text{for } v \in \Gamma^-, \\ \sum_{\Lambda_w \subset \partial L_v} x_w & \text{for } v \in \Gamma^+. \end{cases}$$

Note that $HW^*(L) = 0$ if X is subcritical, or more generally a flexible Weinstein manifold.

Proof We construct an A_∞ -functor $\mathbf{e} = (\mathbf{e}_i)_{i \geq 1}$ where

$$\mathbf{e}_i : (CF^*(L))^{\otimes_{\mathbf{k}} i} \rightarrow LA_{\parallel}^*(\Lambda)$$

\mathbf{e}_i are constructed by dualizing the components of the map \mathcal{U}' . More explicitly, given c_0 and $\mathbf{x}_0 = x_{0;n}, \dots, x_{0;1}$, write $\mathbf{c} = c_0 \mathbf{x}_0$ as in (28), and define

$$\mathbf{e}_i(\mathbf{x}_0) = \sum_{c_0 \in \mathcal{R}} |\mathcal{M}^{\text{fi}}(\mathbf{c})|_{c_0}$$

The proof that $(\mathbf{e}_i)_{i \geq 1}$ is an A_∞ -functor follows as in the proof of Proposition 49.

Now, we need to check that \mathbf{e}_1 is a quasi-isomorphism. In the case that $L = L^-$, this is a consequence of the exact sequence for wrapped Floer cohomology induced by the subdivision of the complex into high and low energy generators:

$$0 \longrightarrow CW_0^*(L) \longrightarrow CW^*(L) \longrightarrow CW_+^*(L) \longrightarrow 0,$$

as follows. In terms of the version of wrapped Floer cohomology presented in Section B.1.1, the low energy sub-complex $CW_0^*(L)$ is generated by Lagrangian intersection points in $L_0 \cap L_1$, where $L = L_0$ and L_1 is the first parallel copy of L , shifted by a positive Morse function f that increases in the end, see Section 3.3. The differential on $CW_0^*(L)$ counts holomorphic strips which is incoming along L_1 and outgoing along L_0 at the output puncture. Similarly, the high energy quotient $CW_+^*(L)$ is generated by Reeb chords connecting Λ_1 to Λ_0 , the differential counts holomorphic strips interpolating between such Reeb chords. Since $CW^*(L)$ is acyclic it follows that the connecting homomorphism $HW_+^*(L) \rightarrow HW_0^{*+1}(L)$ is an isomorphism. In order to connect this to $CF^*(L)$ and $LA_{\parallel}^*(\Lambda)$, renumber the parallel copies so that L_1 now lies in the negative Reeb direction of L_0 at infinity and the shifting function f is replaced by $-f$. Then note that since L is labeled by $-$ the following holds.

- The linear dual of $CW_+^{*-1}(L)$ is canonically identified with $LA_{\parallel}^*(\Lambda)$ as a chain complex (Note that $CW_+^{*-1}(L)$ also has an A_∞ -coalgebra structure as defined in [29] and this should dualize to the A_∞ -algebra structure on LA^* but we do not need that here).
- The linear dual of $CW_0^*(L)$ is canonically identified with $CF^*(L)$ as a chain complex.
- The linear dual of the connecting homomorphism can be canonically identified with the map $\mathbf{e}_1 : \mathbf{k}_- \oplus CF^*(L) \rightarrow LA_{\parallel}^*(\Lambda)$ on critical points that counts strips with an input puncture at $L_0 \cap L_1$ and output puncture at a Reeb chord and is the canonical map on \mathbf{k}_- .

Since the connecting homomorphism is an isomorphism so is its linear dual. (The argument here is originally due to Seidel, compare [28, Theorem 7.2].)

In the case that $L^+ \neq \emptyset$ the argument just given applies after certain deformation that we describe next. For components in L^+ , $CF^*(L)$ and LA_{\parallel}^* are defined via parallel copies shifted in the positive Reeb direction at infinity. To connect to the previous case we consider a Lagrangian cobordism of two cylinders $\mathbf{R} \times \Lambda_0$ which is constant and $\mathbf{R} \times \Lambda_1$ which is the trace of an isotopy pushing Λ_1 across Λ_0 in the negative Reeb direction. Then the intersection points of the two cylinders are in natural 1-1 correspondence between the short Reeb chords between Λ_0 and Λ_1 . Furthermore, it is straightforward to show that there is exactly one transverse holomorphic strip connecting each intersection point to the corresponding Reeb chord at the negative end of the cobordism.

Adding these cylinders to L^+ we get a 1-parameter family of pairs of Lagrangian submanifolds \hat{L}_0^ρ and \hat{L}_1^ρ , where $\rho > 0$ is a gluing parameter that measures the length of the trivial cobordism between L^+ and the added cylinders. The wrapped Floer cohomology $CW^*(\hat{L}^\rho) := CW^*(L_0^\rho, L_1^\rho)$ is acyclic and is generated by the set of long Reeb chords C^+ from L_0 to L_1 , the set of intersection points I between the cylinders, and intersection points P in L . Let C^- denote the short Reeb chords connecting L_0 to L_1 and recall the natural 1-1 correspondence $C^- \approx I$ above. Let $\rho > 0$. We claim that the following sets are in natural 1-1 correspondence for all sufficiently large ρ :

- (i) Rigid strips of \hat{L}^ρ and L with input puncture at $c \in C^+$ and output puncture at $p \in P$.
- (ii) Rigid strips of \hat{L}^ρ with input puncture at $c \in C^+$ and output puncture at $q \in I$ and rigid strips of $\mathbf{R} \times \Lambda$ with input puncture in $c \in C^+$ and output puncture at $q \in C^-$.
- (iii) Rigid strips of \hat{L}^ρ with input puncture at $p \in P$ and output puncture at $q \in I$ and rigid strips of L with input puncture at $p \in P$ and positive puncture at $q \in C^-$.

To see this note first that the strips in (i) are unaffected by adding the almost trivial cobordism. For (ii), in the limit $\rho \rightarrow \infty$ the disks limits to an anchored disk with a positive puncture and a puncture at the intersection point. Gluing the basic strip connecting the intersection point to the short Reeb chord to it gives a 1-dimensional moduli space. The other boundary component of this moduli space consists of a rigid strip in the cobordism and a disk of dimension one in either symplectization end. Since the only rigid strips in the cobordism with negative ends at Reeb chords are either close to trivial strips or a basic strip it follows that the other end must be a trivial strip followed by a strip connecting c to q in the negative symplectization end. For (iii) we

note that every rigid strip must break under stretching into two rigid strips. Since the only rigid strips in the upper part are the basic strips the claim follows.

Observe that the strips in (i) and (iii) contribute to ι' and the strips in (ii) to the differential on $LC_*^{\parallel}(\Lambda)$. The vanishing of the wrapped Floer cohomology of \hat{L}^p then implies that ϵ_1 is a quasi-isomorphism. The last statement follows from Proposition 49. \square

Proof (of Theorem 51) Note that the A_∞ -quasi isomorphism given in Theorem 53 induces a quasi-isomorphism of DG-algebras

$$\Phi: \mathbf{B}(\mathbf{k}_- \oplus CF^*(L)) \rightarrow \mathbf{B}(LA_{\parallel}^*(\Lambda))$$

by an application of [21, Thm. 7.4] with respect to length filtrations on the bar construction.

By the local finiteness and simply-connectedness assumptions, each of these bar constructions are locally finite. So, we can apply the linear dual operation to get a quasi-isomorphism of DG-algebras:

$$(30) \quad \Phi^\#: \mathbf{B}(LA_{\parallel}^*(\Lambda))^\# \rightarrow \mathbf{B}(\mathbf{k}_- \oplus CF^*(L))^\#$$

The result then follows as in Theorem 18 where locally finiteness of the grading enabled us to appeal to Lemma 10. Therefore, the quasi-isomorphism given in (30) induces the quasi-isomorphism:

$$\Omega(LC_*^{\parallel}(\Lambda)) \rightarrow \Omega(\mathbf{k}_- \oplus CF_*(L))$$

as required. \square

Note that the A_∞ -algebras $\mathbf{k}_- \oplus CF^*(L)$ and $LA_{\parallel}^*(\Lambda)$ are finite rank \mathbf{k} -bimodules (in particular, they are locally finite), thus we can consider their \mathbf{k} -duals which are, by definition, the A_∞ -coalgebras $\mathbf{k}_- \oplus CF_*(L)$ and $LC_*^{\parallel}(\Lambda)$. However, unless we have the simply-connectedness assumption, the A_∞ -quasi isomorphism

$$\epsilon: \mathbf{k}_- \oplus CF^*(L) \rightarrow LA_{\parallel}^*(\Lambda)$$

does not necessarily dualize to an A_∞ -comap

$$\mathfrak{f}: LC_*^{\parallel}(\Lambda) \rightarrow \mathbf{k}_- \oplus CF_*(L)$$

because A_∞ -comaps are required to factorize through inclusion of the corresponding direct sum into the direct product as in (9). This is to ensure that A_∞ -coalgebra maps \mathfrak{f} induces a DG-algebra map $\Omega\mathfrak{f}$ on the cobar construction. If we drop this condition, we can observe that the A_∞ -quasi isomorphism dualizes to DG-algebra map

$$CE_{\parallel}^*(\Lambda) = \Omega LC_*^{\parallel}(\Lambda) \rightarrow \widehat{\Omega}(\mathbf{k}_- \oplus CF_*(L))$$

and this is just the map (29) induced by the twisting cochain \mathfrak{t} .

By the universal property of completion, this induces a DG-algebra map

$$\widehat{\Omega}(\mathfrak{f}): \widehat{CE}_{\parallel}^*(\Lambda) \rightarrow \widehat{\Omega}(\mathbf{k}_- \oplus CF_*(L))$$

Now, since \mathfrak{f} is a quasi-isomorphism, by using the length filtration on $\widehat{\Omega}$, and appealing to [21, Thm. 7.4], we can conclude the following.

Theorem 54 *Suppose that $HW^*(L) = 0$, then there exists a quasi-isomorphism of DG-algebras*

$$\widehat{CE}_{\parallel}^*(\Lambda) \rightarrow \widehat{\Omega}(\mathbf{k}_- \oplus CF_*(L)).$$

Note that the completion $\widehat{CE}_{\parallel}^*$ is in general a cruder invariant than both $CE_{\parallel}^*(\Lambda)$ and $CE^*(\Lambda)$. Though, we always have a map

$$CE^*(\Lambda) \rightarrow \widehat{CE}_{\parallel}^*(\Lambda).$$

Theorem 54 can be used to compute $\widehat{CE}_{\parallel}^*$ in a variety of cases. For example, if L is connected Lagrangian filling decorated by $-$ of a Legendrian Λ which is diffeomorphic to a sphere S^{n-1} , then writing $\bar{L} = L \cup_{\partial} D^n$, then we have a quasi-isomorphism $\mathbf{k} \oplus CF_*(L) \simeq C_*(\bar{L})$ since \bar{L} is an exact Lagrangian. Hence, we have a map

$$CE^*(\Lambda) \rightarrow \widehat{\Omega}(C_*(\bar{L}))$$

Here the right hand side can often be computed, in particular $H^0(\widehat{\Omega}(C_*(\bar{L})))$ is the group ring of the unipotent completion of the fundamental group $\pi_1(L)$ (cf. [16]).

In particular, any information on the completion map $CE^*(\Lambda) \rightarrow \widehat{CE}_{\parallel}^*(\Lambda)$ can help to obtain information about $CE^*(\Lambda)$. We will see an application of this idea in the next section.

We end this section with a discussion of the twisting cochains constructed above from an after surgery perspective. Assume that all components of Λ^- are spheres and recall the surgery isomorphism

$$\Theta: CW^*(C) \rightarrow CE^*(\Lambda)$$

of Conjecture 76. Let $\widetilde{\Theta} = \phi \circ \Theta$, where ϕ is the identity map on components in Λ_- and the map ϕ of Theorem 43. We next note that there is a natural A_{∞} -algebra map

$$\Psi: CW^*(C) \rightarrow \mathbf{B}(\mathbf{k} \oplus CF^*(L))^{\#} = \mathcal{A}' \rightarrow \mathcal{A} = \mathcal{A}' / \mathcal{I},$$

where we identify $\mathbf{k}_- \oplus CF^*(L)$ with the Floer cohomology $CF^*(L')$ of the manifold after surgery obtained by capping off all boundary spheres in Λ^- by disks. (Note that

in the simply connected case, the shifting Morse function then extends with a unique minimum in each disk which gives an idempotent corresponding to e_v .)

The map Ψ is defined by a curve count. Fix systems of parallel copies \bar{C} of C and \bar{L}' of L' . Let $\mathbf{c}' = c_1 \dots c_i$ be a word of Reeb chords of C and let $\mathbf{x}_0 = x_{0;1} \dots x_{0;j}$ be a word of intersection points of L' . Let

$$\mathbf{c} = \mathbf{c}'_{z_v \mathbf{x}_0 z_w}$$

and define

$$\Psi(\mathbf{c}') = \sum_{|\mathbf{x}_0|=|\mathbf{c}'|+(1-i)} |\mathcal{M}^{\bar{c}0}(\mathbf{c})|_{\mathbf{x}_0}.$$

Remark 55 We require here that the parallel copies \bar{L}' give a system of parallel copies of Λ near the surgery region such that for the components of Λ_v labeled by $-$ (resp. $+$) the induced parallel copies $\bar{\Lambda}_v$ lies in the negative (resp. positive) Reeb direction. Compare Figure 5.

Theorem 56 The map Ψ is a map of A_∞ -algebras and $\Psi(z_v) = u'_v$, where z_v is the strict unit in $CW^*(C_v)$ and u'_v is the dual of the unit in $\mathbb{K}e_v \oplus CF^*(L_v)$ for each v .

Proof This follows as usual by identifying terms contributing to the A_∞ -relations with the oriented boundary of an oriented 1-manifold and Theorems 63, 64, 65.

To compute $\Psi(z_v)$ note that we can represent z_v as the minimum of the shifting Morse function of C and note that there is a unique flow line from this minimum to the intersection point $C_v \cap L'_v$ and a unique flow line in L'_v from the intersection point to the minimum in L_v . The corresponding holomorphic disk starts at the intersection point between C_0 and C_1 , has two corners at $C_0 \cap L'_0$ and $C_1 \cap L'_1$, and ends at the intersection point in $L'_0 \cap L'_1$ corresponding to the minimum of the shifting Morse function. \square

The pre-twisting cochain $\mathfrak{t}': LC_*^{\parallel}(\Lambda) \rightarrow \mathcal{A}$ can now be seen to arise via SFT-stretching as follows. Consider the first component Ψ_1 of the A_∞ -map above and a holomorphic disk contributing to it. Now stretch the lower end of the cobordism. Then by SFT-compactness the curves contributing to Ψ_1 breaks up into a curve contributing to the map $\phi \circ \Theta$ followed by the twisting cochain at each negative puncture. Noting that z_v are the only low energy generators, it follows that the map induces a map of the high energy quotient into $\text{coker}(\eta)$ and we can write the induced map Ψ_1^+ as $\Psi_1^+ = (\Omega \mathfrak{t}) \circ \phi \circ \Theta^+$.

6 Examples and Applications

6.1 Concrete calculations

In this section we compute Legendrian and Lagrangian invariants in a number of concrete examples.

6.1.1 The unknot

Consider the Legendrian unknot $\Lambda \subset S^{2n-1}$ for $n > 1$ with its standard filling $L = D^n \subset D^{2n}$. Then Λ can be represented as a standard unknot in a small Darboux chart which has effectively only one Reeb chord with respect to the standard Reeb flow on S^{2n-1} , see [11, Section 7.1].

We work over a field \mathbb{K} . Consider first the case when L is decorated by $-$. We then have

$$LC_*(\Lambda) = \mathbb{K} \cdot 1 \oplus \mathbb{K} \cdot c, \quad |c| = -n$$

with all coalgebra maps $(\Delta_i)_{i \geq 1}$ trivial, except Δ_2 for which we have:

$$\Delta_2(1) = 1 \otimes 1, \quad \Delta_2(c) = 1 \otimes c + c \otimes 1$$

by the co-unitality. Using a Morse function on D^{2n} with a unique local maximum a and which decreases in the end corresponding to shift in the negative Reeb direction, we have

$$\mathbb{K} \oplus CF_*(L) = \mathbb{K} \cdot 1 \oplus \mathbb{K} \cdot a, \quad |a| = -n$$

Let $\mathcal{A} = \Omega(\mathbb{K} \oplus CF_*(L))$ be the Adams-Floer algebra, where degree of a is now shifted up by 1. Then, we have the twisting cochain

$$t: LC_*(\Lambda) \rightarrow \mathcal{A}, \quad t(c) = a.$$

Here the disk with input c and output a corresponds to the family of disks with positive puncture at c that sweeps L once.

(The case of $n = 2$ is drawn in Figure (7)).

The Koszul complex is generated over \mathbb{K} by a^k , $a^k c$ for $k \geq 0$. We can compute the non-trivial part of the differential to be:

$$d^t(a^k c) = a^{k+1}, \quad \text{for all } k \geq 0$$

hence t is acyclic. This is consistent with the classical Koszul duality between the algebras $C^*(S^n)$ and $C_{-*}(\Omega S^n)$ for $n > 1$.

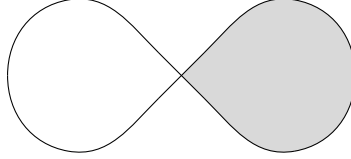


Figure 7: Computation of \mathfrak{t} for the Legendrian unknot, the disk drawn lies in a $(n - 1)$ -dimensional family that sweeps the filling once.

Consider next the case when L is decorated by $+$. Since S^{n-1} is simply connected $CE^*(\Lambda) \approx CE_{\parallel}^*(\Lambda)$ and we will use the parallel copies version in our calculation. Choose a Morse function on Λ with a single minimum and a single maximum. Denote the corresponding Reeb chords x (the co-unit chord) and y . Then

$$LC_{*}^{\parallel}(\Lambda) = \mathbb{K} \cdot x \oplus \mathbb{K} \cdot y \oplus \mathbb{K} \cdot c, \quad |x| = 0, \quad |y| = -(n - 1), \quad |c| = -n.$$

Here

$$\Delta_1(c) = y, \quad \Delta_2(x) = x \otimes x + (-1)^{n-1}x \otimes y + y \otimes x + (-1)^n x \otimes c + c \otimes x,$$

and all other operations are trivial. It follows that

$$CE_{\parallel}^*(\Lambda) = \Omega(LC_{*}^{\parallel}(\Lambda)) \simeq \mathbb{K}.$$

This is in line with Conjecture 3 which says that $CE^*(\Lambda) \simeq CE_{\parallel}^*(\Lambda)$ is isomorphic to $CW^*(C)$, where C is the cotangent fiber in the manifold obtained by attaching a cotangent end $T^*(S^{n-1} \times [0, \infty))$ to the ball along Λ . The manifold that results from this attachment is simply $T^*\mathbf{R}^n$ and the wrapped Floer cohomology of the cotangent fiber C has rank 1 and is generated by the minimum in the disk C , in accordance with the above calculations.

Finally, the twisting co-chain \mathfrak{t} in the $+$ case is the canonical map $\mathbb{K} \rightarrow \mathbb{K}$, and again the Koszul complex is acyclic. As explained in Section 5 this map is induced by a map $\mathfrak{t}' : LC_{*}^{\parallel}(\Lambda) \rightarrow (BCF^*(L))^{\#}$. To define $CF^*(L)$ we use a Morse function on L with a single local minimum and which is increasing in the end corresponding to a shift in the positive Reeb direction. Denote the generator of $CF^*(L)$ corresponding to the minimum u , $|u| = 0$. Then

$$\mathfrak{t}'(x) = u',$$

where u' is the dual of u and the holomorphic strip is the thin strip corresponding to a rigid Morse flow line from the minimum u to the minimum y in the boundary.

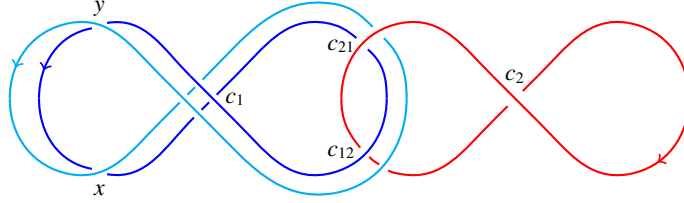


Figure 8: Hopf link with one (blue) marked + and one (red) marked – components.

6.1.2 Geometric twisting cochain for the Hopf link

In this section we carry out the geometric calculation of the twisting cochain for the Hopf link. As explained we cannot directly calculate the twisting cochain into the Legendrian coalgebra with coefficients in chains of the based loop space. We can however calculate the corresponding twisting cochain when we replace chains on the based loop space with the Morse theory of parallel copies for the components decorated by a positive sign. To carry out the calculation we pick a Morse function on the component Λ^+ with on minimum x and a maximum y . We place them on the circle and choose parallel copies as shown in Figure 8. The parallel copies algebra $CE_{\parallel}^*(\Lambda)$ is

$$\mathbf{k}\langle x, y, c_1, c_{12}, c_{21}, c_2 \rangle,$$

where we use notation for Reeb chords and Floer cohomology generators as in Section 1.4. The differential is

$$\begin{aligned} dc_1 &= xc_1 + c_1x + y + c_{12}c_{21}, \\ dx &= xx, \\ dy &= xy - yx, \\ dc_{12} &= -xc_{12}, \\ dc_{21} &= c_{21}x, \\ dc_2 &= c_{21}c_{12}. \end{aligned}$$

Passing to $CE_{\parallel}^*(\Lambda)$ means dividing out by the cokernel of the coaugmentation that takes e_1 to x . This gives the algebra

$$\mathbf{k}\langle y, c_1, c_{12}, c_{21}, c_2 \rangle,$$

and the differential becomes

$$\begin{aligned} dc_1 &= y + c_{12}c_{21}, \\ dc_2 &= c_{21}c_{12}. \end{aligned}$$

The twisting cochain \mathfrak{t} is induced from a map $\mathfrak{t}' : LC_*^{\parallel}(\Lambda) \rightarrow (\mathbf{B}(\mathbf{k}_- \oplus CF^*(L)))^{\#}$ that counts holomorphic disks with one positive puncture and boundary on L , and with several punctures at Lagrangian intersection points in the compact part, see (28). In the current example it is straightforward to find these disk. Note first that by general properties, see Proposition 49,

$$\mathfrak{t}'(x) = a_1^{\vee},$$

where a_1 is the idempotent corresponding to the minimum of the shifting Morse function on L_1 . The holomorphic disk corresponds to a Morse flow line connecting x to u_1 . We next consider $\mathfrak{t}'(c_1)$ and $\mathfrak{t}'(c_2)$. Consider first the moduli spaces $\mathcal{M}^{\text{fi}}(c_j)$ of holomorphic disks with a positive puncture at c_j and boundary on L_j . As in the case of the unknot this moduli space sweeps L_j . On both L_1 and L_2 the shifting functions have one critical point, on L_1 it is a minimum and on L_2 a maximum. The maximum is constraining for the map into the linear dual of $CF^*(L_j)$, whereas the minimum is not. We find that

$$\mathfrak{t}'(c_1) = 0, \quad \mathfrak{t}'(c_2) = a_2^{\vee}.$$

The space $\mathcal{M}^{\text{fi}}(c_j)$ give further information of the twisting co-chain as follows. Note that as the evaluation map hits the double point one can glue on a constant disk. These broken disks are also the boundary of the 1-dimensional moduli spaces $\mathcal{M}^{\text{fi}}(c_1 a_{12} a_{21})$ and $\mathcal{M}^{\text{fi}}(c_2 a_{21} a_{12})$. The other end of these moduli spaces correspond to broken disks with one level in the symplectization, a disk in $\mathcal{M}^{\text{sy}}(c_1 c_{12} c_{21})$ in the former case and in $\mathcal{M}^{\text{sy}}(c_2 c_{21} c_{12})$ in the latter, and two disks one in $\mathcal{M}^{\text{fi}}(c_{12} a_{12})$ and one in $\mathcal{M}^{\text{fi}}(c_{21} a_{21})$ attached at its negative end. The last disks contribute to \mathfrak{t}' and we conclude that

$$\mathfrak{t}'(c_{12}) = a_{12}^{\vee}, \quad \mathfrak{t}'(c_{21}) = a_{21}^{\vee}.$$

Finally, we compute $\mathfrak{t}'(y)$. Since y is a small chord corresponding to a critical point at infinity of the shifting Morse function the only contributions to $\mathfrak{t}'(y)$ comes from small holomorphic disks that are controlled by the Morse theory. It is straightforward to check that the only rigid disk corresponds to a flow line in L_1 connecting y to the intersection point and that this flow line corresponds to a holomorphic triangle in $\mathcal{M}^{\text{fi}}(y a_{12} a_{21})$. It follows that

$$\mathfrak{t}'(y) = a_{21}^{\vee} a_{12}^{\vee}.$$

The actual twisting co-chain takes the cokernel $\overline{LC}_*(\Lambda)$ of the co-augmentation into the cokernel of the co-units. Concretely, this means disregarding x and a_1^{\vee} , and we get

$$\mathfrak{t}(c_1) = 0, \quad \mathfrak{t}(c_2) = a_2^{\vee}, \quad \mathfrak{t}(c_{12}) = a_{12}^{\vee}, \quad \mathfrak{t}(c_{21}) = a_{21}^{\vee}, \quad \mathfrak{t}(y) = a_{12}^{\vee} a_{21}^{\vee}.$$

Remark 57 We point out that the parallel copies algebra $CE_{\parallel}^*(\Lambda)$ is defined using a fixed augmentation (in the current example the zero augmentation) on the one copy version of $CE^*(\Lambda)$. Here this is reflected in the change of variables $t - e_1 = y$.

6.1.3 Products of spheres

We will consider a Legendrian embedding $\Lambda \subset \mathbf{R}^{2(m+k)-1}$, where the ambient space is standard contact $(2(m-k)-1)$ -space with coordinates $(x, y, z) \in \mathbf{R}^{m+k-1} \times \mathbf{R}^{m+k-1} \times \mathbf{R}$ and contact form $dz - y \cdot dx$. We will define it by describing its front in $\mathbf{R}^{m+k-1} \times \mathbf{R}$. To this end consider first the following construction of the front of the Legendrian unknot in $\mathbf{R}^n \times \mathbf{R}$. Take a disk D^n lying in \mathbf{R}^n . Think of it as having multiplicity two and lift one of the sheets up in the auxiliary \mathbf{R} -direction (with coordinate z) keeping it fixed along the boundary. In this way we construct the front of the standard unknot with Reeb chord at the maximum distance between the two sheets and a cusp edge along the boundary of D^n . Consider now instead the base \mathbf{R}^{m+k-1} and the standard embedding of the k -dimensional sphere S^k into this space. A tubular neighborhood of this embedding has the form $S^k \times D^{m-1}$ with fibers D^{m-1} . Now take two copies of this tubular neighborhood and repeat the above construction for the D^{m-1} in each fiber. The result is the front of a Legendrian $S^k \times S^{m-1}$ with an S^k Bott-family of Reeb chords corresponding to the maxima in fibers. Figure 9 shows this front after Morse perturbation. The resulting Legendrian Λ has only two Reeb chords. We denote them a of grading $|a| = -(m+k)$ and b of grading $|b| = -m$. Note also that Λ has an exact Lagrangian filling $L \approx D^m \times S^k$.

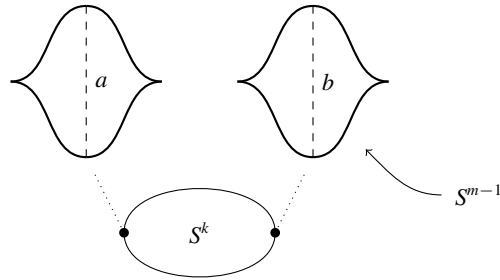


Figure 9: Front of products of spheres

Consider first the case when L is decorated by $-$. If d is the differential in $CE^*(\Lambda)$ we have

$$da = 0, \quad db = 0.$$

The Floer cohomology of L is defined by choosing a shifting function which is decreasing at infinity and we find that CF^* has two generators M , $|M| = m + k$ and S , $|S| = m$. As in the case of the unknot $\mathcal{M}^{\text{fi}}(a)$ sweeps L and $\mathcal{M}^{\text{fi}}(b)$ sweeps $D^m \times \text{pt}$. It follows that the twisting co-chain is

$$\mathfrak{t}(a) = M^\vee, \quad \mathfrak{t}(b) = S^\vee$$

and duality holds.

Consider second the case when L is decorated by $+$. In this case there are additional generators of $LC_*^\parallel(\Lambda)$ corresponding to the Morse theory of Λ . We have in addition to a and b above also

$$x, |x| = 0, \quad s_1, |s_1| = -k, \quad s_2, |s_2| = -(m-1), \quad y, |y| = -(m+k-1).$$

It follows that $CE_*^\parallel(\Lambda)$ is generated by s_1 , s_2 , y , a , and b . Using the flow tree description of moduli spaces \mathcal{M}^{sy} one verifies that if d is the differential on $CE_*^\parallel(\Lambda)$ then

$$\begin{aligned} da &= y - ((-1)^{km}bs_1 + s_1b), \quad dy = s_1s_2 + (-1)^{k(m-1)}s_2s_1, \\ db &= s_2, \quad ds_1 = 0, \quad ds_2 = 0. \end{aligned}$$

The Floer cohomology of L is defined by choosing a shifting function which is increasing at infinity and we find that CF^* has two generators M , $|M| = 0$ and S , $|S| = k$, where M is the unit. It follows that $(BCF^*(L))^\# \simeq \Omega CF_*(L)$ is generated by S^\vee with $|S^\vee| = -k$ and the twisting co-chain is

$$\mathfrak{t}(a) = 0, \quad \mathfrak{t}(y) = 0, \quad \mathfrak{t}(b) = 0, \quad \mathfrak{t}(s_2) = 0, \quad \mathfrak{t}(s_1) = S^\vee,$$

and duality holds.

6.1.4 Plumblings of simply-connected cotangent bundles

Let \mathcal{T} be a tree with vertex set Γ . For each $v \in \Gamma$, let M_v be a compact simply-connected manifold of dimension $n \geq 3$. We will see that duality holds between the wrapped and the compact Fukaya categories of the symplectic manifold $X_{\mathcal{T}}$ obtained by plumbing the collection of T^*M_v according to the tree \mathcal{T} . As usual, we take the pre-surgery perspective. Hence, consider a handle decomposition of each M_v with a unique top-dimensional n -handle. Removing this handle, we get manifolds L_v with spherical boundary Λ_v . Let $W_{\mathcal{T}}$ be the subcritical Weinstein manifold obtained by plumbing the cotangent bundles T^*L_v according to the tree \mathcal{T} . We take the plumbing region to be away from the boundary of L_v . Write $\Lambda = \bigsqcup_v \Lambda_v$ for the Legendrian in the boundary of the subcritical Weinstein manifold $W_{\mathcal{T}}$ which is filled by the Lagrangian $L = \bigcup_v L_v$. Equip the components of Λ with either $+$ or $-$ labeling.

Theorem 58 *If $n \geq 3$ and L_v is simply-connected for each $v \in \Gamma$, then $CE^*(\Lambda)$ and $CF^*(L)$ are Koszul dual.*

Proof Consider first the case $n \geq 5$. Pick a handle decomposition of L_v without 1- and $(n-1)$ -handles. The existence of such a handle decomposition is equivalent to simply-connectivity in high dimensions by the work of Smale. Consider the corresponding Weinstein handle decomposition of T^*L_v . Attaching a k -handle alters the Legendrian boundary by surgery and adds Reeb chords in the co-core sphere of the handle of grading $\leq -(n-k)$ and it follows that all Reeb chords of Λ_v have grading ≤ -2 . To construct $\Lambda \subset \partial W_{\mathcal{T}}$, we perform a version of boundary connected sum as follows. For each edge in \mathcal{T} we pick a $2n$ -ball B with two transversely intersecting Lagrangian disks $D \subset B$ that intersect the boundary sphere ∂B in a standard Legendrian Hopf link Δ . We then make boundary connected sum adding (B, Δ) to join the T^*L_v according to the tree. This adds Reeb chords of index $\leq -(n-1)$ in the boundary connected sum handles and further Reeb chords corresponding to the Reeb chords of each Hopf link, which effectively has four Reeb chords: two Reeb chords connecting the unknot components to themselves of grading $-n$ and two mixed Reeb chords connecting distinct components. We can pick gradings so that the gradings of these two mixed chords are $-d$ and $-(n-d)$ for any d , where the first Reeb chord goes from the component closest to priori fixed root of the tree \mathcal{T} to the one further from it and the other in the opposite direction. Taking d between 2 and $n-2$ we find that $LC_*(\Lambda)$ is simply connected as is Λ . The result then follows from Theorem 51.

For $n = 4$, we can stabilize by multiplying by \mathbf{R}^N as described in Remarks 41 and 44 to get 1-reduced versions of the Legendrian and Lagrangian algebras and then use the above argument.

Finally for $n = 3$, the assumptions of Theorem 51 does not hold but we recall from Remark 20 that to apply the duality result from Theorem 51 all we really need is that $B(LA^*)$ is locally finite. This is easily seen to be the case in our case, since the plumbing is according to a tree (which by definition has no cycles) and for any word of Reeb chords that do not consist of connecting Reeb chords going away from the root of the tree the grading has to increase with the size of the word. Alternatively, in this case we know by Perelman's theorem that we can take Λ as a link of standard Legendrian spheres linked according to the tree as Hopf links and $B(LA^*)$ can be seen to be locally finite directly. \square

Remark 59 *The case $n = 2$ corresponds to plumbing of copies of T^*S^2 . This case was studied in [31] and a version of the duality result still holds, at least when*

$\text{char}(\mathbb{K}) = 0$. However, the above argument fails in that case and a more complicated argument using an additional grading is used in [31]. Also a set of examples for $n = 1$ are studied in [45], where the plumbing tree is a star and the corresponding symplectic manifolds is a punctured torus. The duality still holds in this case, even though this is a plumbing of T^*S^1 's (not simply-connected). The proof of duality given in [45] uses homological mirror symmetry.

6.1.5 The trefoil

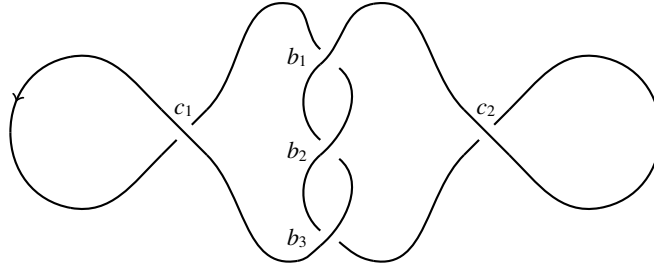


Figure 10: Trefoil

Consider the standard Legendrian trefoil $\Lambda \subset S^3$ described in Figure 10. Let us first consider the case when Λ is marked $-$. With respect to the standard choice of orientation datum, the Chekanov-Eliashberg DG-algebra CE^* is then given by the free algebra

$$\mathbb{K}\langle c_1, c_2, b_1, b_2, b_3 \rangle, \quad |c_1| = |c_2| = -1, |b_1| = |b_2| = |b_3| = 0$$

and the non-trivial part of the differential can be read from Figure 10 as follows:

$$\begin{aligned} dc_1 &= 1 + b_1 + b_3 + b_3b_2b_1 \\ dc_2 &= -1 - b_1 - b_3 - b_1b_2b_3. \end{aligned}$$

It is well known that Λ has an exact Lagrangian torus filling. (In fact, there are at least five of them, see [28]). Any of these can be obtained by doing surgery (pinch move) at b_1 , b_2 and b_3 in some order. Corresponding augmentations $\epsilon_L: CE^* \rightarrow \mathbb{K}$ are given by their values $\epsilon(b_1), \epsilon(b_2), \epsilon(b_3) \in \mathbb{K}$ subject to the condition:

$$1 + \epsilon(b_1) + \epsilon(b_3) + \epsilon(b_1)\epsilon(b_2)\epsilon(b_3) = 0.$$

Note that since CE^* is not simply-connected, Theorem 51 does not apply here. In fact, duality does not hold in this example. However, Theorem 54 shows that there is

a quasi-isomorphisms of completions

$$\widehat{CE}^* \cong \widehat{\Omega}C_*(T^2) \cong \mathbb{K}[[u, v]]$$

where the latter is a power series ring in two commuting variables concentrated in degree 0.

Thus, the completion map composed with the twisting cochain gives an algebra map:

$$H^0(CE^*) = \mathbb{K}\langle b_1, b_2, b_3 \rangle / \langle 1 + b_1 + b_3 + b_3b_2b_1, 1 + b_1 + b_3 + b_1b_2b_3 \rangle \rightarrow \mathbb{K}[[u, v]].$$

We claim that this map is injective. Indeed, observe that $H^0(CE^*)$ is a commutative algebra:

$$d(c_1 + c_2 + b_3b_2c_1 + c_1b_2b_3) = b_3b_2 - b_2b_3,$$

and thus, b_2 commutes with b_3 in homology. Using this, one shows similarly that b_1 commutes with b_2 and b_3 , see [15]. Hence, we have a completion map from a commutative ring to its completion

$$H^0(CE^*) = \mathbb{K}[b_1, b_2, b_3] / (1 + b_1 + b_3 + b_1b_2b_3) \rightarrow \mathbb{K}[[u, v]].$$

It is a well known theorem, known as Krull intersection theorem, in commutative algebra that for any commutative Noetherian ring which is an integral domain the completion map is injective. Thus, even though duality fails in this example, partial information can still be obtained by considering completions.

We next describe the twisting cochain $t: LC_*(\Lambda) \rightarrow BCF^*(L)^\#$ for one of the Lagrangian fillings L of Λ . More precisely, we choose the filling which is obtained by pinching first at b_1 , then at b_2 , then at b_3 , and finally filling the resulting unknots with Lagrangian disks, see [28, Section 8.1]. We think of the Lagrangian filling in as two disks connected by three twisted bands corresponding to the three pinchings. We next consider moduli spaces of holomorphic disks with boundary in L . Here [28, Sections 4 and 5] gives a description in terms of Morse flow trees which gives the following:

- $\mathcal{M}^{\text{fi}}(b_1)$ consists of one disk, δ_1^1 .
- $\mathcal{M}^{\text{fi}}(b_2)$ consists of two disks, δ_2^1 and δ_2^2 .
- $\mathcal{M}^{\text{fi}}(b_3)$ consists of two disks, δ_3^1 and δ_3^2 .

The boundaries of these disks are as follows.

- $\partial\delta_1^1$ is fiber in the first twisted band.
- $\partial\delta_2^1$ is a fiber in the second twisted band, and $\partial\delta_2^2$ runs across the first and the second twisted band.

- $\partial\delta_3^1$ is a fiber in the third twisted band, and $\partial\delta_3^2$ runs across the second and the third twisted bands.

We next describe the moduli spaces $\mathcal{M}^{\text{fi}}(c_2)$, $\mathcal{M}^{\text{fi}}(c_1)$ in a completely analogous manner. The space has four components, all diffeomorphic to intervals, as follows:

- θ_1 with one boundary point the disk in $\mathcal{M}^{\text{sy}}(c_2b_1)$ with δ_1^1 attached, and the other the disk in $\mathcal{M}^{\text{sy}}(c_2b_1b_2b_3)$ with δ_1^1 , δ_2^1 , and δ_3^2 attached.
- θ_2 with one boundary point the disk in $\mathcal{M}^{\text{sy}}(c_2b_3)$ with δ_3^1 attached, and the other the disk in $\mathcal{M}^{\text{sy}}(c_2b_1b_2b_3)$ with δ_1^1 , δ_2^2 , and δ_3^1 attached.
- θ_3 with one boundary point the disk in $\mathcal{M}^{\text{sy}}(c_2b_3)$ with δ_3^2 attached, and the other the disk in $\mathcal{M}^{\text{sy}}(c_2b_1b_2b_3)$ with δ_1^1 , δ_2^2 , and δ_3^2 attached.
- θ_4 with one boundary point the disk in $\mathcal{M}^{\text{sy}}(c_2)$ and the other the disk in $\mathcal{M}^{\text{sy}}(c_2b_1b_2b_3)$ with δ_1^1 , δ_2^1 , and δ_3^1 attached.

Here the disks in θ_4 sweeps the right hand disk of L , whereas the evaluation maps of θ_j , $1 \leq j \leq 3$ does not map surjectively onto either disk. The moduli space $\mathcal{M}^{\text{fi}}(c_1)$ also has four components only one of which sweeps the left hand disk of L .

In order to compute the Floer cohomology $CF^*(L)$ we pick a Morse function with one maximum M and two saddle points S_1 and S_2 . Letting the maximum lie in the right hand disk we find:

$$\mathfrak{t}(c_1) = 0,$$

since the only way to rigidify a disk of dimension one is that its boundary passes the maximum M .

The twisting cochain can now in principle be computed from the moduli spaces \mathcal{M}^{fi} described above by attaching flow trees. To get an algebraically feasible twisting co-chain we first arrange the perturbation scheme so that

$$\mathfrak{t}(b_1) = S_1^\vee + S_1^\vee S_2^\vee,$$

where the first term is δ_1^1 with a flow line and the second with a flow tree, and next so that

$$\mathfrak{t}(b_3) = -S_1^\vee,$$

the disk δ_2^2 with a flow line contributes, other contributions cancel (the disk δ_2^2 with two flow lines and the same disk with a flow tree, the two distinct disks with one flow lines). Remaining parts of the twisting cochain are now determined from the co-product in the Floer homology:

$$dM^\vee = S_1^\vee S_2^\vee - S_2^\vee S_1^\vee,$$

the twisting cochain equation, and the sweeping property of θ_4 , as follows:

$$\mathfrak{t}(c_2) = M^\vee \left(\frac{1 + S_1^\vee}{1 + S_1^\vee + S_1^\vee S_2^\vee} \right).$$

and

$$\mathfrak{t}(b_2) = \frac{S_2^\vee}{(1 + S_1^\vee + S_1^\vee S_2^\vee)}.$$

It is possible to check that these power series agree with the geometric count. We leave out the details but describe the mechanism. In order to arrange that only one flow line can be attached to δ_1^1 we order the stable manifolds of the flow line of the parallel copies so that if a flow line (or flow tree) between copy j and $j - l$ is attached then following δ_1^1 we already passed all intersections with stable manifolds between copy $j - l$ and k for $k < j - l$. Now, if the disk intersects the collection of stable manifolds in the opposite direction this means that we can jump down in all ways, which then gives the desired power series.

We finish this section by a brief discussion of the case L decorated by $+$ and parallel copies. As usual this introduces two extra generators in addition to the Reeb chords above in LC_* , x , $|x| = 0$ and y , $|y| = -1$, where x is the counit corresponding to the minimum of the shifting function and y is the maximum. In the reduced coalgebra (disregarding x) we get the new differential (using the augmentation ϵ_L above to change coordinates):

$$\begin{aligned} dc_1 &= b_1 + b_3 - b_3 b_2 + b_3 b_2 b_1 \\ dc_2 &= -y - b_1 - b_3 + b_2 b_3 - b_1 b_2 b_3. \end{aligned}$$

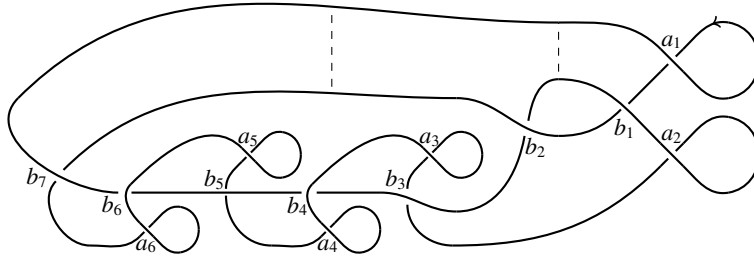
In this case CF^* is defined instead by choosing a shifting function that increases at infinity, and CF^* is generated by the unit u , $|u| = 0$ and s_1, s_2 with $|s_j| = 1$. The new twisting cochain is

$$\mathfrak{t}(c_1) = \mathfrak{t}(c_2) = 0, \quad \mathfrak{t}(y) = (s_1^\vee s_2^\vee - s_2^\vee s_1^\vee) \left(\frac{1 + s_1^\vee}{1 + s_1^\vee + s_1^\vee s_2^\vee} \right),$$

and $\mathfrak{t}(b_j)$ is exactly as above, after the substitutions $s_j \rightarrow S_j, j = 1, 2$.

6.1.6 Mirror of 7_2

We next discuss an example where Koszulity fails and also the completion map fails to be injective, even though CE^* is supported in non-positive degrees. This example was shown to us by Lenhard Ng.


 Figure 11: Mirror of 7_2

Consider the Legendrian knot Λ drawn in Figure 11 decorated $-$. It is easy to see that two pinch moves indicated by dashed lines in the Figure 11 gives a Legendrian unknot, hence Λ has a Lagrangian torus filling, call it L . Thus, we again have a completion map:

$$H^0(CE^*) \rightarrow \mathbb{K}[[u, v]]$$

where $K[[u, v]]$ is the commutative powerseries algebra in two variables.

CE^* is given by the free algebra:

$$\mathbb{K}\langle a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6, b_7 \rangle, |a_i| = -1, |b_i| = 0$$

The differential is given by

$$da_1 = -1 + (1 + b_1b_2)b_7 + b_1(1 + b_4b_3)(1 + b_6b_5)$$

$$da_2 = 1 - b_3(1 + b_2b_1)$$

$$da_3 = 1 + b_3b_4$$

$$da_4 = 1 + b_5b_4$$

$$da_5 = 1 + b_5b_6$$

$$da_6 = 1 + b_7b_6$$

Taking the quotient of $H^0(CE^*)$ by letting $b_4 = b_6$, $b_3 = b_5 = b_7$, $b_1 = 1$ and $b_2 = -1 - b_4$ gives

$$\langle b_3, b_4 \rangle / \langle 1 + b_3b_4 \rangle$$

which is a non-commutative algebra.

Thus, the completion map cannot be injective in this case. Otherwise, $H^0(CE^*)$ and thus any quotient of it would have been commutative.

6.2 Simply connected Legendrian submanifolds

Let $\Lambda \subset Y$ be a Legendrian $(n - 1)$ -submanifold with $\pi_1(\Lambda) = 1$ in the boundary Y of a Weinstein $2n$ -manifold X that bounds an exact Lagrangian $L \subset X$. Assume that $c_1(X) = 0$ and that the Maslov class of L vanishes and that L is relatively spin. Decorate L by $-$. Our next result shows that if the symplectic homology of X vanishes and if all Reeb chords of Λ have negative grading as generators of $CE^*(\Lambda)$ then $CE^*(\Lambda)$ is determined by the topology of L and conversely. If Λ is a sphere then we write $\bar{L} = L \cup_{\partial} D^n$ for the closed manifold obtained by adding a disk to L along Λ .

Theorem 60 *Suppose that $\Lambda = \Lambda^-$ is simply-connected. Assume that $SH^*(X) = 0$ and that $CE^*(\Lambda)$ is supported in degrees ≤ -1 . Then L is simply connected. Moreover, if Λ is a sphere then $CE^*(\Lambda)$ is isomorphic to $C_{-*}(\Omega\bar{L})$.*

Proof Consider the wrapped Floer cohomology $HW_{\pi_1}^*(L)$ of L with coefficients in $\mathbb{Z}[\pi_1(L)]$. Using our model for wrapped Floer cohomology in Section B.1, a chain complex $CW_{\pi_1}^*(L)$ which calculates $HW_{\pi_1}^*(L)$ can be described as follows. Let $L = L_0$ and let L_1 be a parallel copy of L shifted in the negative Reeb direction at infinity. The complex $CW_{\pi_1}^*(L)$ is then generated over $\mathbb{Z}[\pi_1]$ by the intersection points in $L_0 \cap L_1$ which we call the Morse generators and the Reeb chords starting on Λ_0 and ending on Λ_1 . The differential of a generator counts the usual rigid holomorphic strips, keeping track of the homotopy class of the loop obtained from the boundary component of the disk in L_1 completed by the reference paths connecting Reeb chord endpoints and intersection points to the base point. We point out that since there are no Reeb chords of degree 0 the augmentation induced by L is trivial and the high energy part of the differential on $CW_{\pi_1}^*$ counts honest holomorphic strips (without extra negative punctures at augmented Reeb chords). As for usual wrapped Floer homology, $HW_{\pi_1}^*(L)$ is naturally a module over symplectic cohomology $SH^*(X)$ and hence vanishes.

We next describe a geometric version of the complex $CW_{\pi_1}^*(L)$ that we call $CW_{\tilde{p}}^*$ and that also computes $HW^{\pi_1}(L)$. Let $\tilde{p}: \tilde{L} \rightarrow L$ denote the universal covering of L and let $\tilde{\Lambda} = \tilde{p}^{-1}(\Lambda)$. Pick a Morse function $f: \tilde{L} \rightarrow \mathbf{R}$ with the following properties

- f has exactly one local maximum M .
- f has no index $n - 1$ critical points.
- f has no local minima.
- f is constant on $\tilde{\Lambda}$ where it attains its global minimum and if ν is the unit normal vector field along $\tilde{\Lambda}$ then $df(\nu) = -1$.

The generators of $CW_{\tilde{p}}^*$ are of two types

- (i) The preimages of end points of Reeb chords $L_0 \rightarrow L_1$ in $L \approx L_1$ under \tilde{p} graded as the corresponding Reeb chord in $CE^*(\Lambda)$.
- (ii) The critical points of the Morse function $f: \tilde{L} \rightarrow \mathbf{R}$ graded by the negative of the Morse index.

Let M_{-*} denote the Morse chain complex of f with cohomological grading, with generators as in (ii) and differential δ which counts negative gradient flow lines. Then M_{-*} is supported in degrees d with $-n \leq d \leq -1$ and $M_{-(n-1)} = 0$. Let C^* denote the complex generated by the generators of type (i) and equip it with the differential ∂ that counts lifts of the boundary of holomorphic strips in the symplectization interpolating between Reeb chords. (This corresponds naturally to the high-energy part of the differential on $CW_{\pi_1}^*$.) By our assumption on Reeb chord grading the grading of C^* supported in degrees d where $d \leq -1$.

We now define the complex $CW_{\tilde{p}}^* = C^* \oplus M_{-*}$ with differential

$$d = \begin{pmatrix} \partial & \phi \\ 0 & \delta \end{pmatrix},$$

where δ and ∂ are the differentials on M^* and C^* and where ϕ counts rigid lifts of disks with flow lines of f attached. (This is the linear part of the map ϕ in (26).)

The homology of d is then isomorphic to $HW_{\pi_1}^*(L)$. To see this note that we can describe $CW_{\pi_1}^*(L)$ exactly as $CW_{\tilde{p}}^*(L)$ just replacing the Morse function f above with a Morse function $h \circ \tilde{p}$, where \tilde{p} is a Morse function on L without minimum and with the required boundary behavior. Thus passing from $CW_{\pi_1}^*(L)$ to $CW_{\tilde{p}}^*(L)$ corresponds to deforming the Morse function on \tilde{L} and it is well known that this induces a homotopy of complexes. In particular $CW_{\tilde{p}}^*(L)$ is acyclic.

We next want to show that $\pi_1(L) \approx 1$ or equivalently that the map $\tilde{L} \rightarrow L$ has degree one. To show this we first observe that since there are no Reeb chords of grading 0 the augmentation of $CE^*(\Lambda)$ is trivial and the differential on C^* counts honest holomorphic strips in the symplectization. This in turn means that the whole boundary of any holomorphic strip contributing to ∂ actually lies in $\Lambda \times \mathbf{R}$ and therefore cannot pick up any non-trivial $\mathbb{Z}[\pi_1]$ -coefficient.

Consider the following part of the chain complex $C^* \oplus M_{-*}$:

$$\dots \longrightarrow C^{-(n+1)} \longrightarrow C^{-n} \oplus M_{-n} \longrightarrow C^{-(n-1)} \longrightarrow \dots,$$

where we use that $M_{-k} = 0$ for $k = n - 1$ and $k > n$. It follows from the above discussion and the vanishing of the wrapped homology $HW^*(L)$ with trivial coefficients

that the cohomology $HW_{\pi_1}^{-n}(L)$ in degree $-n$ has one generator for each non-trivial element in π_1 . On the other hand $HW_{\pi_1}^{-n}(L) = 0$ and we conclude that $\pi_1 = 1$.

The statement about the isomorphism class of $CE^*(\Lambda)$ then follows from Corollary 52. \square

A Basic results for moduli spaces

Consider as above a Weinstein manifold with an exact Lagrangian submanifold (X, L) , which outside a compact agrees with the positive part of the symplectization of the contact manifold with Legendrian submanifold (Y, Λ) . We assume that the Maslov class of L vanishes and that L is relatively spin. We will consider several versions of punctured holomorphic spheres and disks with boundary on L . The most basic disks we consider will lie either in X or in the symplectization $\mathbf{R} \times Y$. We call the former *filling curves* and the latter *symplectization curves*. We will also consider a more general setting where, like in the symplectization, disks may have both positive and negative punctures. Here we assume that (W, K) is a Weinstein cobordism with negative end $(-\infty, 0] \times (Y, \Lambda)$ and positive end $[0, \infty) \times (Z, \Gamma)$, where Z is a contact manifold and Γ is a Legendrian submanifold. More specifically we will consider a Lagrangian $C \subset W$ with positive end Γ and empty negative end and a Lagrangian L with negative end Λ and empty positive end. We further assume that there is a natural one-to-one correspondence between the components of L^v of L and C^v of C , $v \in Q_0$, and that corresponding components L^v and C^v intersect transversely in one point z^v and that $L^v \cap C^w = \emptyset$ if $v \neq w$. We call disks in W *cobordism curves*.

We will describe a geometric setup that covers the cases considered below. Let Y be the contact boundary of the Weinstein manifold X , where $c_1(X) = 0$. Let $\Lambda \subset Y$ be a Legendrian link with connected components $\Lambda_1, \dots, \Lambda_m$. Let D denote the unit disk in \mathcal{C} and let z_1, \dots, z_r be boundary punctures and ζ_1, \dots, ζ_k be interior punctures. Let each component of $\partial D \setminus \{z_1, \dots, z_r\}$ be decorated with a component Λ_j . The boundary punctures come in two types, positive and negative, all interior punctures are negative. Following [23] we make the further requirement that the disk be *admissible*:

Any arc in D that connects two boundary arcs in $\partial D \setminus \{z_1, \dots, z_r\}$ subdivides the boundary punctures into two subsets. If both these subsets contain positive punctures then the labels of the two boundary segments at the endpoints of arc are different.

We also assume that the disk has one distinguished boundary puncture. Note that using a conformal model where the distinguished positive puncture lies at $1 \in \partial D$ and an

interior puncture ζ_j at the origin the positive real axis determines an asymptotic marker at ζ_j for each j .

A.1 Moduli spaces of spheres for anchoring and compactifications of moduli spaces of disks

Following [11], all symplectization and cobordism disks we consider will be anchored. This means that the actual disks we consider have, aside from their boundary punctures, also additional interior punctures where the maps are asymptotic to Reeb orbits at the negative end. An anchored disk is such a disk completed by holomorphic planes in X at all its negative interior punctures, see Figure 12. We will count rigid disks as well as use moduli spaces of disks to parameterize chains of boundary paths which then requires studying anchored disks. Although, standard arguments using classical methods allow us to prove transversality for the disks with boundary punctures that we consider, anchored disks requires transversality and gluing also for holomorphic planes in (X, L) and that requires a more general perturbation scheme. Necessary perturbations for such curves were constructed in [24].

To state the relevant result let Y denote the contact boundary of X and consider a Reeb orbit γ in Y with a point p fixed on it. Below p will be fixed throughout and suppressed in the notation. Let $\mathcal{M}(\gamma)$ denote the moduli space of holomorphic planes in X with positive puncture with an asymptotic marker where the curve is asymptotic to γ with the marker mapping to p . Then [24, Theorem 1.1] shows that there exists a perturbation λ which replaces the original $\bar{\partial}_J$ -equation with a domain dependent version $\bar{\partial}_{J_\lambda} u = 0$ (where the domain dependent almost complex structure also depends on the location of u in the space of maps) such that the following hold: The λ -perturbed moduli space $\mathcal{M}^\lambda(\gamma)$ is transversely cut out and admits a compactification as a manifold with boundary with corners, where the boundary configurations correspond to several level spheres. Here levels are of two forms: spheres with one positive puncture and possibly several negative punctures in the symplectization $\mathbf{R} \times Y$ and spheres with one positive puncture in X . It is important to note that the asymptotic marker at the positive puncture induces asymptotic markers at all other punctures in all levels and that the images of asymptotic markers at a negative and matching positive puncture agree.

Consider now a moduli space of disks $\mathcal{M}^{\text{neg}}(\mathbf{c}, \gamma)$ as above in a symplectization $(\mathbf{R} \times Y, \mathbf{R} \times \Lambda)$ or a cobordism (Z, K) with negative end (Y, Λ) with boundary punctures mapping to Reeb chords in the word \mathbf{c} and with a distinguished boundary puncture and additional negative interior punctures at ζ_1, \dots, ζ_k mapping to Reeb orbits $\gamma_1, \dots, \gamma_k$ in the word γ . Note that the distinguished boundary puncture determines an asymptotic

marker at each interior puncture ζ_j that determines a point on the corresponding γ_j . Below we will show that relevant moduli spaces of disks with interior negative punctures with markers are transversely cut out for generic almost complex structure. We define the anchored moduli space $\mathcal{M}^{\text{anc}}(\mathbf{c})$ as

$$\mathcal{M}^{\text{anc}}(\mathbf{c}) = \bigcup_{\gamma} (\mathcal{M}^{\text{neg}}(\mathbf{c}, \gamma) \times \prod_{\gamma_j \in \gamma} \mathcal{M}^{\lambda}(\gamma_j)).$$

Consider now a symplectic filling (X, L) of the negative end of the cobordism above. If all punctures mapping to chords in \mathbf{c} are positive then we can consider the moduli space $\mathcal{M}(\mathbf{c})$ of holomorphic disks in (X, L) with boundary punctures in \mathbf{c} . Since all moduli spaces $\mathcal{M}^{\text{neg}}(\mathbf{c}, \gamma)$ are transversely cut out, the corresponding moduli spaces $\mathcal{M}^{\text{neg}, \lambda}(\mathbf{c}, \gamma)$, where a small perturbation near the negative ends corresponding to the perturbation λ of holomorphic planes with asymptotic marker has been turned on, is canonically diffeomorphic to $\mathcal{M}^{\text{neg}}(\mathbf{c}, \gamma)$. Gluing the curves in $\mathcal{M}^{\text{neg}, \lambda}(\mathbf{c}, \gamma)$ to the curves in $\mathcal{M}^{\lambda}(\gamma_j)$, and extending the perturbation, we get a compactification of the moduli space $\mathcal{M}(\mathbf{c})$ with boundary given by tree configurations of anchored disks. For example if $\mathbf{c} = c$ is a single Reeb chord then $\mathcal{M}(c)$ has a natural compactification with boundary of the form

$$\mathcal{M}^{\text{anc}}(c, \mathbf{b}) \times \prod_{b_j \in \mathbf{b}} \mathcal{M}^{\text{anc}}(b_j).$$

All moduli spaces of disks considered in this paper will be anchored and we will drop the superscript *anc* from the notation.

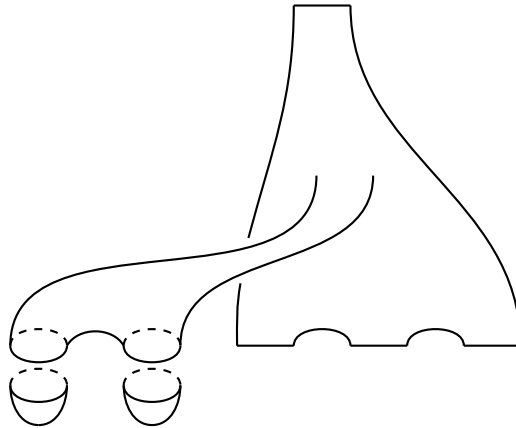


Figure 12: Anchored disks

A.2 Moduli spaces of anchored disks

Consider a system of parallel copies $\bar{L} = \{L_j\}_{j=0}^\infty$, where $L_0 = L$, as in Section 3.3. We first discuss numberings that determine the boundary conditions of our holomorphic disks. Let D_m denote the unit disk in the complex plane with m boundary punctures ζ_1, \dots, ζ_m . One of the boundary punctures is distinguished. We choose notation so that ζ_1 is distinguished. The m punctures subdivides the boundary of D_m into m boundary arcs. We will consider disks with numbered boundary arcs where the numbers correspond to the parallel copies L_j . We will consider two types of numberings *increasing* and *decreasing*: traversing the boundary of the disk in the positive direction starting from the distinguished puncture, the numbering increases or remain constant respectively decreases or remain constant as we pass the other punctures. We call punctures where the numbering changes non-constant and punctures where it does not change constant. If the boundary numbering has no constant puncture then we call it strictly increasing or strictly decreasing.

We next consider asymptotic conditions at the boundary punctures. There are two basic forms of asymptotics, a puncture is either asymptotic to a Lagrangian intersection point or to a Reeb chord. The former case is the standard form of asymptotics on Lagrangian Floer theory and the latter in Legendrian DG-algebras. More precisely, we choose an almost complex structure on X which in the cylindrical end $\mathbf{R} \times Y$ is invariant under \mathbf{R} -translation, leaves the contact planes invariant and is compatible with the symplectic form induced on the contact planes by the contact form. Furthermore, it pairs the \mathbf{R} -direction with the Reeb direction. This means in particular that the Reeb chord strip which is the product of a Reeb chord and \mathbf{R} is holomorphic and we study holomorphic disks with boundary punctures that are asymptotic to these Reeb chord solutions.

Consider a disk D_m as above with strictly increasing or decreasing numbering $\kappa = (\kappa_1, \dots, \kappa_m)$ and let $\mathbf{a} = a_1 \dots a_m$ be a word of Reeb chords and Lagrangian intersection points in $L_0 \cap L_1$. We let $\mathcal{M}^{\text{h}}(\mathbf{a}; \kappa)$ denote the moduli space of holomorphic disks $u: (D_m, \partial D_m) \rightarrow (X, \bar{L})$ that satisfies the following conditions:

- u takes the boundary component labeled by κ_j to the Lagrangian L_{κ_j} .
- u is asymptotic to the unique Reeb chord or Lagrangian intersection point of L_{κ_j} and $L_{\kappa_{j+1}}$ near a_j at ζ_j , where we let $\kappa_{m+1} = \kappa_1$.

We next consider disks in the symplectization. Consider again the disk D_m with boundary numbering κ and punctures ζ_1, \dots, ζ_m . We next note that in the symplectization there are two possible Reeb chord asymptotics, positive or negative according to the sign of the t -coordinate near the puncture. Let $\mathbf{c} = c_1^{\sigma_1} \dots c_m^{\sigma_m}$ be a word of signed

Reeb chords of $\Lambda_0 \cup \Lambda_1$, where $\sigma \in \{+, -\}$ is a sign. If $m > 1$ then we require that the Reeb chords c_r at all constant punctures connect Λ_0 to Λ_0 and that their signs are all negative, $\sigma_r = -1$. We let $\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)$ denote the moduli space of holomorphic disks $v: (D_m, \partial D_m) \rightarrow (\mathbf{R} \times Y, \mathbf{R} \times \Lambda)$ that satisfy the following conditions:

- v takes the boundary components labeled by κ_j to the Lagrangian Λ_{κ_j} .
- v is asymptotic at positive or negative infinity according to the sign of σ_j to the unique Reeb chord between Λ_{κ_j} and $\Lambda_{\kappa_{j+1}}$ near c_j at a puncture ζ_j .

We write $\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)$ for the moduli space of holomorphic disks v satisfying these conditions.

Finally, we consider disks in the cobordism and in the filled cobordism. We start with the cobordism disks without filling. Consider a system of parallel copies $\bar{C} = \{C_j\}_{j=0}^\infty$. Consider the disk D_{i+j+2} where we fix two punctures that subdivide the boundary of the disk into two arcs, upper and lower. Let κ be a decreasing boundary numbering of the boundary components in the upper arc and extend it to a constant numbering in the lower arc. Let $\mathbf{c}_0 = c_{0;1} \dots c_{0;j}$ be a composable word of Reeb chords connecting Λ_v to Λ_w , and let $\mathbf{c}' = c_i \dots c_1$ be a word of Reeb chords of Γ . Consider the word of Reeb chords and intersection points

$$\mathbf{c} = c_{0;1} \dots c_{0;j} z^w c_i \dots c_1 z^v$$

and let

$$\mathcal{M}^{\text{co}}(\mathbf{c}; \kappa)$$

denote the moduli space of holomorphic disks $u: (D_{i+j+2}, \partial D_{i+j+2}) \rightarrow (W, \bar{C} \cup L)$ such that the following holds.

- u is asymptotic to the Reeb chord $c_{0;r}$ at its r^{th} constant puncture, takes adjacent boundary arcs to L , and neighboring punctures to the unique intersection point near z^w in $L \cap C_{\kappa_i}^w$ and near z^v in $L \cap C_{\kappa_i}^v$, respectively.
- On remaining boundary arcs and punctures the boundary maps as described by \mathbf{c} and the numbering κ , exactly as above.

The disks in the filled cobordism are entirely analogous. Here we assume that L is a Lagrangian submanifold in $X = X_0 \cup W$. Consider again a system of parallel copies $\bar{C} = \{C_j\}_{j=0}^\infty$ and also a system of parallel copies $\bar{L} = \{L_j\}_{j=0}^\infty$ of L . Consider the disk D_{i+j+2} where we fix two punctures that subdivide the boundary of the disk into two arcs, upper and lower. Let κ be a decreasing boundary numbering of the boundary components in the upper and lower arcs. Let $\mathbf{x}_0 = x_{0;1} \dots x_{0;j}$ be a word of intersection

points of L and let $\mathbf{c}' = c_i \dots c_1$ be a word of Reeb chords of Γ . Consider the word of Reeb chords and intersection points

$$\mathbf{c} = x_{0;1} \dots x_{0;j} z^w c_i \dots c_1 z^v$$

and let

$$\mathcal{M}^{\overline{\text{co}}}(\mathbf{c}; \kappa)$$

denote the moduli space of holomorphic disks $u: (D_{i+j+2}, \partial D_{i+j+2}) \rightarrow (X, \bar{C} \cup \bar{L})$ such that the following holds.

- u is asymptotic to the Reeb chord $x_{0;r}$ at its r^{th} puncture in the lower arc, takes adjacent boundary arcs to L , and neighboring punctures to the unique intersection point near z^w in $L_{\kappa_j}^w \cap C_{\kappa_{j+1}}^w$ and the unique intersection point near z^v in $K_{\kappa_1} \cap C_{\kappa_{i+j}}^v$.
- On remaining boundary arcs and punctures the boundary maps as described by \mathbf{c} and the numbering κ , exactly as above.

The formal dimension of the moduli spaces above is computed in terms of the negative of a Conley-Zehnder index CZ of the Reeb chords. Recall $\text{CZ}(a)$ of a Reeb chord a of Λ as defined for example in [11, Section 2.1]: we pick paths connecting base points in the boundary of the various components of L , and paths connecting Reeb chord endpoints to the base points. We define $\text{CZ}(a)$ to be the Maslov index of this path closed up by a positive rotation in the contact plane, and the grading $|a| = -\text{CZ}(a)$. For a Lagrangian intersection x between L^1 and L^2 we similarly pick paths connecting to the base points and use these to form a loop γ starting in L^2 and ending in L^1 and define $\text{CZ}(x)$ to be the Maslov index of the loop of Lagrangian planes that results from closing up the path of Lagrangian planes along γ by a positive rotation, and $|x| = -\text{CZ}(x)$.

Remark 61 *The above gradings are related to the grading $|\cdot|_{\text{Leg}}$ in the Legendrian contact homology algebra, see e.g. [27, 11] as follows:*

$$|c|_{\text{Leg}} = -|c| - 1.$$

Lemma 62 *The formal dimension of the moduli space $\mathcal{M}^{\text{fi}}(\mathbf{a}; \kappa)$ equals*

$$\dim(\mathcal{M}^{\text{fi}}(\mathbf{a}; \kappa)) = (n-3) - \sum_{j=1}^m (|a_j| - (n-2)).$$

The formal dimension of the moduli space $\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)$ equals

$$\dim(\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)) = (n-3) + \sum_{\sigma_j=-1} (|c_j| + 1) - \sum_{\sigma_j=+1} (|c_j| - (n-2)).$$

The formal dimension of the moduli space $\mathcal{M}^{\text{co}}(\mathbf{c}; \kappa)$ equals

$$\dim(\mathcal{M}^{\text{co}}(\mathbf{c}; \kappa)) = 1 - \sum_{r=1}^i (|c_r| - (n-2)) + \sum_{s=1}^j (|c_{0;s}| + 1).$$

The formal dimension of the moduli space $\mathcal{M}^{\overline{\text{co}}}(\mathbf{c}; \kappa)$ equals

$$\dim(\mathcal{M}^{\overline{\text{co}}}(\mathbf{c}; \kappa)) = 1 - \sum_{r=1}^i (|c_r| - (n-2)) - \sum_{s=1}^j (|x_{0;s}| - (n-2)).$$

Proof See [17, Theorem A.1]. □

We next study topological properties of the moduli spaces just defined. It turns out to be comparatively simple because of two key features. First, since we require our disks to switch copies at punctures “in the same direction” they cannot be multiply covered, and second, for the same reason there can be no boundary splitting. As in [23], the first property allows us to prove transversality by perturbing the almost complex structure and the second shows that the moduli spaces admit compactifications consisting only of punctured curves joined at Reeb chords. Precise formulations of these results are as follows.

Theorem 63 *For generic almost complex structure J the moduli spaces $\mathcal{M}^{\text{fi}}(\mathbf{a}; \kappa)$, $\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)$, $\mathcal{M}^{\text{co}}(\mathbf{c}; \kappa)$, and $\mathcal{M}^{\overline{\text{co}}}(\mathbf{c}; \kappa)$ are transversely cut out manifolds of dimensions $\dim(\mathcal{M}^{\text{fi}}(\mathbf{a}; \kappa))$ and $\dim(\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)$, $\dim(\mathcal{M}^{\text{co}}(\mathbf{c}; \kappa)$, and $\dim(\mathcal{M}^{\overline{\text{co}}}(\mathbf{c}; \kappa)$, respectively.*

Proof To see this we note first that if u is a disk in one of the moduli spaces then it cannot be multiply covered. We use the argument of [27]. Fix a puncture of u . Consider first the more difficult case when this puncture maps to a Lagrangian intersection. Pick coordinates so that the intersection point lies at the origin in \mathbf{C}^n and so that the two Lagrangians correspond to \mathbf{R}^n and $i\mathbf{R}^n$. Let $(x_1 + iy_1, \dots, x_n + iy_n)$ be standard coordinates on \mathbf{C}^n . Consider the complex hyperplanes $H_{\pm\epsilon} = \{x_1 + iy_1 = \pm\epsilon(1+i), x_2 + iy_2 = \dots = x_n + iy_n = 0\}$. Looking at the Fourier expansion of u near the puncture it is clear that for suitable coordinates (such that the leading Fourier coefficient of u lies in direction of the first coordinate) the number of intersection points of the image of u and $H_{\pm\epsilon}$ near the puncture have different parities depending on the sign of ϵ . By analytic continuation, other disks and half disks mapping there either have images agreeing completely in the ball or there are injective points of the disk near the puncture. In the case where they agree completely we note that the parity of the

number of intersection points in each local sheet not containing the puncture with $H_{\pm\epsilon}$ is independent of sign. We find that we can achieve transversality by perturbing the complex structure near $H_{\pm\epsilon}$: because the sheet with the puncture intersects only one of $H_{\pm\epsilon}$, if the contributions of the sheets mapping to H_ϵ cancel then those mapping to $H_{-\epsilon}$ cannot cancel and vice versa, by unique continuation. \square

Theorem 64 *The moduli space $\mathcal{M}^{\text{fi}}(\mathbf{a}; \kappa)$ admits a compactification consisting of several level disks joined at Reeb chords and intersection points, where some levels may lie in the symplectization.*

The moduli space $\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)$ admits a compactification consisting of several level disks joined at Reeb chords.

The moduli space $\mathcal{M}^{\text{co}}(\mathbf{a}; \kappa)$ admits a compactification consisting of several level disks joined at Reeb chords. There is one level of disks in the cobordisms and remaining levels in the symplectization ends.

The moduli space $\mathcal{M}^{\overline{\text{co}}}(\mathbf{a}; \kappa)$ admits a compactification consisting of several level disks joined at Reeb chords and intersection points.

Proof Note that the boundary condition on our punctured disks have the following property: any arc that subdivides the source into two components with a positive puncture in each must connect boundary components numbered with distinct numbers. This shows that there can be no boundary splitting. The theorem then follows from SFT compactness [12]. \square

We next discuss orientations of moduli spaces following [26]. We fix capping operators at all Reeb chords and Lagrangian intersection points so that the two capping operators there glue to a disk with the Fukaya orientation, see [34]. Recall the relative spin structure on the Lagrangian submanifold induces an orientation on the determinant bundle over the space of disks with boundary condition in the Lagrangian, see. As in [26] we see that these choices then induces a system of coherent orientations on the moduli spaces.

We will use one more property of the moduli spaces which says that they are effectively independent of the increasing or decreasing boundary labeling κ .

Theorem 65 *Let κ and κ' be two increasing (decreasing) boundary numberings. Then there are canonical orientation preserving diffeomorphisms*

$$\begin{aligned}\mathcal{M}^{\text{fi}}(\mathbf{a}; \kappa) &\approx \mathcal{M}^{\text{fi}}(\mathbf{a}; \kappa'), \\ \mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa) &\approx \mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa'), \\ \mathcal{M}^{\text{co}}(\mathbf{c}; \kappa) &\approx \mathcal{M}^{\text{co}}(\mathbf{c}; \kappa')\end{aligned}$$

Proof Let $\mathcal{M}(\kappa)$ denote either one of the above moduli spaces. This moduli space is the transverse zero set of a Fredholm section in a Banach bundle. Changing the numbering from κ to κ' corresponds to an arbitrarily small isotopy which induces an arbitrarily small deformation of the section. The theorem follows. \square

B Wrapped Floer cohomology and Legendrian surgery

In this section we present the argument that establishes the isomorphism between $CE^*(\Lambda)$ and $CW^*(C)$ where C is the co-core disk of the surgery. Our proof is a generalization of the corresponding result under Lagrangian handle attachment exposed in [11] and uses the technical results on relevant moduli spaces in [24].

We first define a version of wrapped Floer cohomology using only purely holomorphic disks and show that the resulting theory agrees with the usual version defined in terms of holomorphic disks with a Hamiltonian term. Second we discuss the surgery isomorphism in [11], and third we discuss how to generalize that argument to partially wrapped Floer cohomology calculations.

B.1 Wrapped Floer cohomology without Hamiltonian

Let X be a Weinstein manifold and L be an exact Lagrangian. Fix a system of shifting Morse functions that are positive at infinity and let $\bar{L} = \{L_j\}_{j=0}^\infty$ be the corresponding family of parallel Lagrangian submanifolds. Define $CW^*(L)$ to be the chain complex generated by Reeb chords of L and intersection points $L_0 \cap L_1$. We define operations m_i on $CW^*(L)$ using what we call *partial holomorphic buildings*.

We start in the simplest case when the output of m_i is an intersection point c_0 . Consider i generators $c_i \dots c_1$ and consider a disk D_{i+1} with a decreasing boundary numbering κ , distinguished negative (output) puncture and remaining punctures positive (inputs). Let $\mathbf{c}' = c_i \dots c_1$ and $\mathbf{c} = c_0 c_i \dots c_1$. Define

$$m'_i(\mathbf{c}') = \sum_{|c_0|=|\mathbf{c}'|+(2-i)} |\mathcal{M}^{\text{fl}}(\mathbf{c}; \kappa)|_{c_0}.$$

Here we use the temporary notion m'_i to denote the summand of the full operation m_i that takes values in intersection points. We next turn to the more complicated definition of the part m''_i of the operation that takes values in Reeb chord generators and to this end we introduce the notion of a partial holomorphic building.

The domain of partial holomorphic building is a possibly broken disk D_{i+1} with decreasing boundary numbering κ . The partial holomorphic buildings we consider always have exactly one disk in the symplectization. We call it the primary disk of the building. We require that the distinguished puncture which is increasing is a negative puncture of this primary disk. If the distinguished puncture is the only negative puncture of the primary disk then the partial building consist only of its primary component. If on the other hand the primary disk has additional negative punctures then we require that at each additional puncture (which is decreasing or constant) is a disk in the cobordism with decreasing boundary condition is attached at its distinguished increasing or constant puncture to the additional negative puncture. We call these disks the secondary disks of the partial building. The resulting partial holomorphic building is then a disk with domain a broken D_{i+1} , with distinguished puncture a negative puncture at a Reeb chord and with remaining i punctures either Reeb chords or intersection points. See Figure 13.

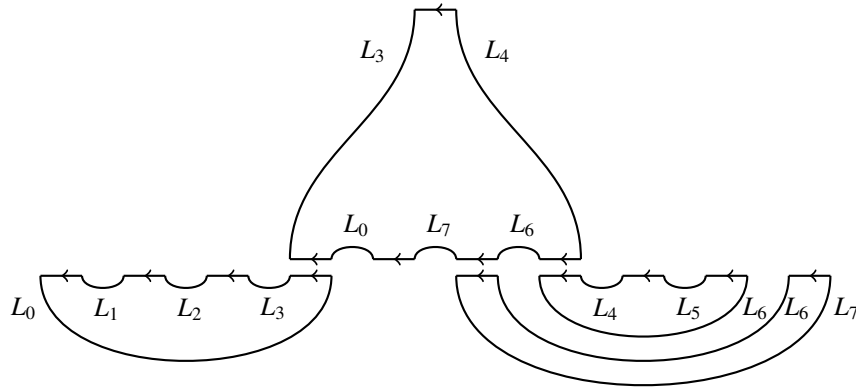


Figure 13: The domain of a partial building contributing to the operation m_7 of $CW^*(L)$ with a possible decoration. The map sends the tensor product of chords $L_0 \leftarrow L_1, L_1 \leftarrow L_2, \dots, L_6 \leftarrow L_7$ to a chord $L_0 \leftarrow L_7$.

Remark 66 *At additional negative punctures there may be holomorphic disks with one positive puncture and boundary on L attached. These are the usual augmentation disks, or disks on L used as anchoring disks, compare Section 3.4.*

We write the punctures of the partial holomorphic disk building as $\mathbf{c} = c_0 c_i \dots c_1$, where c_0 is the distinguished puncture. Write

$$\mathcal{M}^{\text{pb}}(\mathbf{c}; \kappa)$$

for the moduli space of partial holomorphic disk buildings with boundary condition on \bar{L} according to κ . Using this we define for generators $\mathbf{c}' = c_i \dots c_1$ the operation

$$\mathfrak{m}_i''(\mathbf{c}') = \sum_{|c_0|=|\mathbf{c}'|+(2-i)} |\mathcal{M}^{\text{pb}}(\mathbf{c}; \kappa)|_{c_0},$$

where the sum ranges over Reeb chords c_0 with grading as indicated. Finally we define the total operation \mathfrak{m}_i as the sum

$$\mathfrak{m}_i(\mathbf{c}') = \mathfrak{m}_i'(\mathbf{c}') + \mathfrak{m}_i''(\mathbf{c}').$$

Lemma 67 *The A_∞ -relations hold for the operations \mathfrak{m}_i .*

Proof First, Theorem 65 shows that the operations compose and that they are independent of the choice of decreasing boundary numbering. To see that the relations hold we will as usual identify the terms contributing to them with the boundary of an oriented 1-dimensional compact manifold.

To this end we first consider 1-dimensional moduli spaces \mathcal{M}' of the form $\mathcal{M}' = \mathcal{M}^{\text{fi}}(\mathbf{c}; \kappa)$, where the distinguished puncture c_0 is an intersection point. As usual, the boundary numbering precludes boundary bubbling and we find that the boundary consists of broken disk that either break at an intersection point in which case the holomorphic parts both have dimension zero, or break into a partial holomorphic building with a rigid disk attached at its negative puncture, in which case the primary component of the partial building has dimension one. We find the boundary points of \mathcal{M}' are in 1-1 correspondence with disks contributing to compositions of \mathfrak{m}_i' and \mathfrak{m}_j' (disks breaking at intersection points) and disks contributing to \mathfrak{m}_i'' and \mathfrak{m}_j'' .

Remaining contributions to the A_∞ -relations correspond to compositions of \mathfrak{m}_i'' and \mathfrak{m}_j'' . We identify also these with the boundary of an oriented 1-manifold. To this end we consider the boundary of the moduli space $\mathcal{M}'' = \mathcal{M}^{\text{sy}}(\mathbf{b}; \kappa)$ of dimension two with distinguished negative puncture. (After we divide out the natural \mathbf{R} -action this is 1-dimensional space.) The boundary consists of two level broken disks and completing all negative punctures except the distinguished puncture with rigid disks in $\mathcal{M}^{\text{co}}(\mathbf{a})$ we find also the composition of \mathfrak{m}_i'' and \mathfrak{m}_j'' expressed as the boundary of an oriented compact 1-manifold. The lemma follows. \square

B.1.1 Isomorphism with the Hamiltonian version

We next show that the above definition of wrapped Floer cohomology agrees with the standard theory. We keep the geometric setting as above and write $CW_{\text{Ham}}^*(L)$ for the

usual version of Hamiltonian wrapped Floer cohomology. We give a brief recollection of the definition.

We define the wrapped Floer cohomology complex $CW_{\text{Ham}}^*(L)$ of L as follows. Write $X = \bar{X} \cup [0, \infty) \times Y$, where \bar{X} is a compact domain and $[0, \infty) \times Y$ the positive end of the Weinstein manifold X . Consider time dependent Hamiltonians $H_a: X \times [0, 1] \rightarrow \mathbf{R}$ which are perturbations of functions that equal 0 on \bar{X} and are linear of the form

$$(t, y) \mapsto ae^t + b,$$

on $[0, \infty) \times Y$, where a is not in the chord and orbit spectrum or the contact form on Λ . We choose these Hamiltonians in such a way that if $a_0 < a_1$ then $H_{a_0} < H_{a_1}$ on X .

Define the chain complex $CW_{\text{Ham}}^*(L, H_a)$ to be generated by Hamiltonian chords $\gamma: [0, 1] \rightarrow X$ of C of action

$$\mathfrak{a}(\gamma) = \int_0^1 (\lambda(\dot{\gamma}(t)) - H_a(\gamma(t))) dt < a$$

The differential on $CW^*(L, H_a)$ is defined by counting solutions of the perturbed Cauchy-Riemann equation over the strip with coordinates $s + it \in \mathbf{R} \times [0, 1]$:

$$(du + X_{H_a} \otimes dt)^{0,1} = 0.$$

Choosing a decreasing interpolation between H_{a_0} and H_{a_1} we get continuation maps

$$CW_{\text{Ham}}^*(L, H_{a_0}) \mapsto CW_{\text{Ham}}^*(L, H_{a_1})$$

and we define the wrapped Floer cohomology complex as the direct limit

$$CW_{\text{Ham}}^*(L) = \varinjlim_a CW_{\text{Ham}}^*(L, H_a).$$

The wrapped Floer cohomology $HW_{\text{Ham}}^*(L)$ is the homology of this complex. Writing $HW_{\text{Ham}}^*(L, H_a)$ for the homology of $CW_{\text{Ham}}^*(L, H_a)$ we then have

$$HW_{\text{Ham}}^*(L) = \varinjlim_a HW_{\text{Ham}}^*(L, H_a),$$

by exactness of direct limits.

A well known argument [28] shows that that $CW^*(L)$ with differential m_1 is quasi-isomorphic to the wrapped Floer cohomology by a geometrically defined chain map. We recall the argument:

The filling L of Λ gives an augmentation and we define $CW^*(L)$ (as a chain complex disregarding higher product operations) without Hamiltonian as the 'Morse extended linearized Chekanov-Eliashberg complex' with respect to this augmentation as generated by Reeb chords and the critical point of a Morse function on L with a unique

minimum, and take the differential to count unperturbed augmented and anchored holomorphic strips. We also introduce the subcomplexes $CW^*(L, a)$ generated by chords of action $< a$. Then by definition

$$CW^*(L) = \varinjlim_a CW^*(L, a).$$

The isomorphism is now constructed by interpolating exactly as in the continuation maps above from the 0 Hamiltonian (ordinary Cauchy-Riemann equation) to the Hamiltonians H_a above. Choosing the interpolations compatibly, we get the following commutative diagram

$$\begin{array}{ccccccc} CW^*(L, a_0) & \longrightarrow & CW^*(L, a_1) & \longrightarrow & \dots & \longrightarrow & CW^*(L, a_j) & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \dots & & \downarrow & & \\ CW_{\text{Ham}}^*(L, H_{a_0}) & \longrightarrow & CW_{\text{Ham}}^*(L, H_{a_0}) & \longrightarrow & \dots & \longrightarrow & CW_{\text{Ham}}^*(L, H_{a_j}) & \longrightarrow & \dots \end{array}$$

Here each vertical arrows are chain isomorphisms by the standard argument and taking limits we find a chain isomorphism

$$CW^*(L) \rightarrow CW_{\text{Ham}}^*(C).$$

We extend this chain map to an A_∞ -map, then the standard spectral sequence argument establishes the desired A_∞ quasi-isomorphism.

We follow the approach in [29] where similar isomorphisms between contact and symplectic differential graded algebras were constructed. More precisely, as there we construct a splitting compatible non-negative field of 1-forms with values in Hamiltonian vector fields, and further a 1-parameter family of such forms interpolating between the zero Hamiltonian at the positive end and the Hamiltonian used to define wrapped Floer cohomology at the negative end, see [29, Section 2]. We then define the corresponding moduli spaces over the deformation interval. Keeping the notation from [29] we denote them

$$\mathcal{F}_{\mathbf{R}}(\mathbf{a}, b).$$

In order for the asymptotics at infinity of these maps to make sense we need to include the parallel copies of the Lagrangians according to boundary numbering, and in particular also to incorporate this in the description of wrapped Floer cohomology. More precisely, as in the case above we will have moduli spaces of Floer holomorphic disks with boundary in distinct Lagrangians that are arbitrarily close. The analogue of Theorem 65 holds by the same argument and the corresponding moduli spaces are

canonically isomorphic for sufficiently small perturbations. Using these observations we then define $\Phi: CW^*(L) \rightarrow CW_{\text{Ham}}^*(L)$ by

$$\Phi(\mathbf{a}) = \sum_{\dim(\mathcal{F}_{\mathbf{R}}(\mathbf{a}, b))=0} |\mathcal{F}_{\mathbf{R}}(\mathbf{a}, b)|b.$$

Lemma 68 *The map Φ is an A_∞ homomorphism.*

Proof To see this we note again that the disks which contributes to the A_∞ relations correspond exactly to the ends of 1-dimensional moduli space. \square

Lemma 69 *The map Φ is a quasi-isomorphism.*

Proof The map respects the word length filtration and is the standard isomorphism from the linearized Legendrian cohomology to the wrapped Floer cohomology, discussed above, on the E_2 -page. \square

B.2 Wrapped Floer cohomology and Lagrangian handle attachment

In this subsection we prove the results in [11] which gives a Legendrian surgery description of the wrapped Floer cohomology of a co-core disk in a Weinstein manifold obtained by Lagrangian handle attachment along a Legendrian sphere, referring to [24] for the results on holomorphic curves missing in [11]. To state this result we first introduce notation.

Let X_0 be a Weinstein $2n$ -manifold with ideal boundary the contact $(2n - 1)$ -manifold Y_0 . Let $\Lambda = \Lambda_0 \cup \dots \cup \Lambda_m$ be a Legendrian submanifold such that all of its components Λ_j are parameterized $(n - 1)$ -spheres. Let X be the Weinstein manifold that results from attaching Lagrangian handles H to Λ . Here $H = H_1 \cup \dots \cup H_m$, where each component H_j is a disk sub-bundle of the cotangent bundle T^*D of the n -disk D , and where H_j is attached to Λ_j . Then X contains m co-core disks corresponding to the cotangent fibers at the center of the disk in each H_j . We let $C_j \subset X$ denote co-core disk in H_j , $\Gamma_j \subset Y$ denote its Legendrian boundary inside the contact boundary Y of X , and write $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_m$.

As a first step in the calculation of the wrapped Floer cohomology of C we describe the generators of the underlying chain complex. By the results in Section B.1, generators of $CW^*(C)$ are of two kinds Lagrangian intersection points and Reeb chords. Here the Lagrangian intersection points are easily understood: pick the shifting Morse function

so that it has one minimum on each component of C and no other critical points, then there is exactly one intersection point for each component of C . We denote the intersection point of C_j by m_j and we denote the subcomplex generated by the m_j by $CW_0^*(\Gamma)$. Remaining generators are Reeb chords of Γ we write $CW_+^*(\Gamma)$ for the quotient complex $CW^*(\Gamma)/CW_0^*(\Gamma)$ and note that CW_+^* is generated by Reeb chords.

Consider the link Λ and let all components be decorated by minus, $\Lambda^- = \Lambda$. Consider $CE^*(\Lambda)$ as a chain complex, generated by composable words of Reeb chords with differential d and with product \cdot given by concatenation if the words are composable and zero otherwise. Let $\epsilon > 0$ denote the size of the attaching region (i.e., the size of the tubular neighborhood of Λ where H is attached), we then have the following.

Lemma 70 *For any $A > 0$ there exists $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$ then there is a natural one to one correspondence between the generators of $CW_+^*(\Gamma)$ of action $< A$ and the generators of $CE^*(\Lambda)$ of action $< A$.*

Proof This is [24, Theorem 1.2]. □

We will next define the surgery map which is an A_∞ -morphism

$$\Phi: CW^*(\Gamma) \rightarrow CE^*(\Lambda),$$

that counts certain holomorphic disks. See Figure 14.

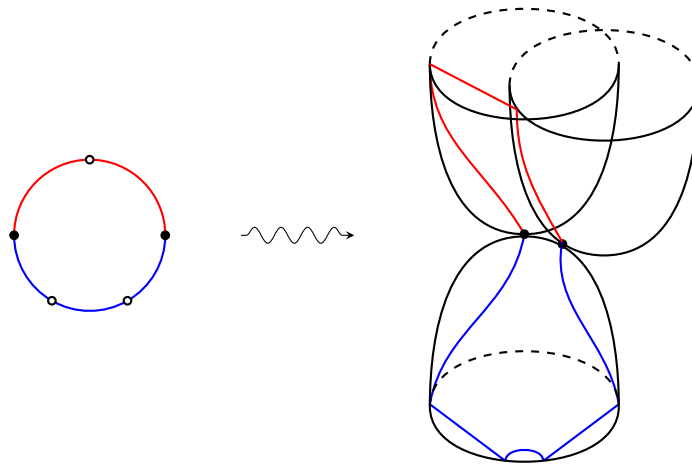


Figure 14: A picture illustrating a curve contributing to Φ^1 of the A_∞ -functor Φ

As in Section A, consider the disk D_{i+j+2} with two special punctures subdividing the boundary into an upper and a lower arc with i and j punctures respectively, and with a boundary numbering in the upper arc. Let $\mathbf{c}_0 = c_{0;1} \dots c_{0;j}$ be a composable word of Reeb chords connecting Λ_v to Λ_w , and let $c_i \dots c_1$ be a word of generators of $CW^*(C)$. Consider the word of Reeb chords and intersection points

$$\mathbf{c} = c_{0;1} \dots c_{0;j} z^w c_i \dots c_1 z^v.$$

Define $\Phi_i: CW^*(C)^{\otimes i} \rightarrow CE^*(\Lambda)$,

$$\Phi_i(\mathbf{c}') = \sum_{|\mathbf{c}_0|=|\mathbf{c}'|+i(n-2)} |\mathcal{M}^{\mathbf{c}_0}(\mathbf{c})| \mathbf{c}_0.$$

Remark 71 Note that if m^v is the minimum of the Morse function on C^v as above then

$$\Phi_1(m^v) = e_v,$$

because of the unique holomorphic disk corresponding to the flow line from the minimum to the intersection point between $C^v \cap L$, for the parallel copies this gives a triangle with corners at $m^v = C_0^v \cap C_1^v$, at $C_0^v \cap L$, and at $C_1^v \cap L$, and since there are no negative punctures the output is e_v . Also, and if a word \mathbf{c}' of generators of $CW^*(C)$ contains a generator m^v and has length $i > 1$ then

$$\Phi_i(\mathbf{c}') = 0,$$

as this corresponds to a holomorphic disk with a flow line from the minimum attached and such a configuration cannot be rigid unless the disk is constant.

Theorem 72 The maps Φ_i gives an A_∞ -map $CW^*(C) \rightarrow CE^*(\Lambda)$ which is an A_∞ quasi-isomorphism.

Proof In order to see the A_∞ -relations we study the boundary of the moduli space $\mathcal{M}^{\text{ha}}(\mathbf{c})$. As usual the boundary numbering guarantees that there is no boundary splitting on C . The boundary of the moduli space thus consists of the following configurations.

- (i) Two level curves with one level in the cobordism and one in either symplectization end.
- (ii) Curves which split at the intersection point $C \cap L$.

Splitting (i) corresponds to the map followed by the operation d in $CE^*(\Lambda)$. Splitting (ii) corresponds to the tensor product of the map followed by the product operation \cdot in $CE^*(\Lambda)$. The A_∞ -relations follow.

To see that Φ is a quasi-isomorphism we argue as follows. We first fix an action cut off $A > 0$ and show that Φ_1 induces an isomorphism on homology below action A by constructing algebraically one holomorphic disks interpolating between a Reeb chord of Γ and the corresponding word of Reeb chords of Λ . The existence of such disks implies that the map Φ_1 has a triangular matrix with respect to the action filtration and hence is a chain isomorphism. (Compare [11, Section 6.2].)

The required algebraically one interpolating disks were constructed in [24, Theorem 1.3]. The map being an A_∞ -quasi-isomorphism below action $A > 0$ then follows from the word length spectral sequence.

To see that we get an isomorphism on the full complex we show that the isomorphisms discussed are compatible with action limits. To this end consider attaching a handle of size $\delta > 0$ to Λ and denote the resulting new Weinstein manifold X_δ and the co-core disk $C_\delta \subset X_\delta$.

If $\delta_0 > \delta_1$ then there is a cobordism map

$$CW_+(C_{\delta_0}) \rightarrow CW_+(C_{\delta_1})$$

which is a quasi-isomorphism. Moreover, by the surgery description of chords for any $a > 0$ there exists $\delta_1 > 0$ such that the above map has ± 1 on the diagonal for all chords of action $< a$.

Consider the directed system

$$(31) \quad CW_+(C_{\delta_0}) \rightarrow CW_+(C_{\delta_1}) \rightarrow \cdots \rightarrow CW_+(C_{\delta_j}) \rightarrow \cdots,$$

where $\delta_j \rightarrow 0$, and let

$$\overline{CW}_+(C) = \varinjlim_\delta CW_+(C_\delta).$$

Then the homology $\overline{HW}_+(C)$ of $\overline{CW}_+(C)$ satisfies

$$\overline{HW}_+(C) = \varinjlim_\delta HW_+(C_\delta) = HW_+(C_{\delta_j}), \text{ for any fixed } j.$$

Here the last equality follows since all the arrows in the directed system of homology groups of (31) are isomorphisms.

Consider next the Chekanov-Eliashberg algebra $CE^*(\Lambda)$ of Λ . We define the action truncated subcomplex $CE^*(\Lambda, a)$ generated by words of chords of total action $< a$. Viewing $CE^*(\Lambda)$ as chain complex generated by words of chords we then have

$$CE^*(\Lambda) = \varinjlim_a CE^*(\Lambda, a).$$

The surgery map gives for each $a_j, \delta_j > 0$ such that the map

$$\overline{CW}_+^*(C_{\delta_j}) \rightarrow CE^*(\Lambda, a_j)$$

is a chain isomorphism with ± 1 on the diagonal below action a_j . By definition of surgery and cobordism maps the following diagram commutes

$$(32) \quad \begin{array}{ccccccc} \overline{CW}_+^*(C_{\delta_0}) & \longrightarrow & \overline{CW}_+^*(C_{\delta_1}) & \longrightarrow & \dots & \longrightarrow & \overline{CW}_+^*(C_{\delta_j}) & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \dots & & \downarrow & & \\ CE^*(\Lambda, a_0) & \longrightarrow & CE^*(\Lambda, a_1) & \longrightarrow & \dots & \longrightarrow & CE^*(\Lambda, a_j) & \longrightarrow & \dots, \end{array}$$

where $\delta_{j+1} < \delta_j$ and $a_j < a_{j+1}$. Taking limits of the sequences we get a chain map

$$(33) \quad \overline{CW}_+^*(C) \rightarrow CE^*(\Lambda).$$

Taking the limits of the sequence (32) on the homology level and using that all vertical arrows are homology isomorphisms then gives homology isomorphisms in the limit and (33) is a quasi-isomorphism inducing an isomorphism

$$\overline{HW}^*(C) \approx HCE^*(\Lambda).$$

The above gives a homology isomorphism of chain complexes. To consider also products one uses the exact same argument. The product operation on $\overline{CW}^*(C)$ is induced from the action truncated version

$$\overline{CW}^*(C, a_1) \otimes \dots \otimes \overline{CW}^*(C, a_m) \rightarrow \overline{CW}^*(C, a_1 + \dots + a_m)$$

and similarly on CE^* :

$$CE^*(C, a_1) \otimes CE^*(C, a_2) \rightarrow CE^*(C, a_1 + a_2).$$

□

Remark 73 *There is also an “upside-down” perspective on the surgery just described. Namely, one can start from the contact manifold Y and produce the contact manifold Y_0 by doing so called +1-surgery on Γ . In complete analogy with the above one shows that Reeb chords on Λ are in natural one to one correspondence with words of Reeb chords on Γ and one can construct an upside down surgery map of A_∞ -coalgebras:*

$$BCW^*(C) \rightarrow LC_*(\Lambda).$$

A similar argument also shows that this map is a quasi-isomorphism. Alternatively, one can prove this from the original surgery map using only algebra as follows. First write $CE^(\Lambda) = \Omega LC_*(\Lambda)$ then*

$$BCW^*(C) \simeq B\Omega LC_*(\Lambda) \simeq LC_*(\Lambda),$$

since $LC_(\Lambda)$ is co-nilpotent, see Section 2.2.2.*

B.3 Legendrian surgery and stopped wrapping

In this section we outline a surgery approach to the computation of wrapped Floer homology in a Weinstein manifold X with wrapping stopped by a Legendrian Λ in its boundary. We will use the following model for the ambient manifold. Fix a tubular neighborhood of Λ in the contact boundary Y of X and attach a disk-bundle neighborhood of the 0-section in $T^*([0, \infty) \times \Lambda)$ along the boundary $T^*([0, \infty) \times \Lambda)|_{0 \times \Lambda}$ just like in Lagrangian handle attachment. We use a Liouville vector field on this domain that agrees with the standard Liouville vector field pointing outwards along fibers in the cotangent bundle over $[T, \infty) \times \Lambda$ for some $T > 0$. Let the components of Λ be denoted Λ_v , $v \in Q_0$. Fix a base point $p_v \in \Lambda_v$ for each v . Let $C^{v;\tau}$ denote the cotangent fiber $T^*_{(p_v, \tau)}([0, \infty) \times \Lambda)$. We will compute the wrapped Floer cohomology of $C^\tau = \bigcup_{v \in Q_0} C^{v;\tau}$ for sufficiently large τ using a surgery approach.

We first consider the surgery map into the Chekanov-Eliashberg algebra with loop space coefficients. Consider all components of Λ decorated by a positive sign $\Lambda = \Lambda^+$ and consider $CE^*(\Lambda)$ which now involves, except for Reeb chords, also chains $C_{-*}(\Omega\Lambda)$ on the based loop space. We define an A_∞ -map

$$\Phi: CW^*(C) \rightarrow CE^*(\Lambda),$$

where A_∞ -structure on the left hand side is the standard DG-algebra structure induced by concatenation and the Pontryagin product. As in Section A, consider a disk D_{i+j+2} with two dividing punctures that subdivides the boundary into two arcs, lower and upper. Let the upper arc contain i boundary punctures and is equipped with a decreasing boundary numbering κ and the lower arc j boundary punctures. Let $\mathbf{c}' = c_i \dots c_1$ be Reeb chords of C and let $\mathbf{c}_0 = c_{0;1} \dots c_{0;j}$ be Reeb chords of Λ . Let

$$\mathbf{c} = c_{0;1} \dots c_{0;j} z^y c_i \dots c_1 \dots z^w$$

and consider $\mathcal{M}^{\text{co}}(\mathbf{c}; \kappa)$, again as in Figure 14.

Theorems 63 and 64 show that this moduli space carries a fundamental chain. We view this chain as parametrizing chains of paths in Λ connecting the Reeb chord endpoints in \mathbf{c}_0 . We write $[\mathcal{M}^{\text{co}}(\mathbf{c})]$ for the alternating word of chains of loops and Reeb chords and view it as an element in $CE^*(\Lambda)$. Define the maps

$$\Psi_i: CW^*(C)^{\otimes k^i} \rightarrow CE^*(\Lambda)$$

as

$$\Psi_i(\mathbf{c}') = \sum_{\mathbf{c}_0} [\mathcal{M}^{\text{co}}(\mathbf{c})].$$

Theorem 74 *The map $\Psi : CW^*(C) \rightarrow CE^*(\Lambda)$ is an A_∞ -map.*

Proof To see that the A_∞ -relation holds we look at the boundary of the moduli space $\mathcal{M}^{\text{co}}(\mathbf{c})$ of dimension d . The codimension one boundary consists of three splittings:

- (i) A one-dimensional curve splits off in the positive symplectization end.
- (ii) A curve splits off at the negative end.
- (iii) Splitting at one of the intersection points z^v .

In order for splittings of the form (i) to contribute to the codimension one boundary of the moduli space the part of the holomorphic building in W consists of rigid disks with only positive punctures attached at one puncture to a negative puncture of the disk in the positive end and a disk of dimension $d - 1$ in $\mathcal{M}^{\text{co}}(\mathbf{b})$ attached at the remaining negative puncture. Assembling the rigid disks and the one-dimensional disk we get a partial holomorphic disk building that contributes to the A_∞ -operations in $CW^*(L)$ followed by the map Ψ . Splittings of type (ii) corresponds to the map Ψ followed by the differential μ_1 in $CE^*(\Lambda)$. Finally splittings of type (iii) correspond to the map Ψ followed by the product μ_2 on $CE^*(\Lambda)$. We conclude that the terms contributing to A_∞ -relations express the codimension one boundary of $[\mathcal{M}^{\text{co}}(\mathbf{c})]$ in two different ways and hence Ψ is an A_∞ -map. \square

We will use slight generalizations of the map Ψ below. More precisely, if $p_j, j = 1, \dots, m$ are points in $[0, \infty) \times \Lambda$ and if F_j is the cotangent fiber at p_j then we have a similar surgery map

$$\Psi^{p_i p_j} : CW^*(F_i, F_j) \rightarrow CE_{p_i p_j}^*(\Lambda),$$

which counts holomorphic disks with a positive Reeb chord connecting F_i to F_j , two Lagrangian intersection punctures at p_i and at p_j , and a word of chains of loops in Λ and Reeb chords of Λ as output, and where CE_{ij}^* is directly analogous to CE^* but where the first chain of loops in a word is replaced by a chain of paths from p_i to the base point and the last is replaced by a chain of paths from the base point to p_j . In this setup the counterpart of the second component Ψ_2 is

$$(34) \quad \Psi^{p_i p_j p_k} : CW^*(F_j, F_k) \otimes CW^*(F_i, F_j) \rightarrow CE_{p_i p_k}^*(\Lambda),$$

and counts disks with two positive punctures at Reeb chords and two Lagrangian intersection punctures at p_i and p_k . The counterpart of the A_∞ -equations in this setup is then

$$(35) \quad d \circ \Psi^{p_i p_k} + \Psi^{p_j p_k} \cdot \Psi^{p_i p_j} + \Psi^{p_i p_j p_k} \circ (1 \otimes \mu_1 + \mu_1 \otimes 1) + \Psi^{p_i p_k} \circ \mu_2 = 0,$$

where d is the differential on CE_{ik}^* and where \cdot is the (Pontryagin) product $CE_{p_i p_k}^* \otimes CE_{p_i p_j}^* \rightarrow CE_{p_i p_k}^*$. The proofs of these statements are word by word repetitions of the proof of Theorem 74.

We next sketch a proof that the map Ψ in Theorem 74 is in fact a quasi-isomorphism, or, in other words, that its first component Ψ_1 induces an isomorphism on homology. We filter $CW^*(C)$ by action of its Reeb chord generators. To get a corresponding filtration on $CE^*(\Lambda)$ we use action on Reeb chords in combination with the energy on the loops. We start with a discussion of the energy of loops, following [49].

Equip Λ with a Riemannian metric and $\Omega = \Omega(\Lambda)$ denote the space of based loops in Λ with the supremum norm: for two loops $\gamma, \beta: [0, 1] \rightarrow \Lambda$,

$$d^*(\gamma, \beta) = \sup_{t \in [0, 1]} \rho(\gamma(t), \beta(t)),$$

where ρ is the metric on Λ induced by the Riemannian structure. Then the metric topology on Ω agrees with the standard compact open topology.

Let $\Omega' = \Omega'(\Lambda)$ denote the space of piecewise smooth paths with metric

$$d(\gamma, \beta) = d^*(\gamma, \beta) + \int_0^1 (|\dot{\gamma}| - |\dot{\beta}|)^2 dt,$$

where $\dot{\gamma}$ denotes the derivative of γ . The natural inclusion $\Omega' \rightarrow \Omega$ is a homotopy equivalence, [49, Theorem 17.1]. We will use finite dimensional approximations to study Ω' . The energy of a piece-wise smooth loop in Λ is

$$E(\gamma) = \int_0^1 |\dot{\gamma}|^2 dt.$$

For $c > 0$, let $\Omega^c \subset \Omega'$ denote the subset of loops of energy $E < c$. The space Ω^c can be approximated by piece-wise geodesic loops. More precisely, fixing a subdivision $0 = t_0 < t_1 < \dots < t_m = 1$ of $[0, 1]$ we consider the space B^c of loops of energy $E < c$ that are geodesic on each interval $[t_i, t_{i+1}]$. Then [49, Lemma 16.1] shows that for all sufficiently fine subdivisions B^c is a finite dimensional manifold (a submanifold of the product $\Lambda^{\times m}$ in a natural way). Moreover [49, Theorem 16.2], all critical points of $E|_{\Omega^c}$ lie in B^c which is a deformation retract of Ω^c and for generic metric $E|_{B^c}$ is a Morse function.

With these preliminaries established we turn to the actual proof. The first step will be to describe the Reeb chords of C^τ . Let g be a Riemannian metric on Λ as above and let $f: [0, \infty) \rightarrow \mathbf{R}$ be a positive function with $f(0) = 1$, $f'(0) = -1$, and $f'(t) < 0$ monotone increasing. Define the metric h on $\Lambda \times \mathbf{R}$ by

$$h = dt^2 + f(t)g.$$

Then if x and y are points in Λ and $\gamma: [0, s] \rightarrow \Lambda$ is a geodesic with $\gamma(0) = x$ and $\gamma(s) = y$ then there is a unique geodesic $(\gamma(t), r(t)) \in \Lambda \times [0, \infty)$ such that

- $(\gamma(0), 0) = (x, 0)$ and $(\gamma(s), r(s)) = (y, 0)$;
- $r(t)$ is Morse with has a unique maximum at an interior point $t = t_0$.

Note that the Reeb flow in the unit disk bundle is the natural lift of the geodesic flow. Assume next as above that the metric g on Λ is generic in the sense that the length functional for curves connecting any two Reeb chord endpoints in Λ has only Morse critical points. Concretely, this means that the index form of any geodesic connecting two Reeb chord endpoints is non-degenerate. As in Lemma 70, this allows us to control the Reeb chords of C^τ below a given action for all sufficiently thin handles. More precisely, let ϵ denote the size of the tubular neighborhood of Λ in Y where we attach $T^*(\Lambda \times [0, \infty))$ and we introduce the following notion of a geodesic-Reeb chord word. A *geodesic-Reeb chord word* is a word

$$\gamma_1 c_1 \gamma_2 c_2 \cdots c_m \gamma_m,$$

where γ_1 is a geodesic from one of the base points p_v to the start point of the Reeb chord c_1 , where γ_2 is a geodesic from the endpoint of c_1 to the start point of c_2 , etc, until finally γ_m is a geodesic from the endpoint of the Reeb chord c_m to one of the base points p_w . We define the action of a geodesic-Reeb chord word to be the sum of actions of its Reeb chords and the energies of its geodesics.

Lemma 75 *For any $A > 0$ there exists $\epsilon_0 > 0$ and $\tau_0 > 0$ such that for any $\epsilon < \epsilon_0$ and any $\tau > \tau_0$ there is a natural one to one correspondence between Reeb chords of C^τ of action $< A$ and geodesic-Reeb chord words of Λ of action $< A$.*

Sketch of proof: The proof uses the transversality of the Reeb chords and of the geodesics. The basic observation is that the point in the normal fiber of Λ where the Reeb flow hits determines the direction of the geodesic in $\Lambda \times [0, \infty)$. After introducing a concrete smoothing of corners the lemma then follows from the finite dimensional inverse function theorem.

To show that Ψ_1 is a quasi-isomorphism we will show that it is represented by a triangular matrix with ones on the diagonal with respect to the action/energy filtration. To this end we will use the Morse theoretic finite dimensional model for the chain complex underlying the homology of the based loop space described above. In order to have $[\mathcal{M}^{\text{co}}(\mathbf{c})]$ defined as a chain in this model we need to assure that the paths on the boundary of the holomorphic disk are sufficiently well behaved. We only sketch the construction. On holomorphic disk with unstable domains, we fix gauge using

small spheres surrounding Reeb chord endpoint, compare [30, Section A.2]. As in [30, Section A.1] we use a configuration space for holomorphic curves consisting of maps with two derivatives in L^2 . This means that the restriction to the boundary has $3/2$ derivatives in L^2 and in particular the projection to Λ has bounded energy. Since the action of the positive puncture in a holomorphic disk contributing to the differential controls the norm of the solution it follows that we can use configuration spaces of bounded energy to study the disks in the differential: we approximate the boundary curves uniformly by a piecewise geodesic curve by introducing a uniformly bounded number of subdivision points and straight line homotopies in small charts.

Conjecture 76 *The chain-map $\Psi_1: CW^*(C) \rightarrow CE^*(\Lambda)$ induces an isomorphism on homology.*

Sketch of proof: Consider a word of the form

$$\gamma_0 c_1 \gamma_1 c_2 \dots c_m \gamma_m,$$

where γ_j are geodesics in $\Lambda \times [0, \infty)$ and where c_j are Reeb chords. We aim to construct algebraically one disk connecting the Reeb chord a of C , corresponding to this word (see Lemma 75), to the word itself. We use an inductive argument and energy filtration. To start the argument we pick additional fiber disks F_{c^+} and F_{c^-} in $T^*(\Lambda \times [0, \infty))$ at (c^+, ϵ) and (c^-, ϵ) for very small $\epsilon > 0$ near all Reeb chord end points c_+ and c_- in Λ . We use the natural counterparts of the correspondence between mixed words of geodesics and Reeb chords before surgery and Reeb chords after, for mixed wrapped Floer cohomologies. For example, there is a straightforward analogue of Lemma 75: Reeb chord generators of $CW^*(F_{c^+}, C)$ correspond before the surgery words of the form

$$\gamma_1 c_1 \gamma_2 \dots c_m \gamma_m,$$

where γ is a geodesic connecting the base point of F_{c^+} to the initial point of c_1 , γ_2 from the endpoint of c_1 to the start point of c_2 , etc. To start the argument we note that it is straightforward to construct holomorphic strips corresponding to the short geodesics starting at F_{c^-} followed by the chord c and then the short geodesic to F_{c^+} and to show that they are unique. This corresponds to a generator of $CW^*(F_{c^-}, F_{c^+})$. Likewise, it is immediate to construct the holomorphic disk connecting a Reeb chord generator of $CW^*(F_{c^+}, C)$ corresponding to a geodesic, and show that it is unique, compare Lemma 72.

We now use these two to construct algebraically one disk from the Reeb chord generator of $CW^*(F_{c^-}, C)$ corresponding to the short geodesic, the chord c , and a geodesic

connecting the endpoint of c to the base point of C . To this end we consider the natural map

$$\Psi^{p_v c^+ c^-} : CW^*(F_{c^+}, C) \otimes CW^*(F_{c^-}, F_{c^+}) \rightarrow CE^*(\Lambda),$$

see (35). For the two Reeb chords a connecting F_{c^-} to F_{c^+} corresponding to the chord c of Λ , and b connecting F_{c^+} to C corresponding to the geodesic we then have (with p_v denoting the base point):

$$\begin{aligned} & d(\Psi^{p_v c^+ c^-}(b, a)) + (\Psi^{p_v c^-}(b)) \cdot (\Psi^{c^+ c^-}(a)) \\ & + \Psi^{p_v c^-}(\mathfrak{m}_2(b, a)) + (-1)^{|a|-1} \Psi^{p_v c^+ c^-}(\mathfrak{m}_1(b), a) + \Psi^{p_v c^+ c^-}(b, \mathfrak{m}_1(a)) = 0. \end{aligned}$$

Here we know that the terms containing \mathfrak{m}_1 and d involves non-trivial holomorphic disks or Morse flows in the finite dimensional approximation and hence lowers action/energy by an amount bounded below by some $\delta > 0$ which we assume is much larger than $\epsilon > 0$ above. Therefore, if we restrict attention to a small action window we find

$$(\Psi^{p_v c^-}(b) \cdot \Psi^{c^+ c^-}(a)) + \Psi^{p_v c^-}(\mathfrak{m}_2(b, a)) = 0.$$

Here the first term is simply the Pontryagin product at the common endpoint of the curves, which is homologous to the word $\epsilon' c \gamma$ of the small geodesic, the Reeb chord and then the longer geodesic, by rounding the corner at c^+ . It follows that $\mathfrak{m}_2(a, b) = r$, where r is a Reeb chord with action between the sum of the actions of a and b and the action of $\epsilon' c \gamma$ and that $\Psi^{c^- p_v}(r)$ contains this word with coefficient ± 1 . Noting that there is only one Reeb chord in the action window studied we find that the desired coefficient equals ± 1 . It is now clear how to continue the induction, in each step we add one more geodesic or Reeb chord to any word. Using already constructed curves and (35) in a small action window we find that the map Ψ_1 has a triangular action matrix with ± 1 on the diagonal, hence it is a quasi-isomorphism.

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