Symplectic Cohomology of cDV singularities

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Theme

Let $X = \{f = 0\} \subset \mathbb{C}^N$ be a hypersurface with an isolated singularity at 0.

Geometric spaces of interest:

$X = \{f = 0\}$, the singularity,
$\tilde{X}$, a resolution (preferably, a crepant one),
$V = f^{-1}(\epsilon) \cap \{|z| \leq \delta\}$ for $0 < \epsilon \ll \delta \ll 1$, the Milnor fiber
$L = \partial V$, link of the singularity

When f is quasi-homogeneous with respect to a good $\mathbb{C}^\times$ action, we can simply define the Milnor fiber as $V = \{f = 1\}$.

Our theme is to consider V as an open symplectic manifold, and L as its contact boundary, and deduce properties of X and $\tilde{X}$ from the wrapped Fukaya category of V.

Kleinian singularity

$$\mathbf{f}(x,y,z) = \begin{cases} x^{n+1} + y^2 + z^2 & A_n \\ x^{n-1} + xy^2 + z^2 & D_n \\ x^4 + y^3 + z^2 & E_6 \\ x^3 + xy^3 + z^2 & E_7 \\ x^5 + y^3 + z^2 & E_8 \end{cases}$$

For $\Gamma = A_n, D_n, E_6, E_7, E_8$, let $V_\Gamma = \mathbf{f}^{-1}(1)$ Milnor fiber.

Classical beautiful fact (McKay) : X_Γ has a minimal (crepant) resolution $\pi : \tilde{X}_\Gamma \to X_\Gamma$ such that $\pi^{-1}(0)$ is a configuration of rational curves whose intersection pattern is encoded by the Dynkin diagram associated to Γ.
Modern beautiful fact (derived McKay): $D^b\mathrm{Coh}(\tilde{X}_\Gamma) \simeq \mathrm{Perf}(\Pi_{\widehat{\Gamma}})$, where $\Pi_{\widehat{\Gamma}}$ is the preprojective algebra, i.e. the 2-Calabi-Yau completion of $\mathbb{C}\widehat{\Gamma}$ for the **extended** quiver.

Symplectic invariants of Kleinian singularity

Theorem. (Etgü -L. '15)

$$\mathcal{W}(V_\Gamma) \simeq \mathrm{Perf}(\mathcal{G}_\Gamma)$$

$\mathcal{G}_\Gamma = \Pi_2(\mathbb{C}\Gamma)$ is the 2-Calabi-Yau completion of $\mathbb{C}\Gamma$.

(cf. Kalck-Yang '16) $\mathrm{Perf}(\mathcal{G}_\Gamma) \simeq D^b\mathrm{Coh}(\tilde{X}_\Gamma)/\langle \mathcal{O}_{\tilde{X}_\Gamma} \rangle$

Theorem. (L. - Ueda '20)

$$\mathrm{SH}^*(V_\Gamma) = \mathrm{HH}^*(\mathcal{W}(V_\Gamma)) = \begin{cases} \mathbb{C}^\mu & \text{if } * \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

where $\mu = n$ for A_n and D_n and $6, 7, 8$ for E_6, E_7, E_8.

Gorenstein terminal singularity = isolated cDV

These are hypersurface singularities that play an important role in MMP. They are given, in suitable analytic co-ordinates, by

$$\mathbf{w}(x, y, z, w) = f(x, y, z) + wg(x, y, z, w)$$

$w = 0$ hyperplane section yields a Kleinian singularity defined by $f(x, y, z)$.

Instinctively, we would like to consider resolutions $\pi : \tilde{X} \to X$. It turns out this is not always the right thing to do. In order to run MMP, we should learn to work with singular varieties.

Reid: "I do not wish to go at present into the various interesting questions concerned with resolving the cDV points; for many purposes it seems natural to leave them alone!"

Small resolutions

Any general cDV singularity admits a partial crepant resolution $\pi\colon \tilde{X} \to X$ called a $\mathbb{Q}$-factorial terminalisation. Such a resolution contracts some number of rational curves and has $\mathbb{Q}$-factorial terminal singularities.

For today, let us make the simplifying (and very special assumption) that the $\mathbb{Q}$-factorial terminalisation is smooth.

A small resolution is a resolution whose exceptional set has codimension at least 2. By a theorem of Reid, a resolution of an isolated cDV singularity is small if and only if it is crepant.

Examples

Here are some examples which admit small resolutions together with the number of curves that contracted in a small resolution. Any two small resolutions are related by a sequence of flops, and so the number of contracted curves in $\tilde{X}$ is invariant of $\tilde{X}$.

	Singularity	ADE type	# curves in $\tilde{X}$
1.	$x^2 + y^2 + z^{\ell+1} + w^{k(\ell+1)}$	A_ℓ	ℓ
2.	$x^2 + y^2 + zw(z^{\ell-1} + w^{k(\ell-1)})$	A_ℓ	ℓ
3.	$x^2 + y^3 + z^3 + w^{6k}$	D_4	4
4.*	$x^3 + xy^{2k+1} + yz^2 + w^2$	D_4	1
5.	$x^2 + y^3 + z^4 + w^{12k}$	E_6	6
6.	$x^2 + y^3 + z^5 + w^{30k}$	E_8	8

A theorem and a conjecture

Theorem. (Evans-L.)

	Singularity	ADE type	$\mathrm{SH}^{*\leq 1}(V;\mathbb{C})$
1.	$x^2 + y^2 + z^{\ell+1} + w^{k(\ell+1)}$	A_ℓ	ℓ
2.	$x^2 + y^2 + zw(z^{\ell-1} + w^{k(\ell-1)})$	A_ℓ	ℓ
3.	$x^2 + y^3 + z^3 + w^{6k}$	D_4	4
4.*	$x^3 + xy^{2k+1} + yz^2 + w^2$	D_4	1
5.	$x^2 + y^3 + z^4 + w^{12k}$	E_6	6
6.	$x^2 + y^3 + z^5 + w^{30k}$	E_8	8

$\mathrm{SH}^3(V;\mathbb{C}) = \mathbb{C}^\mu$, μ Milnor number, $\mathrm{SH}^*(V;\mathbb{C}) = 0$ in all other degrees.

Conjecture. (Evans-L.)
Suppose that $P \in X$ is a cDV singularity and let V be the Milnor fibre of the singularity. Then $P \in X$ admits a small resolution such that the exceptional set has ℓ irreducible components if and only if $\mathrm{SH}^*(V;\mathbb{C})$ has rank ℓ in every negative degree.

A non-example

Consider the family of cA_1 singularities

$$A_\ell := \{x^2 + y^2 + z^2 + w^{\ell+1} = 0\}, \qquad \ell \geq 1.$$

In fact, any cA_1 singularity is equivalent to one of these. The link is either S^5 (if ℓ is even) or $S^2 \times S^3$ (if ℓ is odd).
If ℓ is even then SH^* is given by

$$\begin{cases} \mathbb{C}^\ell & \text{if } * = 3 \\ \mathbb{C} & \text{if } * = -q(\ell+3) - r \text{ for } r \in \{0, \dots, \ell-1\},\ r = q (\mathrm{mod} 2) \\ \mathbb{C} & \text{if } * = -q(\ell+3) - r + 1 \text{ for } r \in \{0, \dots, \ell-1\},\ r = q (\mathrm{mod} 2) \\ 0 & \text{otherwise,} \end{cases}$$

for $q \in \mathbb{N}$. We see that SH^* can be either 0 or $\mathbb{C}$ for $* < 0$.
If ℓ is odd then

$$\mathrm{SH}^*(V_\ell; \mathbb{C}) = \begin{cases} \mathbb{C}^\ell & \text{if } * = 3 \\ \mathbb{C} & \text{if } * = 1 \text{ or } * < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Symplectic Cohomology

Given a Liouville manifold V, one defines *symplectic cohomology*

$$\mathrm{SH}^*(V)$$

as a Hamiltonian Floer cohomology group associated with a time-dependent Hamiltonian with quadratic growth.[1]

Symplectic Cohomology is a generalization of Quantum Cohomology to non-compact symplectic manifolds.

Very roughly, in addition to Morse critical points capturing $\mathrm{H}^*(V)$, there are generators corresponding to Reeb orbits along ∂V.

[1] A good reference is P. Seidel-A biased view of symplectic cohomology

Symplectic Cohomology as Hochschild Cohomology

-Wrapped Fukaya category

Let $\mathcal{W}(V)$ denote the wrapped Fukaya category. This has objects exact Lagrangians L with controlled behaviour at infinity. In analogy with $\mathrm{SH}^*(V)$

$$hom(L_1, L_2)$$

has generators not only the intersection points between L_1 and L_2 but also Reeb chords from L_1 to L_2.

$$\mathrm{SH}^*(V) = \mathrm{HH}^*(\mathcal{W}(V), \mathcal{W}(V))$$

This is a culmination of many people's work. Notably, Bourgeois-Ekholm-Eliashberg, Abouzaid, Ganatra, Chantraine-Dimitroglou Rizell-Ghiggini-Golovko,...

Invertible polynomials

A weighted homogeneous polynomial $\mathbf{w} \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ with an isolated critical point at the origin is *invertible* if there is an integer matrix $A = (a_{ij})_{i,j=1}^{n+1}$ with non-zero determinant such that

$$\mathbf{w} = \sum_{i=1}^{n+1} \prod_{j=1}^{n+1} x_j^{a_{ij}}.$$

The *transpose* of $\mathbf{w}$ is defined as

$$\check{\mathbf{w}} = \sum_{i=1}^{n+1} \prod_{j=1}^{n+1} x_j^{a_{ji}},$$

For example, the transpose of

$$x^{n-1} + xy^2 + z^2 \quad \text{is} \quad x^{n-1}y + y^2 + z^2$$

(The latter is equivalent to $x^{2n-2} + y^2 + z^2$).

The group

$$\Gamma_{\mathbf{w}} := \{(t_0, t_1, \ldots, t_{n+1}) \in (\mathbb{G}_m)^{n+2} | \\ t_1^{a_{1,1}} \cdots t_{n+1}^{a_{1,n+1}} = \cdots = t_1^{a_{n+1,1}} \cdots t_{n+1}^{a_{n+1,n+1}} = t_0 t_1 \cdots t_{n+1}\}$$

acts naturally on $\mathbb{A}^{n+2} := \mathrm{Spec}\mathbb{C}[x_0, \ldots, x_{n+1}]$.

$\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 \cdots x_{n+1})$ denote the idempotent completion of the dg category of $\Gamma_{\mathbf{w}}$-equivariant coherent matrix factorizations of $\mathbf{w} + x_0 \cdots x_{n+1}$ on $\mathbb{A}^{n+2}$

Conjecture (L.-Ueda '19) For any invertible polynomial **w**, one has a quasi-equivalence

$$\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 x_1 \cdots x_{n+1}) \simeq \mathcal{W}(\check{\mathbf{w}}^{-1}(1)).$$

In what follows, $n = 3$ for today.

Lemma: If $\mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})) = 0$, then $\mathbf{w} + x_0 x_1 \cdots x_{n+1}$ is right-equivalent to $\mathbf{w}$ by a formal change of variables.

Hence, we have (by Orlov)

$$\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 x_1 \cdots x_{n+1}) \simeq \mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})$$

Finally, [LU'20], we have

$$\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}) \simeq \Pi_n(\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w}))$$

Thus, under the assumption that $\mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})) = 0$, the mirror symmetry conjecture reads

$$\Pi_n \mathrm{mf}(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w}) \simeq \mathcal{W}(\check{\mathbf{w}}^{-1}(1)).$$

On the other hand, we can check for all of our examples except the Laufer family, we have

$$\mathcal{W}(\check{\mathbf{w}}^{-1}(1)) = \Pi_3 \mathcal{FS}(\mathbf{w})$$

Thus, it suffices to know that the Fukaya-Seidel category $\mathcal{FS}(\mathbf{w})$ is equivalent to $\mathrm{mf}(\mathbb{A}^4, \Gamma_{\mathbf{w}}, \mathbf{w})$ which is known in the Brieskorn-Pham case (Futaki-Ueda '11), and in the case of double stabilization (Habermann-Smith '19).

Example: Consider $\check{\mathbf{w}} = x^2 + y^2 + z^m + w^n$ defining a cA_ℓ singularity for some ℓ. In this case, $\mathcal{FS}(\check{\mathbf{w}}) \simeq \mathrm{Perf}(A_{m-1} \otimes A_{n-1})$. It follows that in this case

$$\mathcal{W}(\check{\mathbf{w}}^{-1}(1)) = \Pi_3(A_{m-1} \otimes A_{n-1})$$

Summary of calculation

In the case of our examples, we can identify

$$\mathcal{W}(\check{\mathbf{w}}^{-1}(1)) \simeq \mathrm{mf}(\mathbb{A}^5, \Gamma_{\mathbf{w}}, \mathbf{w}) \simeq \Pi_3(\mathrm{mf}(\mathbb{A}^4, \Gamma_{\mathbf{w}}, \mathbf{w}))$$

Hence, compute

$$\mathrm{HH}^*(\mathcal{W}(\check{\mathbf{w}}^{-1}(1)) \simeq \mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^5, \Gamma_{\mathbf{w}}, \mathbf{w}))$$

There is an explicit formula for computing the latter as a sum of $\Gamma_{\mathbf{w}}$-invariant pieces of twisted Koszul homologies (Dyckerhoff, Bavard-Favero-Katzarkov, ...).

This allows us to compute $\mathrm{SH}^*(V)$ for all our examples (and many more!).

Restatement of the calculation

Theorem. (Evans-L.)

	Singularity	ADE type	$\mathrm{SH}^{*\leq 1}(V;\mathbb{C})$
1.	$x^2+y^2+z^{\ell+1}+w^{k(\ell+1)}$	A_ℓ	ℓ
2.	$x^2+y^2+zw(z^{\ell-1}+w^{k(\ell-1)})$	A_ℓ	ℓ
3.	$x^2+y^3+z^3+w^{6k}$	D_4	4
4.*	$x^3+xy^{2k+1}+yz^2+w^2$	D_4	1
5.	$x^2+y^3+z^4+w^{12k}$	E_6	6
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$\mathrm{SH}^3(V;\mathbb{C})=\mathbb{C}^\mu$, μ Milnor number,

$\mathrm{SH}^*(V;\mathbb{C})=0$ in all other degrees.

Application to contact topology (if time permits)

	Singularity	Link	Contact structure
1.	$x^2 + y^2 + z^{\ell+1} + w^{k(\ell+1)}$	$\sharp_\ell(S^2 \times S^3)$	$\alpha_{\ell,k}$
2.	$x^2 + y^2 + zw(z^{\ell-1} + w^{k(\ell-1)})$	$\sharp_\ell(S^2 \times S^3)$	$\beta_{\ell,k}$
3.	$x^2 + y^3 + z^3 + w^{6k}$	$\sharp_4(S^2 \times S^3)$	$\delta_{4,k}$
4.	$x^3 + xy^{2k+1} + yz^2 + w^2$	$S^2 \times S^3$	$\lambda_{1,k}$
5.	$x^2 + y^3 + z^4 + w^{12k}$	$\sharp_6(S^2 \times S^3)$	$\epsilon_{6,k}$
6.	$x^2 + y^3 + z^5 + w^{30k}$	$\sharp_8(S^2 \times S^3)$	$\epsilon_{8,k}$

Let Ξ_ℓ denote the list of all contact structures on $\sharp_\ell(S^2 \times S^3)$ from this table. For example,

$$\Xi_1 = (\alpha_{1,1}, \alpha_{1,2}, \ldots, \lambda_{1,1}, \lambda_{1,2}, \ldots)$$
$$\Xi_4 = (\alpha_{4,1}, \alpha_{4,2}, \ldots, \beta_{4,1}, \beta_{4,2}, \ldots, \delta_{4,1}, \delta_{4,2}, \ldots).$$

Theorem. (Evans-L.) For each ℓ, the contact structures in the list Ξ_ℓ are pairwise nonisomorphic except for $\alpha_{\ell,1} \cong \beta_{\ell,1}$.

End