

# Symplectic cohomology of compound Du Val singularities

Jonny Evans

Yankı Lekili

## Abstract

We compute symplectic cohomology for Milnor fibres of certain compound Du Val singularities that admit small resolution by using homological mirror symmetry. Our computations suggest a new conjecture that the existence of a small resolution has strong implications for the symplectic cohomology and conversely. We also use our computations to give a contact invariant of the link of the singularities and thereby distinguish many contact structures on connected sums of  $S^2 \times S^3$ .

## 1 Introduction

### 1.1 Links

Let  $X \subset \mathbb{C}^N$  be a normal  $n$ -dimensional algebraic variety over  $\mathbb{C}$  and let  $P \in X$  be a point; we will write  $[P \in X]$  for the germ of  $X$  at  $P$  considered up to local analytic equivalence. Recall that the *link* of  $P \in X$ , written  $\text{Link}(P)$ , is the intersection of a small Euclidean sphere centred at  $P$  with  $X$ . If  $P$  is a smooth point or isolated singularity then the link is a smooth, compact  $(2n-1)$ -dimensional manifold; we will focus on hypersurface singularities, whose link is  $(n-2)$ -connected. How much information do we retain about  $[P \in X]$  if we only remember the manifold  $\text{Link}(P)$ ?

Mumford [33] proves that if  $n = 2$  then  $\text{Link}(P)$  is a simply-connected 3-manifold if and only if  $P \in X$  is a smooth point. By contrast, in higher dimensions, the topology of the link exerts less influence. For example, if  $\Sigma$  is any homotopy 7-sphere, Brieskorn [8] constructs singular complex 4-folds  $P_k \in X_k$ ,  $k \in \mathbb{N}$ , with  $[P_i \in X_i] \neq [P_j \in X_j]$  for  $i \neq j$  and  $\text{Link}(P_k) \cong \Sigma$ . More generally, when  $n \geq 3$ , surgery theory tells us there are not very many  $(n-2)$ -connected  $(2n-1)$ -manifolds<sup>1</sup>, but there are lots of singularities.

The field of complex tangencies  $\xi$  forms a contact distribution on  $\text{Link}(P)$  [45]. McLean [31] demonstrates that the contact manifold  $(\text{Link}(P), \xi)$  retains much more information about  $[P \in X]$ . For example, he shows that  $(\text{Link}(P), \xi)$  is contactomorphic to the standard

---

<sup>1</sup>For a classification, see Wall [47].

contact 5-sphere if and only if  $P \in X$  is a smooth point, and that the minimal discrepancy of a canonical  $\mathbb{Q}$ -Gorenstein singularity  $P \in X$  is determined by  $(\text{Link}(P), \xi)$ .

An interesting corollary of McLean's work relates the purely algebro-geometric notion of *terminal* singularities to the purely contact geometric notion of *dynamical convexity*.

- A singularity is called terminal if its minimal discrepancy is positive. Terminal singularities emerged in the work of Reid [36] as a natural class of singularities that should appear on minimal models of smooth 3-folds. The 3-fold terminal singularities were classified by Mori [32].
- A Reeb flow on a contact manifold is called *dynamically convex* if every closed Reeb orbit  $\gamma$  satisfies  $\mu_{CZ}(\gamma) + n - 3 > 0$ , where  $\mu_{CZ}$  is the Conley-Zehnder index. A contact manifold which admits a dynamically convex Reeb flow is called *index positive*.

**Theorem 1.1** ([31]). *Suppose that  $P \in X$  is an isolated  $\mathbb{Q}$ -Gorenstein singularity with  $H^1(\text{Link}(P); \mathbb{Q}) = 0$  (e.g. a hypersurface singularity of dimension  $n \geq 3$ ). The singularity  $P \in X$  is terminal if and only if its link  $(\text{Link}(P), \xi)$  is index positive.*

*Proof.* If  $P \in X$  is terminal then the minimal discrepancy is positive, so McLean's theorem implies that the highest minimal SFT index of the link is positive, which is precisely the statement that there is a dynamically convex Reeb flow on  $(\text{Link}(P), \xi)$ . If there is no dynamically convex Reeb flow then the highest minimal SFT index is nonpositive so, by McLean's theorem, the minimal discrepancy is also nonpositive; therefore  $P \in X$  is not terminal.  $\square$

Invariants of contact manifolds (like contact homology or symplectic field theory) are notoriously difficult to define because of bubbling of pseudoholomorphic curves in symplectisations. The condition of index positivity allows us to bypass many of these problems to get useful contact invariants. For example, if  $Y$  is a contact manifold and  $V$  is a strong symplectic filling of  $Y$  with  $c_1(V) = 0$ , then we can define symplectic cohomology  $\text{SH}^*(V; \mathbb{C})$ . This is a  $\mathbb{Z}$ -graded  $\mathbb{C}$ -vector space (with various additional algebraic structures on it) which usually depends on  $V$ . However, if  $Y$  is simply-connected and index positive then the dependence of  $\text{SH}^*(V; \mathbb{C})$  on the filling is very mild. The positive symplectic cohomology  $\text{SH}_+^*(V; \mathbb{C})$ , constructed as the cohomology of a quotient complex of the cochain complex of  $\text{SH}^*(V; \mathbb{C})$  by the cochains coming from the interior of the filling, is known to be a contact invariant [12, Proposition 9.17]. This has been used successfully by Uebele to distinguish some contact structures on  $S^2 \times S^3$  [42].

We explore a refinement of this in Corollary 4.5. In particular, for  $n = 3$ , we are able prove by a standard neck-stretching technique that the Lie algebra structure on  $\text{SH}^1(V; \mathbb{C})$  and its Lie algebra representation on  $\bigoplus_{d < 0} \text{SH}^d(V; \mathbb{C})$  is a contact invariant.

Our goal in this paper is to compute symplectic cohomology for some further examples of links of terminal 3-fold hypersurface singularities, observe some patterns which emerge,

and use it to distinguish a variety of links.

## 1.2 Compound Du Val (cDV) singularities

It is a theorem of Reid [36, Theorem 1.1] that the Gorenstein terminal 3-fold singularities are precisely the isolated *compound Du Val (cDV) singularities*. These are hypersurface singularities which (in suitable local analytic coordinates  $(w, x, y, z)$ ) are cut out by an equation of the form

$$f(x, y, z) + wg(x, y, z, w) = 0$$

where  $f$  is one of the following polynomials:

$$\begin{aligned} A_\ell &: x^2 + y^2 + z^{\ell+1}, \\ D_\ell &: x^2 + y(z^2 + y^{\ell-2}), \\ E_6 &: x^2 + y^3 + z^4, \\ E_7 &: x^2 + y(y^2 + z^3), \\ E_8 &: x^2 + y^3 + z^5. \end{aligned}$$

The  $w = 0$  hyperplane section has an ADE singularity at 0. If  $\Gamma$  is the ADE type of this hyperplane section, we refer to the 3-fold singularity as a *compound*  $\Gamma$  or  $c\Gamma$  singularity.

As we have explained in Theorem 1.1, the links of these singularities are index positive and so we can use  $\text{SH}^*$  of the Milnor fibre for  $* < 0$  as a contact invariant.

**Remark 1.2.** Observe that if we define  $B \subset \text{Link}(0)$  to be the intersection  $\{w = 0\} \cap \text{Link}(0)$  then we get a Milnor open book

$$w/|w|: \text{Link}(0) \setminus B \rightarrow S^1$$

with binding  $B$ . The page is a copy of the corresponding 4-dimensional ADE Milnor fibre and the contact structure determined by the open book is contactomorphic to  $\xi$ .

**Example 1.3.** Consider the family of  $cA_1$  singularities

$$A_\ell := \{x^2 + y^2 + z^2 + w^{\ell+1} = 0\}, \quad \ell \geq 1.$$

In fact, any  $cA_1$  singularity is equivalent to one of these. The link is either  $S^5$  (if  $\ell$  is even) or  $S^2 \times S^3$  (if  $\ell$  is odd). The page of the Milnor open book is the  $A_1$ -Milnor fibre  $T^*S^2$ , and the monodromy is the  $(\ell + 1)$ st power of a Dehn twist in the zero-section. The symplectic cohomology of the Milnor fibre  $V_\ell$  behaves differently if  $\ell$  is odd or even. If  $\ell$  is even then, by [29, Section 5.2], we have

$$\text{SH}^*(V_\ell; \mathbb{C}) = \begin{cases} \mathbb{C}^\ell & \text{if } * = 3 \\ \mathbb{C} & \text{if } * = -q(\ell + 3) - r \text{ for } r \in \{0, \dots, \ell - 1\}, r = q(\text{mod}2) \\ \mathbb{C} & \text{if } * = -q(\ell + 3) - r + 1 \text{ for } r \in \{0, \dots, \ell - 1\}, r = q(\text{mod}2) \\ 0 & \text{otherwise,} \end{cases}$$

for  $q \in \mathbb{N}$ . In particular, we see that  $\mathrm{SH}^*$  can be either 0 or  $\mathbb{C}$  for  $* < 0$ .

If  $\ell$  is odd then we will see below that

$$\mathrm{SH}^*(V_\ell; \mathbb{C}) = \begin{cases} \mathbb{C}^\ell & \text{if } * = 3 \\ \mathbb{C} & \text{if } * = 1 \text{ or } * < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Write  $\xi_\ell$  for the contact structure on the link of  $0 \in A_\ell$ . Since the contact invariant  $\mathrm{SH}_+^*$  coincides with  $\mathrm{SH}^*$  if  $* < 0$ , this shows that

- (a) the links  $\{(S^5, \xi_\ell) : \ell = 2, 4, 6, \dots\}$  are pairwise nonisomorphic as contact manifolds,
- (b) we cannot distinguish the links  $\{(S^2 \times S^3, \xi_\ell) : \ell = 1, 3, 5, \dots\}$  using  $\mathrm{SH}_+^*$  with coefficients in  $\mathbb{C}$ .

A similar phenomenon was observed by Van Koert [44, Example 3.1.1] for these contact structures on  $S^2 \times S^3$ : they are not distinguished by their cylindrical contact homology. Interestingly, Uebele [42] does distinguish them using  $\mathrm{SH}_+^*$  with coefficients in  $\mathbb{Z}/2$ . We will give a second way to distinguish them below.

From an algebro-geometric perspective, the singularities  $A_\ell$  have different behaviour when  $\ell$  is even/odd. For example, these singularities admit *small resolutions*<sup>2</sup> if and only if  $\ell$  is odd; indeed, if  $\ell$  is odd, there is a resolution whose exceptional set is an irreducible rational curve. However, if  $\ell$  is even then there cannot be a small resolution because the link is not diffeomorphic to a nontrivial connected sum of copies of  $S^2 \times S^3$ .

Inspired by this example, we record an optimistic conjecture, which provides the main motivation for the calculations in this paper. We will establish this conjecture in a range of examples (Theorem 1.8).

**Conjecture 1.4.** *Suppose that  $P \in X$  is a cDV singularity and let  $V$  be the Milnor fibre of the singularity. Then  $P \in X$  admits a small resolution such that the exceptional set has  $\ell$  irreducible components if and only if  $\mathrm{SH}^*(V; \mathbb{C})$  has rank  $\ell$  in every negative degree.*

**Remark 1.5.** In this paper, we have focused on providing evidence for one direction of this conjecture: that the existence of a small resolution constrains the symplectic cohomology. The converse is plausible: we have calculated many examples and found no counterexample. This would give an a priori way of detecting whether a cDV singularity admits a small resolution just by knowing its link.

**Remark 1.6.** If  $P \in X$  admits a small resolution whose exceptional set has  $\ell$  irreducible components then the link is diffeomorphic to  $\sharp_\ell(S^2 \times S^3)$ . Small resolutions can be constructed by thinking of the 3-fold as a 1-parameter deformation of an ADE singularity.

---

<sup>2</sup>Recall that a small resolution is a resolution whose exceptional set has codimension at least 2. Note that, by [36, Theorem 1.14], a resolution of an isolated cDV singularity is small if and only if it is crepant.

This gives a classifying map from the disc to the versal deformation space of the ADE singularity such that the 3-fold is the pullback along the classifying map of the versal family. Brieskorn [9], Tjurina [41] and Pinkham [35] constructed branched coverings of the versal ADE deformation space (branched over the discriminant locus) such that the pullback of the versal family to the branched covering admits a simultaneous (partial) resolution. More precisely, the fundamental group of the complement of the discriminant locus is the ADE Artin braid group; Brieskorn and Tyurina constructed the branched covering corresponding to the kernel of the homomorphism to the ADE Weyl group and found a full simultaneous resolution, while Pinkham constructed simultaneous partial resolutions for intermediate covering spaces. For a specific 3-fold, if the classifying map from the disc lifts (in the sense of algebraic topology) to one of these branched covers, then you get a small resolution by pulling back the simultaneous partial resolution of the versal family. In particular, the existence of a small resolution can be read off from the monodromy of the Milnor open book mentioned in Remark 1.2 (which is the element of the fundamental group of the ADE Artin braid group represented by the boundary of the disc under the classifying map).

**Remark 1.7.** Remark 1.6 provides a sanity check on Conjecture 1.4. Consider what happens if we deform the germ of the singularity at  $P$ . Namely, suppose we have a family  $h_s(w, x, y, z) = f(x, y, z) + g_s(w, x, y, z)$  of cDV singularities parametrised by  $s \in \mathbb{C}$ . Suppose that there are balls  $B \subset \mathbb{C}$  and  $B' \subset \mathbb{C}^4$  such that for  $s \in B$ , the origin is the only singularity of the hypersurface  $h_s^{-1}(0) \cap B'$ . Gray's stability theorem tells us that the contact geometry of the link of the singularity is independent of  $s \in B$ . Moreover, if  $0 \in h_0^{-1}(0)$  admits a small resolution then so do all the singularities  $0 \in h_s^{-1}(0)$  because the monodromy of the Milnor open book is stable under perturbations.

We now summarise our evidence for Conjecture 1.4. These calculations will be explained in Section 3. Throughout, we work over  $\mathbb{C}$ .

**Theorem 1.8.** *The table below summarises our calculations of symplectic cohomology for Milnor fibres of some cDV singularities. The left-most column is a polynomial  $\check{\mathbf{w}}$  and the singularity is defined by  $0 \in \check{\mathbf{w}}^{-1}(0)$ . The columns  $\text{SH}^*$  give the ranks of the various graded pieces of  $\text{SH}(\check{\mathbf{w}}^{-1}(1))$ . In all cases,  $\text{SH}^d(\check{\mathbf{w}}^{-1}(1)) = 0$  if  $d = 2$  or  $d \geq 4$ . The final column gives a reference for the calculation. Case 4 is conditional on Conjecture 2.2 or Conjecture 2.3, so we have marked it with an asterisk.*

<i>Singularity</i>	<i>ADE type</i>	$\text{SH}^3$	$\text{SH}^{d \leq 1}$	<i>See Theorem...</i>
1. $x_1^2 + x_2^2 + x_3^{\ell+1} + x_4^{k(\ell+1)}$	$A_\ell$	$\ell(k(\ell+1) - 1)$	$\ell$	3.7 (1), (2)
2. $x_1^2 + x_2^2 + x_3 x_4 (x_3^{\ell-1} + x_4^{k(\ell-1)})$	$A_\ell$	$(k\ell + 1)(\ell - 1)$	$\ell$	3.13
3. $x_1^2 + x_2^3 + x_3^3 + x_4^{6k}$	$D_4$	$24k - 4$	4	3.7 (3)
4*. $x_1^3 + x_1 x_2^{2k+1} + x_2 x_3^2 + x_4^2$	$D_4$	$6k + 5$	1	3.10
5. $x_1^2 + x_2^3 + x_3^4 + x_4^{12k}$	$E_6$	$72k - 6$	6	3.7 (4)
6. $x_1^2 + x_2^3 + x_3^5 + x_4^{30k}$	$E_8$	$240k - 8$	8	3.7 (5)

**Remark 1.9.** In all cases, these singularities admit small resolutions and the number of exceptional curves in the resolution equals the rank of  $\text{SH}^d$  for  $d \leq 1$ ; this is explained case-by-case in Section 3. In particular, this establishes Conjecture 1.4 for these examples.

**Remark 1.10.** The examples in Theorem 1.8 are all *invertible polynomials* (see Section 2), and our strategy for calculating symplectic cohomology uses mirror symmetry for invertible polynomials to relate SH with the Hochschild cohomology of a mirror dg-category of equivariant matrix factorisations. In all cases except case 4, the required mirror symmetry conjecture is proven. Case 4 is only proved conditionally (see Section 2.2). This example is the base of the Laufer flop [26].

**Remark 1.11.** Theorem 1.8 seems to indicate that symplectic cohomology (over  $\mathbb{C}$ ) of the Milnor fibre is not a useful invariant for distinguishing contact structures on links. We are nonetheless able to distinguish all these examples by studying a certain bigrading on symplectic cohomology, as we discuss in Section 1.3. Note that Uebele's work (discussed in Example 1.3 above) shows that Conjecture 1.4 breaks down if we work over a field of characteristic  $\neq 0$ , which gives an alternative way to distinguish contact structures on links.

### 1.3 Families of inequivalent contact structures

We introduce the following notation for the contact structures on the links of our singularities:

Table 1: Contact structures on links of our cDV singularities.

Singularity	Link	Contact structure
1. $x_1^2 + x_2^2 + x_3^{\ell+1} + x_4^{k(\ell+1)}$	$\sharp_\ell(S^2 \times S^3)$	$\alpha_{\ell,k}$
2. $x_1^2 + x_2^2 + x_3 x_4 (x_3^{\ell-1} + x_4^{k(\ell-1)})$ ( $\ell \geq 2$ )	$\sharp_\ell(S^2 \times S^3)$	$\beta_{\ell,k}$
3. $x_1^2 + x_2^3 + x_3^3 + x_4^{6k}$	$\sharp_4(S^2 \times S^3)$	$\delta_{4,k}$
4. $x_1^3 + x_1 x_2^{2k+1} + x_2 x_3^2 + x_4^2$	$S^2 \times S^3$	$\lambda_{1,k}$
5. $x_1^2 + x_2^3 + x_3^4 + x_4^{12k}$	$\sharp_6(S^2 \times S^3)$	$\epsilon_{6,k}$
6. $x_1^2 + x_2^3 + x_3^5 + x_4^{30k}$	$\sharp_8(S^2 \times S^3)$	$\epsilon_{8,k}$

**Remark 1.12.** Note that  $\alpha_{\ell,1} \cong \beta_{\ell,1}$ : the two singularities are related by a change of variables.

Let  $\Xi_\ell$  denote the list of all contact structures on  $\sharp_\ell(S^2 \times S^3)$  from this table. For example,

$$\begin{aligned}\Xi_1 &= (\alpha_{1,1}, \alpha_{1,2}, \dots, \lambda_{1,1}, \lambda_{1,2}, \dots) \\ \Xi_4 &= (\alpha_{4,1}, \alpha_{4,2}, \dots, \beta_{4,1}, \beta_{4,2}, \dots, \delta_{4,1}, \delta_{4,2}, \dots).\end{aligned}$$

**Theorem 1.13.** *For each  $\ell$ , the contact structures in the list  $\Xi_\ell$  are pairwise nonisomorphic except for  $\alpha_{\ell,1} \cong \beta_{\ell,1}$ .*

**Remark 1.14.** We remind the reader that all results about  $\lambda_{1,k}$  are conditional on a mirror symmetry statement.

**Remark 1.15.** What makes this an interesting theorem is that all of these links have the same positive symplectic cohomology over  $\mathbb{C}$ . We equip  $\text{SH}^*$  with a contact-invariant bigrading to distinguish these contact manifolds. This bigrading will be the weight decomposition of  $\bigoplus_{d < 0} \text{SH}^d$  under the action of the Lie algebra  $\text{SH}^1$ .

**Remark 1.16.** As explained in Example 1.3, the fact that  $\alpha_{1,i} \not\cong \alpha_{1,j}$  if  $i \neq j$  was proved by Uebele [42] using positive symplectic cohomology with coefficients in  $\mathbb{Z}/2$  (rather than a bigrading).

**Remark 1.17.** If one focuses on cDV singularities which do not admit a small resolution, one finds very many more contact structures which can be distinguished by  $\text{SH}_+^*$  already without using the bigrading. This is not so surprising: it is much easier for 5-manifolds to be diffeomorphic than contactomorphic.

*Acknowledgments.* J. E. would like to thank Mark McLean and Michael Wemyss for inspiration and for helpful discussions. Y. L. would like to thank Kazushi Ueda for collaborations [28],[29] from which many ideas are borrowed here.

J. E. was supported by EPSRC Grant EP/P02095X/2. Y. L. was partially funded by the Royal Society URF\R\180024.

## 2 Symplectic cohomology for invertible polynomials

### 2.1 Symplectic cohomology

Let  $V$  be a Liouville manifold with  $c_1(V) = 0$ . Associated to  $V$  we can define an invariant  $\text{SH}^*(V)$  called the symplectic cohomology of  $V$ . Symplectic cohomology was introduced by Cieliebak, Floer, Hofer [22, 14, 11] and Viterbo [46]. An excellent exposition can be found in [39]. More recent results can be learned from [12]. See also [27, Sec 2.1] for a fast review

of our sign and grading conventions. In particular, our conventions are cohomological and the unit lives in degree zero!

Briefly,  $\mathrm{SH}^*(V)$  is an algebra over the homology operad of framed little discs over an arbitrary commutative ring  $\mathbf{k}$  (in this paper  $\mathbf{k} = \mathbb{C}$ ). In particular, it has a (graded) commutative product, a Gerstenhaber bracket  $[\cdot, \cdot]$  (i.e. a Lie bracket of degree  $-1$ ), and a Batalin-Vilkovisky operator  $\Delta$  (i.e. a degree  $-1$  operator whose Hochschild coboundary is the bracket).

In general, symplectic cohomology is rather difficult to compute explicitly. A fruitful approach to do such computations goes via the open string A-model. Namely, we have an isomorphism

$$\mathrm{SH}^*(V) \simeq \mathrm{HH}^*(\mathcal{W}(V))$$

where  $\mathcal{W}(V)$  is the wrapped Fukaya category of  $V$ . An early version of this result based on Legendrian surgery is due to Bourgeois-Ekholm-Eliashberg ([6], elaborated in [13]) which concerned Hochschild homology; a definitive version based on duality appeared in [20, Theorem 1.1] (see also the more recent [10]).

On the other hand, even if one achieved a good understanding of  $\mathcal{W}(V)$ , in general, it is still a difficult algebraic problem to compute Hochschild cohomology of  $A_\infty$  categories.

In [28], [29], a method to compute symplectic cohomology for certain Milnor fibres was given based on the homological mirror symmetry conjecture for *invertible polynomials*.

## 2.2 Invertible polynomials and mirror symmetry

**Definition 2.1.** To an  $(n+1)$ -by- $(n+1)$  integer matrix  $A = (a_{ij})$  with nonzero determinant, we associate the polynomial

$$\mathbf{w}(x_1, \dots, x_{n+1}) = \sum_{i=1}^{n+1} \prod_{j=1}^{n+1} x_j^{a_{ij}}.$$

We write  $\check{\mathbf{w}}$  for the polynomial associated to  $A^T$  (the *Berglund-Hübsch mirror* to  $\mathbf{w}$ , see [4]).

An invertible polynomial is weighted homogeneous, that is there is a uniquely determined weight system  $(d_1, d_2, \dots, d_{n+1}; h)$  satisfying  $\mathrm{gcd}(d_1, d_2, \dots, d_{n+1}, h) = 1$  for which

$$\mathbf{w}(\lambda^{d_1} x_1, \dots, \lambda^{d_n} x_{n+1}) = \lambda^h \mathbf{w}(x_1, \dots, x_{n+1})$$

for all  $\lambda \in \mathbb{G}_m$ . In this paper, we are primarily concerned with the log Fano case, i.e. when

$$h - \sum_{i=1}^{n+1} d_i =: d_0 < 0$$



In fact, there is a finite extension  $\Gamma_{\mathbf{w}}$  of  $\mathbb{G}_m$  acting on  $\mathbb{A}^{n+1}$  which preserves  $\mathbf{w}$ , namely

$$\Gamma_{\mathbf{w}} := \left\{ (t_0, t_1, \dots, t_{n+1}) \in \mathbb{G}_m : \prod_{j=1}^{n+1} t_j^{a_{ij}} = t_0 t_1 \cdots t_{n+1}, i = 1, \dots, n+1 \right\},$$

acting on  $\mathbb{A}^{n+1}$  via  $(x_1, \dots, x_{n+1}) \mapsto (t_1 x_1, \dots, t_{n+1} x_{n+1})$ . This group also acts on  $\mathbb{A}^{n+2}$  via  $(x_0, x_1, \dots, x_{n+1}) \mapsto (t_0 x_0, t_1 x_1, \dots, t_{n+1} x_{n+1})$ , and this  $\Gamma_{\mathbf{w}}$ -action preserves the polynomial

$$\mathbf{w}(x_1, \dots, x_{n+1}) + x_0 \cdots x_{n+1}.$$

With this setup, we can formulate the following mirror symmetry conjectures. A version of Conjecture 2.2 appeared in [16] (see also [28, Conjecture 1.2] and references therein), and Conjecture 2.3 appeared in [28].

**Conjecture 2.2.** *There is a quasi-equivalence of idempotent complete  $A_\infty$ -categories*

$$\mathcal{F}(\check{\mathbf{w}}) \simeq \text{mf}(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w})$$

*between the Fukaya-Seidel category of a Morsification of  $\check{\mathbf{w}}$  and the dg-category of  $\Gamma_{\mathbf{w}}$ -equivariant matrix factorisations of  $\mathbf{w}$ . Moreover, there exists a full collection  $\Delta_1, \dots, \Delta_K$  of vanishing thimbles for the Morsification of  $\check{\mathbf{w}}$  such that the  $A_\infty$ -algebra  $\mathcal{A} := \text{end}_{\mathcal{F}(\check{\mathbf{w}})}(\bigoplus_i \Delta_i)$  is quasi-isomorphic to its cohomology algebra  $A := H(\mathcal{A})$  (so that both  $\mathcal{F}(\check{\mathbf{w}})$  and  $\text{mf}(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w})$  are quasi-equivalent to  $\text{perf}(A)$ ).*

**Conjecture 2.3** ([28, Conjecture 1.4]). *There is a quasi-equivalence of idempotent complete  $A_\infty$  categories*

$$\mathcal{W}(\check{\mathbf{w}}^{-1}(1)) \simeq \text{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 x_1 \cdots x_{n+1})$$

*between the wrapped Fukaya category of the Milnor fibre  $\check{\mathbf{w}}^{-1}(1)$  and the dg-category of  $\Gamma_{\mathbf{w}}$ -equivariant matrix factorisations of  $\mathbf{w} + x_0 \cdots x_{n+1}$ .*

These conjectures are established in the following situations:

- If the matrix  $A$  is diagonal (so  $\mathbf{w}$  defines a Brieskorn-Pham singularity) then Conjecture 2.2 was proved by Futaki and Ueda [17]. More generally, if the matrix  $A$  is block diagonal and its blocks are either 1-by-1 or 2-by-2 equal to  $\begin{pmatrix} 2 & 1 \\ 0 & k \end{pmatrix}$  (so that  $\mathbf{w}$  is a Sebastiani-Thom sum of ADE polynomials of type  $A$  or  $D$ ), Conjecture 2.2 was proved by Futaki and Ueda [18].
- If  $n = 1$ , Conjecture 2.2 was proved by Habermann and Smith [21]. In fact, this means Conjecture 2.2 holds for any invertible polynomial  $\mathbf{w}$  of the form  $\mathbf{w}(x_1, \dots, x_{n+1}) = x_1^2 + \cdots + x_{n-1}^2 + f(x_n, x_{n+1})$ . This is because stabilising  $\mathbf{w}$  and  $\check{\mathbf{w}}$  by adding quadratic terms in extra variables changes neither the Fukaya-Seidel nor the matrix factorisation category.

- In [28], various cases of Conjecture 2.3 were verified. The sequel paper [29] focused on the log Fano case and established Conjecture 2.3 for the Milnor fibres of simple singularities. Conjecture 2.3 was proved in full generality by Gammage [19] in the  $\mathbb{Z}/2$ -graded case using a microlocal sheaf category version of wrapped Fukaya categories. For our purposes, we will need to work with  $\mathbb{Z}$ -graded categories; a careful chase of  $\mathbb{Z}$ -gradings in [19] might allow us to assume Conjecture 2.3 in all cases.

**Remark 2.4.** The main theorem statements from Futaki-Ueda and Habermann-Smith do not mention the formality of  $\mathcal{A}$ , but this is either implicit or proved along the way. For example, in [17] the mirror category is a dg-category with trivial differential (see the definition of  $\mathfrak{A}_p$  on [17, p.437]); in [21, Theorem 2.13] it is explicitly stated that  $\mathcal{A}$  is formal for degree reasons.

**Remark 2.5.** The examples  $\check{\mathbf{w}}$  from Theorem 1.8 all fall into one of these cases except for  $\check{\mathbf{w}} = x_1^3 + x_1x_2^{2k+1} + x_2x_3^2 + x_4^2$ . In this case, our results are conditional on one of the two Conjectures 2.2 or 2.3 holding.

We now explain how knowing one or other of these conjectures can help one to calculate symplectic cohomology.

### 2.3 Using mirror symmetry to compute symplectic cohomology

Pick a Morsification of  $\check{\mathbf{w}}$ . Let  $\mathcal{F}(\check{\mathbf{w}})$  denote the Fukaya-Seidel category of the Morsification, let  $V := \check{\mathbf{w}}^{-1}(1)$  denote the Milnor fibre, and let  $\mathcal{W}(V)$  (respectively  $\mathcal{F}(V)$ ) denote the wrapped (respectively compact) Fukaya category of  $V$ . Choose a collection of vanishing paths for the Morsification and let  $\Delta_1, \dots, \Delta_K$  (respectively  $S_1, \dots, S_K$ ) be the corresponding vanishing thimbles (respectively vanishing cycles). Let  $\mathcal{A} = \text{end}_{\mathcal{F}(\check{\mathbf{w}})}(\bigoplus_i \Delta_i)$  and  $\mathcal{B} = \text{end}_{\mathcal{F}(V)}(\bigoplus_i S_i)$ . Let  $A = H(\mathcal{A})$  and  $B = H(\mathcal{B})$  denote the cohomology algebras of  $\mathcal{A}$  and  $\mathcal{B}$  (considered as  $A_\infty$ -algebras with zero higher products).

**Theorem 2.6.** *Assume that  $\text{HH}^2(\text{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})) = 0$ , that  $d_0 \neq 0$ , and either Conjecture 2.2 or Conjecture 2.3 holds. Then*

$$\text{SH}^*(V) \cong \text{HH}^*(\text{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})) \quad (2.1)$$

as Gerstenhaber algebras.

In the next section, we give a formula to compute  $\text{HH}^*(\text{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$ .

*Proof that Conjecture 2.2 implies Equation (2.1).* If  $d_0 \neq 0$  then [28, Theorem 6.4] implies that the inclusion of categories  $\mathcal{B} \rightarrow \mathcal{W}(V)$  induces an isomorphism on Hochschild cohomology. Since this map comes from a functor, it is a morphism of Gerstenhaber algebras. Ganatra [20, Theorem 1.1] shows that  $\text{SH}^*(V) \cong \text{HH}^*(\mathcal{W}(V))$  as Gerstenhaber algebras. Therefore, we need to show

$$\text{HH}^*(\mathcal{B}) \cong \text{HH}^*(\text{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})). \quad (2.2)$$

As a first step, we calculate  $\mathrm{HH}^*(B)$ , where  $B = H(\mathcal{B})$  is the cohomology algebra of  $\mathcal{B}$ .

**Lemma 2.7.** *We have  $\mathrm{HH}^*(B) \cong \mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$  as Gerstenhaber algebras.*

*Proof.* By [29, Eq. (1.7)], the category  $\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})$  is the  $n$ -Calabi-Yau completion  $\Pi_n(\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w}))$  (defined in [25, Section 4.1]; see also [29, Section 2]). By Conjecture 2.2, this is quasi-equivalent to  $\Pi_n(\mathrm{perf}(A)) \simeq \mathrm{perf}(\Pi_n(A))$ . Thus

$$\mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})) \cong \mathrm{HH}^*(\Pi_n(A)).$$

By construction,  $\Pi_n(A)$  is Koszul-dual to the trivial extension algebra  $T_n(A)$  (defined in [38, Section 4.1] where it is called *cyclic completion*; see also [29, Section 2]), so

$$\mathrm{HH}^*(\Pi_n(A)) \cong \mathrm{HH}^*(T_n(A)).$$

as Gerstenhaber algebras (see for example [24]).

For any Lefschetz fibration with  $(n+1)$ -(complex-)dimensional total space with  $n > 0$ , the Floer cohomology algebra  $B = H(\mathcal{B})$  of the vanishing cycles is isomorphic to the trivial extension algebra  $T_n(A)$  of the Floer cohomology algebra  $A = H(\mathcal{A})$ . Therefore, in this case, we get

$$\mathrm{HH}^*(B) \cong \mathrm{HH}^*(T_n(A)) \cong \mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})). \quad \square$$

**Lemma 2.8.** *The  $A_\infty$ -algebra  $\mathcal{B}$  is quasi-isomorphic to its cohomology algebra  $B$ .*

*Proof.* By Lemma 2.7,  $\mathrm{HH}^2(B) \cong \mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$ , which vanishes by assumption, so  $\mathcal{B}$  is intrinsically formal, and hence quasi-isomorphic to  $B = H(\mathcal{B})$ .  $\square$

Together, these two lemmas show that  $\mathrm{HH}^*(\mathcal{B}) \cong \mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$  as Gerstenhaber algebras, establishing Equation (2.2), so Equation (2.1) follows.  $\square$

*Proof that Conjecture 2.3 implies Equation (2.1).* We will show in Theorem 2.15 below that if  $\mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})) = 0$  then we can make a  $\Gamma_{\mathbf{w}}$ -equivariant formal change of coordinates along the critical locus of  $\mathbf{w} + x_0 \cdots x_{n+1}$  such that the pullback of  $\mathbf{w} + x_0 \cdots x_{n+1}$  in these new coordinates equals  $\mathbf{w}$ . If we can make such a formal change of coordinates, it follows from [34, Theorem 2.10] that  $\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 \cdots x_{n+1})$  is quasi-equivalent to  $\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})$ , so Conjecture 2.3 implies

$$\mathrm{HH}^*(\mathcal{W}(V)) \cong \mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$$

as Gerstenhaber algebras. By [20, Theorem 1.1],  $\mathrm{HH}^*(\mathcal{W}(V)) \cong \mathrm{SH}^*(V)$  as Gerstenhaber algebras, so Equation (2.1) follows.  $\square$

## 2.4 Calculating $\mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$

There is a formula for  $\mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$  which expresses it as a sum of  $\Gamma_{\mathbf{w}}$ -invariant pieces of twisted Koszul cohomologies; this formula appeared in [3, Theorem 1.2], where its context and history are discussed. It is also explained and used in [28, Theorem 3.1] and [29, Section 5.1]. We now briefly describe how to perform calculations in practice with this formula; Theorem 2.14 below summarises the answer. We will use the notation from this section in our calculations in Section 3.

**Definition 2.9.** Define the character

$$\chi: \Gamma_{\mathbf{w}} \rightarrow \mathbb{G}_m, \quad \chi(t_0, \dots, t_{n+1}) = t_0 \cdots t_{n+1}.$$

Its kernel  $\ker \chi$  is the finite group

$$\ker \chi = \left\{ (t_0, \dots, t_{n+1}) \in \mathbb{G}_m : \prod_{j=1}^{n+1} t_j^{a_{ij}} = 1, t_0 = t_1^{-1} \cdots t_{n+1}^{-1} \right\}.$$

**Definition 2.10.** Given an element  $\gamma \in \ker \chi$ , let

$$\{1, \dots, n+1\} = \{i_1, \dots, i_k\} \cup \{j_1, \dots, j_{n+1-k}\}$$

be the partition for which each  $x_{i_m}$  is fixed under the action of  $\gamma$  and each  $x_{j_m}$  is not fixed under the action of  $\gamma$ . Let  $J_\gamma$  be a monomial basis for the Jacobian ring of  $\mathbf{w}|_{x_{j_1}=\dots=x_{j_{n+1-k}}=0}$ .

**Definition 2.11.** The set  $M_\gamma$  of  $\gamma$ -monomials is the union  $M_\gamma = A_\gamma \cup B_\gamma \cup C_\gamma$  where

$$\begin{aligned} A_\gamma &= \begin{cases} \{x_0^\beta p x_{j_1}^\vee \cdots x_{j_{n+1-k}}^\vee : p \in J_\gamma, \beta = 0, 1, 2, \dots\} & \text{if } x_0 \text{ is fixed by } \gamma \\ \emptyset & \text{otherwise.} \end{cases} \\ B_\gamma &= \begin{cases} \{x_0^\beta p x_0^\vee x_{j_1}^\vee \cdots x_{j_{n+1-k}}^\vee : p \in J_\gamma, \beta = 0, 1, 2, \dots\} & \text{if } x_0 \text{ is fixed by } \gamma \\ \emptyset & \text{otherwise.} \end{cases} \\ C_\gamma &= \begin{cases} \emptyset & \text{if } x_0 \text{ is fixed by } \gamma \\ \{p x_0^\vee x_{j_1}^\vee \cdots x_{j_{n+1-k}}^\vee : p \in J_\gamma\} & \text{otherwise.} \end{cases} \end{aligned}$$

**Definition 2.12.** Let  $\zeta: \Gamma_{\mathbf{w}} \rightarrow \mathbb{G}_m$  be a character of  $\Gamma_{\mathbf{w}}$ . We say that a polynomial or formal power series  $p(x_0, \dots, x_{n+1})$  is  $\zeta$ -isotypical if  $p(g\mathbf{x}) = \zeta(g)p(\mathbf{x})$  for all  $g \in \Gamma_{\mathbf{w}}$ . Note that every monomial  $m$  determines a character  $\xi(m)$  such that  $m$  is  $\xi(m)$ -isotypical. The space of formal power series  $K := \mathbb{C}[[x_0, \dots, x_{n+1}]]$  is therefore the completed direct sum of its  $\zeta$ -isotypical summands

$$K = \widehat{\bigoplus_{\zeta \in \hat{\Gamma}_{\mathbf{w}}} K_\zeta}, \quad K_\zeta = \{p \in K : p(g\mathbf{x}) = \zeta(g)p(\mathbf{x}) \ \forall g \in \Gamma_{\mathbf{w}}\}.$$

**Definition 2.13.** Given a  $\gamma$ -monomial  $m$ , we write  $b_k$  for the total exponent of  $x_k$  in  $m$ , where  $x_k^\vee$  contributes  $-1$  to  $b_k$ . The character  $\xi(m)$  is determined by these exponents:

$$\xi(m)(t_0, \dots, t_{n+1}) = t_0^{b_0} \cdots t_{n+1}^{b_{n+1}}.$$

**Theorem 2.14.** *The Hochschild cohomology  $\mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$  is a direct sum of 1-dimensional contributions, one from each pair  $(\gamma, m)$  with  $m \in M_\gamma$  such that  $\chi(m) = \chi^{\otimes u}$  for some  $u \in \mathbb{Z}$ . In this case,  $(\gamma, m)$  contributes to*

$$\begin{aligned} & \mathrm{HH}^{2u+n-k+1} \text{ if } m \in A_\gamma, \\ & \mathrm{HH}^{2u+n-k+2} \text{ if } m \in B_\gamma, \\ & \mathrm{HH}^{2u+n-k+2} \text{ if } m \in C_\gamma. \end{aligned}$$

## 2.5 Formal change of coordinates

Recall that there exist weights  $d_0, d_1, \dots, d_{n+1}$  such that if we give  $x_i$  weight  $d_i$  then both  $\mathbf{w}(x_1, \dots, x_{n+1})$  and  $x_0 \cdots x_{n+1}$  are quasihomogeneous of degree  $h = \sum_{i=0}^{n+1} d_i$  and  $\chi$ -isotypical. Let  $|\cdot|_0$  be the  $x_0$ -valuation on the space  $K = \mathbb{C}[[x_0, \dots, x_{n+1}]]$  of formal power series, i.e.  $|p|_0 = k$  if  $x_0^k$  divides  $p$  but  $x_0^{k+1}$  does not.

**Theorem 2.15.** *Suppose that  $p_0(x_0, \dots, x_{n+1})$  is a  $\chi$ -isotypical formal power series which is quasihomogeneous of degree  $h$  and  $|p_0|_0 > 0$ . If  $\mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})) = 0$  then there is a formal change of variables  $\mathbf{z} = (x_0, z_1(\mathbf{x}), \dots, z_{n+1}(\mathbf{x}))$  such that  $\mathbf{w}(\mathbf{z}) = \mathbf{w}(\mathbf{x}) + p_0(\mathbf{x})$ .*

**Remark 2.16.** In particular, the theorem applies when  $p_0(\mathbf{x}) = x_0 \cdots x_{n+1}$ . To prove Theorem 2.15, we first establish a sequence of lemmas.

**Lemma 2.17.** *Suppose that  $\mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})) = 0$ . Then the image of  $K_\chi$  in the Jacobian ring is trivial.*

*Proof.* If  $m$  is a  $\chi$ -isotypical monomial which is nontrivial in the Jacobian ring then we can use it as part of our monomial basis  $J_\gamma$  for  $\gamma = id$ . It will then contribute as a type  $A$   $id$ -monomial to  $\mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$ . Thus if  $\mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})) = 0$ , we deduce that any monomial  $m \in K_\chi$  is trivial in the Jacobian ring, and hence the image of  $K_\chi$  in the Jacobian ring is zero.  $\square$

**Lemma 2.18.** *If  $p \in K_\chi$  is trivial in the Jacobian ring then  $p = \sum_{i=1}^{n+1} v_i \frac{\partial \mathbf{w}}{\partial x_i}$  for some  $v_1, \dots, v_{n+1} \in K$  where  $v_i \in K_{t_i}$ . Here,  $t_i$  denotes the character of  $\Gamma_{\mathbf{w}}$  which projects to  $t_i$ .*

*Proof.* Consider the map  $\partial: K^{n+1} \rightarrow K$  defined by  $\partial(v_1, \dots, v_{n+1}) = \sum_{i=1}^{n+1} v_i \frac{\partial \mathbf{w}}{\partial x_i}$ . The cokernel of  $\partial$  is the Jacobian ring. Because  $\mathbf{w} \in K_\chi$ , we have  $\partial \mathbf{w} / \partial x_i \in K_{\chi \otimes t_i^{-1}}$  for all  $i$ , so  $\partial(v_1, \dots, v_{n+1}) \in K_\chi$  if and only if  $v_i \in K_{t_i}$  for all  $i = 1, \dots, n+1$ .  $\square$

If  $\mathbf{v} \in K_{t_1} \oplus \cdots \oplus K_{t_{n+1}}$  then we call  $\mathbf{v}$  a  $\Gamma_{\mathbf{w}}$ -equivariant vector field because the components  $v_i$  of  $\mathbf{v}$  transform under  $\Gamma_{\mathbf{w}}$  like the coordinates  $x_i$ . We have now seen that, under the hypotheses of Theorem 2.15,  $p_0 = \partial\mathbf{v}$  for a  $\Gamma_{\mathbf{w}}$ -equivariant vector field  $\mathbf{v}$ .

**Lemma 2.19.** *In the setting of Theorem 2.15, there exists a formal change of variables  $\mathbf{y}$  such that*

$$p_1(\mathbf{y}) := \mathbf{w}(\mathbf{x}) + p_0(\mathbf{x}) - \mathbf{w}(\mathbf{y})$$

is  $\chi$ -isotypical and satisfies  $|p_1|_0 > |p_0|_0$ .

*Proof.* This is a small modification of [2, Section 12.6]. We know that  $p_0 = \partial\mathbf{v}$  for a  $\Gamma_{\mathbf{w}}$ -equivariant vector field  $\mathbf{v}$ . We define  $\mathbf{y}$  implicitly by  $y_0 = x_0$ ,  $x_i = y_i - v_i(\mathbf{y})$ . Since  $\mathbf{v}$  is  $\Gamma_{\mathbf{w}}$ -equivariant, this formal change of coordinates is  $\Gamma_{\mathbf{w}}$ -equivariant. As in the proof<sup>3</sup> of [2, Section 12.6], we find that  $p_1(\mathbf{y}) := \mathbf{w}(\mathbf{x}) + p_0(\mathbf{x}) - \mathbf{w}(\mathbf{y})$  has  $|p_1|_0 > |p_0|_0$ . Moreover, since  $\mathbf{v}$  is  $\Gamma_{\mathbf{w}}$ -equivariant,  $p_1$  is  $\chi$ -isotypical.  $\square$

*Proof of Theorem 2.15.* We can apply Lemma 2.19 iteratively and compose the formal diffeomorphisms we get at each stage. Composition makes sense because  $|v|_0 > 0$ , so the  $x_0^k$ -term in the composition of formal diffeomorphisms only involves summing finitely many terms. In this way, we construct a sequence of perturbation terms  $p_1, p_2, \dots$  with  $|p_1|_0 < |p_2|_0 < \dots$ . In the limit, we obtain a formal change of coordinates  $\mathbf{z}$  with perturbation term  $p_\infty(\mathbf{z}) := \mathbf{w}(\mathbf{x}) + p_0(\mathbf{x}) - \mathbf{w}(\mathbf{z})$  satisfying  $|p_\infty|_0 = \infty$ . Therefore  $p_\infty = 0$  and we have proved the theorem.  $\square$

### 3 Compendium of examples

In this section, we calculate  $\mathrm{HH}^* := \mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$  for the invertible polynomials  $\mathbf{w}$  which are mirror-dual to the polynomials in Theorem 1.8. We now summarise how this leads to a proof of that theorem.

*Proof of Theorem 1.8.* These examples are log Fano, so  $d_0 < 0$ , and, in all cases, we will see that  $\mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})) = 0$ . In Cases 1–3 and 5–6 of Theorem 1.8, Conjecture 2.2 holds, so that Theorem 2.6 applies. As a consequence, we can conclude Theorem 1.8 unconditionally in these cases. Case 4 holds conditionally on Conjecture 2.2 or 2.3.  $\square$

---

<sup>3</sup>In [2], they have no variable  $x_0$  and filter by the weighted degree of the perturbation rather than the  $x_0$ -valuation. Since the perturbation terms are quasihomogeneous of degree  $h$ , the weighted degree of the perturbation term with respect to  $x_1, \dots, x_{n+1}$  is proportional to the  $x_0$ -valuation, so our strategy is equivalent.

### 3.1 Brieskorn-Pham

A Brieskorn-Pham singularity is an isolated hypersurface singularity given by the vanishing of the polynomial

$$\mathbf{w}(x_1, \dots, x_{n+1}) = x_1^{a_1} + \dots + x_{n+1}^{a_{n+1}}$$

for a collection of integers  $a_i \geq 2$ . This is an invertible polynomial with  $\tilde{\mathbf{w}} = \mathbf{w}$ .

Let  $\mu_k$  denote the cyclic group of  $k$ th roots of unity, and let

$$l = \text{lcm}(a_1, \dots, a_{n+1}), \quad \nu = 1 - \sum_{i=1}^{n+1} \frac{1}{a_i}.$$

We have a surjective  $l$ -to-1 homomorphism

$$\begin{aligned} T: \mu_{a_1} \times \dots \times \mu_{a_{n+1}} \times \mathbb{G}_m &\rightarrow \Gamma_{\mathbf{w}}, \\ T(\mu_1, \dots, \mu_{n+1}, \tau) &= (\tau^{l\nu} \mu_1^{-1} \dots \mu_{n+1}^{-1}, \tau^{l/a_1} \mu_1, \dots, \tau^{l/a_{n+1}} \mu_{n+1}, \tau) \end{aligned}$$

**Remark 3.1.** Under  $T$ , the subgroup  $\mu_{a_1} \times \dots \times \mu_{a_{n+1}}$  maps isomorphically onto  $\ker \chi$ ; we will use this identification to write elements of  $\ker \chi$  as  $(n+1)$ -tuples of roots of unity.

Fix an element  $\gamma \in \ker \chi$ . Restricting  $\mathbf{w}$  to the fixed variables  $x_{i_1}, \dots, x_{i_k}$  we get  $\sum_{m=1}^k x_{i_m}^{a_{i_m}}$ , and we pick the monomial basis

$$J_\gamma = \left\{ x_{i_1}^{b_{i_1}} \dots x_{i_k}^{b_{i_k}} : 0 \leq b_{i_m} \leq a_{i_m} - 2 \text{ for } m = 1, 2, \dots, k \right\}$$

for its Jacobian ring.

**Lemma 3.2.** *Let  $m$  be a  $\gamma$ -monomial with total exponents  $b_0, \dots, b_{n+1}$  and suppose that  $(\gamma, m)$  contributes to Hochschild cohomology. Then  $b_i = b_0 \bmod a_i$  for  $i = 1, \dots, n+1$  and  $\xi(m) = \chi^{\otimes(b_0 - \sum_{i=1}^{n+1} m_i)}$ , where the integers  $m_i$  are determined by  $b_0 = b_i + m_i a_i$ .*

*Proof.* The  $\gamma$ -monomial  $m$  with total exponents  $b_0, \dots, b_{n+1}$  has character<sup>4</sup>

$$\xi(m)(\mu_1, \dots, \mu_{n+1}, \tau) = \tau^{b_0 l \nu + \sum_{i=1}^{n+1} b_i l / a_i} \mu_1^{b_1 - b_0} \dots \mu_{n+1}^{b_{n+1} - b_0}$$

This coincides with a power of  $\chi$  if and only if  $b_0 = b_i \bmod a_i$  for  $i = 1, \dots, n+1$ . More precisely, if  $b_0 = b_i + m_i a_i$  for integers  $m_1, \dots, m_n$  then  $\xi(m) = \chi^{\otimes(b_0 - \sum_{i=1}^{n+1} m_i)}$ .  $\square$

**Remark 3.3.** In fact, if  $b_0 \geq 0$ , then  $b_0$  uniquely determines monomials  $m_A(b_0)$  and  $m_B(b_0)$  of types  $A$  and  $B$  respectively which have total exponents  $b_i = b_0 \bmod a_i$ . Namely, we multiply together factors of  $x_i^{b_i}$ ,  $i = 1, \dots, n+1$ , where  $x_i^{-1}$  means  $x_i^\vee$ . To obtain  $m_A(b_0)$  we include a factor of  $x_0^{b_0}$ ; to obtain  $m_B(b_0)$  we include a factor of  $x_0^{b_0+1} x_0^\vee$ . Similarly,  $b_0 = -1$  determines unique monomials  $m_B(-1) = x_0^\vee \dots x_{n+1}^\vee$  of type  $B$  and  $m_C(-1) = x_0^\vee \dots x_{n+1}^\vee$  of type  $C$ .

<sup>4</sup>The characters of  $\Gamma_{\mathbf{w}}$  induce characters of  $\mu_{a_1} \times \dots \times \mu_{a_{n+1}} \times \mathbb{G}_m$  by precomposing with  $T$  and we will often write characters of  $\Gamma_{\mathbf{w}}$  by giving a character of the bigger group which factors through  $T$ .

**Remark 3.4.** By the Sun Zi remainder theorem, given any collection of total exponents  $0 \leq b_1 < a_1, \dots, 0 \leq b_{n+1} < a_{n+1}$ , we can solve this system of congruences for  $b_0$  uniquely modulo  $l$  if and only if  $b_i = b_j \pmod{\gcd(a_i, a_j)}$  for all  $i, j \in \{1, \dots, n+1\}$ .

Our approach to calculating  $\mathrm{HH}^*$  will therefore be to consider each possible value  $b_0$  and find the number of elements  $\gamma \in \ker \chi$  such that  $(\gamma, m_A(b_0))$ ,  $(\gamma, m_B(b_0))$  or  $(\gamma, m_C(b_0))$  is a contributing  $\gamma$ -monomial.

The contributions from  $b_0 = -1$  are easy to calculate.

**Lemma 3.5.** *The contributions from monomials with total exponent  $b_0 = -1$  come from*

$$(\gamma, x_0^\vee \cdots x_{n+1}^\vee) \in \mathrm{HH}^n$$

for all  $\gamma \in (\mu_{a_1} \setminus \{1\}) \times \cdots \times (\mu_{a_{n+1}} \setminus \{1\})$ .

*Proof.* We have  $m_B(-1) = m_C(-1) = x_0^\vee \cdots x_{n+1}^\vee$ . This contributes as a  $\gamma$ -monomial if and only if either:

- $\gamma$  leaves all variables  $x_0, \dots, x_{n+1}$  unfixed. In this case we get a type  $C$  contribution from  $(\gamma, m_C(-1))$ .
- $\gamma$  fixes  $x_0$  and does not fix any other variable. In this case we get a type  $B$  contribution from  $(\gamma, m_B(-1))$ .  $\square$

**Remark 3.6.** In fact, in our examples (but not in general), these will be the only contributions to  $\mathrm{HH}^n$ , which gives  $\dim \mathrm{HH}^n = (a_1 - 1) \cdots (a_{n+1} - 1)$ . Note that this equals the Milnor number of the singularity.

We now proceed to the specific examples of interest to compute the contributions explicitly. These examples are:

$$\begin{aligned} cA_\ell &: x_1^2 + x_2^2 + x_3^{\ell+1} + x_4^{k(\ell+1)}, & k, \ell \geq 1 \\ cD_4 &: x_1^2 + x_2^3 + x_3^3 + x_4^{6k}, & k \geq 1 \\ cE_6 &: x_1^2 + x_2^3 + x_3^4 + x_4^{12k}, & k \geq 1 \\ cE_8 &: x_1^2 + x_2^3 + x_3^5 + x_4^{30k}, & k \geq 1 \end{aligned}$$

In all cases, the  $x_4 = 0$  slice has an ADE singularity at the origin, having the type indicated. The 3-folds admit small resolutions which fully resolve the singularity of the slice; this follows from [9, Satz 0.2] because the exponent of  $x_4$  is a multiple of the Coxeter number of the ADE singularity.

**Theorem 3.7.** *For each  $\mathbf{w}$  below, we will compute  $\mathrm{HH}^* = \mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^5, \Gamma_{\mathbf{w}}, \mathbf{w}))$ . In all cases,  $\mathrm{HH}^d$  vanishes when  $d = 2$  or  $d \geq 4$ .*



1. **Type  $cA_\ell$ :** Let  $\mathbf{w} = x_1^2 + x_2^2 + x_3^{\ell+1} + x_4^{k(\ell+1)}$ . Then

$$\dim \mathrm{HH}^3 = \ell(k(\ell+1) - 1), \quad \dim \mathrm{HH}^d = \ell \text{ for } d \leq 1.$$

2. **Type  $cD_4$ :** Let  $\mathbf{w} = x_1^2 + x_2^3 + x_3^3 + x_4^{6k}$ . Then

$$\dim \mathrm{HH}^3 = 24k - 4, \quad \dim \mathrm{HH}^d = 4 \text{ for } d \leq 1.$$

3. **Type  $cE_6$ :** Let  $\mathbf{w} = x_1^2 + x_2^3 + x_3^4 + x_4^{12k}$ . Then

$$\dim \mathrm{HH}^3 = 72k - 6, \quad \dim \mathrm{HH}^d = 6 \text{ for } d \leq 1.$$

4. **Type  $cE_8$ :** Let  $\mathbf{w} = x_1^2 + x_2^3 + x_3^5 + x_4^{30k}$ . Then

$$\dim \mathrm{HH}^3 = 240k - 8, \quad \dim \mathrm{HH}^d = 8 \text{ for } d \leq 1.$$

*Proof.* In each case, the  $\mathrm{HH}^3$  contributions come from 3.5. We will consider the contributions from  $m_A(b_0)$  with  $b_0 \geq 0$ .

In the various cases we will use Euclid's algorithm to write:

Type	$b_0 =$	$q \in$	$r \in$
$cA_\ell$	$k(\ell+1)p + (\ell+1)q + r$	$\{0, 1, \dots, k-1\}$	$r \in \{0, 1, \dots, \ell\}$
$cD_4$	$6kp + 6q + r$	$\{0, 1, \dots, k-1\}$	$r \in \{0, 1, \dots, 5\}$
$cE_6$	$12kp + 12q + r$	$\{0, 1, \dots, k-1\}$	$r \in \{0, 1, \dots, 11\}$
$cE_8$	$30kp + 30q + r$	$\{0, 1, \dots, k-1\}$	$r \in \{0, 1, \dots, 29\}$

In the following tables, we indicate: the type  $A$  monomials  $m_A(b_0)$ ; the  $\gamma$  for which  $(\gamma, m_A(b_0))$  contribute to  $\mathrm{HH}^*$ ; the number of such  $\gamma$ ; and the degree of  $\mathrm{HH}^*$  to which they contribute. We omit monomials  $m$  for which there are no  $\gamma$  such that  $(\gamma, m)$  contributes.

In every case, we will see that  $\mathrm{HH}^*$  has the rank stated in the theorem in every even degree  $d \leq 0$ . The type  $B$  contributions, other than those appearing in Lemma 3.5, will differ only in replacing  $m_A(b_0)$  with  $x_0 x_0^\vee m_A(b_0)$  and yield the same ranks in every odd degree  $d \leq 1$ .

Table 2: Table for  $cA_\ell$ . Note that  $b_0 = k(\ell + 1)p + (\ell + 1)q + r$  with  $p \geq 0$ ,  $0 \leq q \leq k - 1$ ,  $0 \leq r \leq \ell$ . The top two row give us rank  $\ell$  in every degree  $d \neq -2k \bmod 2(k + 1)$ . The bottom row gives us rank  $\ell$  in degrees  $d = -2k \bmod 2(k + 1)$  ( $\omega$  is a chosen primitive  $(\ell + 1)$ th root of unity and  $a \in \{1, 2, \dots, \ell\}$ ).

$q$	$r$	$m_A(b_0)$	$\gamma$	$\# \gamma$	HH* degree	
any	$< \ell$	$x_0^{b_0} x_3^r x_4^{(\ell+1)q+r}$	$\begin{cases} 1 & \text{if } b_0 \text{ even} \\ x_1^\vee x_2^\vee & \text{if } b_0 \text{ odd} \end{cases}$	$\begin{cases} (1, 1, 1, 1) \\ (-1, -1, 1, 1) \end{cases}$	1	$-2(k + 1)p - 2q$
$k - 1$	$\ell$	$x_0^{b_0} x_3^\vee x_4^\vee$	$\begin{cases} 1 & \text{if } b_0 \text{ even} \\ x_1^\vee x_2^\vee & \text{if } b_0 \text{ odd} \end{cases}$	$\begin{cases} (-1, -1, \omega^a, \omega^{-a}) \\ (1, 1, \omega^a, \omega^{-a}) \end{cases}$	$\ell$	$-2(k + 1)p - 2k$

Table 3: Table for  $cD_4$ . The top three rows give us rank 4 in every degree  $d \neq -2k \bmod 2(k + 1)$ . The bottom row gives us rank 4 in degrees  $d = -2k \bmod 2(k + 1)$  (on the second and fourth rows,  $\omega$  is a chosen primitive 3rd root of unity and  $a, b \in \{1, 2\}$ ).

$q$	$r$	$m_A(b_0)$	$\gamma$	$\# \gamma$	HH* degree
any	0	$x_0^{6kp+6q} x_4^{6q}$	$(1, 1, 1, 1)$	1	$-2(k + 1)p - 2q$
any	2	$x_0^{6kp+6q+2} x_4^{6q+2} x_2^\vee x_3^\vee$	$(1, \omega^a, \omega^{-a}, 1)$ ,	2	$-2(k + 1)p - 2q$
any	4	$x_0^{6kp+6q+4} x_2 x_3 x_4^{6q+4}$	$(1, 1, 1, 1)$	1	$-2(k + 1)p - 2q$
$k - 1$	5	$x_0^{6kp+6k-1} x_1^\vee x_2^\vee x_3^\vee x_4^\vee$	$(-1, \omega^a, \omega^b, -\omega^{-a-b})$	4	$-2(k + 1)p - 2k$

Table 4: Table for  $cE_6$ . The top six rows give us rank 6 in every degree  $d \neq -2k \bmod 2(k + 1)$ . The bottom row gives us rank 6 in degrees  $d = -2k \bmod 2(k + 1)$  ( $\omega$  and  $i$  are chosen primitive 3rd and 4th roots of unity; and  $a \in \{1, 2\}$  and  $b \in \{1, 2, 3\}$ ).

$q$	$r$	$m_A(b_0)$	$\gamma$	$\# \gamma$	HH* degree
any	0	$x_0^{12kp+12q} x_4^{12q}$	$(1, 1, 1, 1)$	1	$-2(k + 1)p - 2q$
any	3	$x_0^{12kp+12q+3} x_4^{12q+3} x_1^\vee x_3^\vee$	$(-1, 1, -1, 1)$	1	$-2(k + 1)p - 2q$
any	4	$x_0^{12kp+12q+4} x_2 x_4^{12q+4}$	$(1, 1, 1, 1)$	1	$-2(k + 1)p - 2q$
any	6	$x_0^{12kp+12q+6} x_3^2 x_4^{12q+6}$	$(1, 1, 1, 1)$	1	$-2(k + 1)p - 2q$
any	7	$x_0^{12kp+12q+7} x_2 x_4^{12q+7} x_1^\vee x_3^\vee$	$(-1, 1, -1, 1)$	1	$-2(k + 1)p - 2q$
any	10	$x_0^{12kp+12q+10} x_2 x_3^2 x_4^{12q+10}$	$(1, 1, 1, 1)$	1	$-2(k + 1)p - 2q$
$k - 1$	11	$x_0^{12kp+12k-1} x_1^\vee x_2^\vee x_3^\vee x_4^\vee$	$(-1, \omega^a, i^b, -\omega^{-a} i^{-b})$	6	$-2(k + 1)p - 2k$

Table 5: Table for  $cE_8$ . The top eight rows give us rank 8 in every degree  $d \neq -2k \pmod{2(k+1)}$ . The bottom row gives us rank 8 in degrees  $d = -2k \pmod{2(k+1)}$  ( $\omega$  and  $\zeta$  are chosen primitive 3rd and 5th roots of unity,  $a \in \{1, 2\}$ , and  $b \in \{1, 2, 3, 4\}$ ).

$q$	$r$	$m_A(b_0)$	$\gamma$	$\# \gamma$	HH* degree
any	0	$x_0^{30kp+30q} x_4^{30q}$	$(1, 1, 1, 1)$	1	$-2(k+1)p - 2q$
any	6	$x_0^{30kp+30q+6} x_3 x_4^{30q+6}$	$(1, 1, 1, 1)$	1	$-2(k+1)p - 2q$
any	10	$x_0^{30kp+30q+10} x_2 x_4^{30q+10}$	$(1, 1, 1, 1)$	1	$-2(k+1)p - 2q$
any	12	$x_0^{30kp+30q+12} x_3^2 x_4^{30q+12}$	$(1, 1, 1, 1)$	1	$-2(k+1)p - 2q$
any	16	$x_0^{30kp+30q+16} x_2 x_3 x_4^{30q+16}$	$(1, 1, 1, 1)$	1	$-2(k+1)p - 2q$
any	18	$x_0^{30kp+30q+18} x_3^3 x_4^{30q+18}$	$(1, 1, 1, 1)$	1	$-2(k+1)p - 2q$
any	22	$x_0^{30kp+30q+22} x_2 x_3^2 x_4^{30q+22}$	$(1, 1, 1, 1)$	1	$-2(k+1)p - 2q$
any	28	$x_0^{30kp+30q+28} x_2 x_3^3 x_4^{30q+28}$	$(1, 1, 1, 1)$	1	$-2(k+1)p - 2q$
$k-1$	29	$x_0^{30kp+30k-1} x_1^\vee x_2^\vee x_3^\vee x_4^\vee$	$(-1, \omega^a, \xi^b, -\omega^{-a}\xi^{-b})$	8	$-2(k+1)p - 2k$ $\square$

### 3.2 Laufer's examples

Let

$$\check{\mathbf{w}} = x_1^3 + x_1 x_2^{2k+1} + x_2 x_3^2 + x_4^2$$

This polynomial defines a  $cD_4$  singularity: the  $x_1 = x_2$  slice has an isolated  $D_4$  singularity at the origin. Laufer [26] showed that this admits a small resolution with a single exceptional curve; the small resolution yields a partial resolution of the  $D_4$  slice (the map from the minimal resolution to the partial resolution collapses the three peripheral curves in the  $D_4$  configuration).

The Berglund-Hübsch transpose is

$$\mathbf{w} = x_1^3 x_2 + x_2^{2k+1} x_3 + x_3^2 + x_4^2$$

which has

$$\Gamma_{\mathbf{w}} = \{(t_0, t_1, t_2, t_3, t_4) : t_1^3 t_2 = t_2^{2k+1} t_3 = t_3^2 = t_4^2 = t_0 t_1 t_2 t_3 t_4\}.$$

**Lemma 3.8.** *There is a 3-to-1 surjective homomorphism*

$$\begin{aligned} T: \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_3 \times \mathbb{C}^\times &\rightarrow \Gamma_{\mathbf{w}}, \\ T(\pm 1, \mu, \tau) &= (\pm \mu^{-1} \tau^{-(4k+4)}, \mu \tau^{4k+1}, \tau^3, \tau^{6k+3}, \pm \tau^{6k+3}). \end{aligned}$$

The composition  $\chi \circ T$  is given by  $(\pm 1, \mu, \tau) \mapsto \tau^{12k+6}$ .

*Proof.* We first show that the stated homomorphism is surjective. Since  $t_2^{2k+1} t_3 = t_3^2$  we get  $t_3 = t_2^{2k+1}$ . Since  $t_4^2 = t_3^2$ , we have  $t_4 = \pm t_2^{2k+1}$ . Since  $t_1^3 t_2 = t_3^2 = t_2^{4k+2}$ , we get  $t_1^3 = t_2^{4k+1}$ . If  $t_2 = \tau^3$  for some  $\tau \in \mathbb{C}^\times$  then  $t_1 = \mu \tau^{4k+1}$  for some cube root  $\mu$  of unity. Finally,  $t_0$  is determined by  $t_0 \cdots t_4 = t_3^2$ , which gives  $t_0 = \pm \mu^{-1} \tau^{-4k-4}$ .

To see that the homomorphism is 3-to-1, observe that its kernel consists of triples  $(s, \mu, \tau)$  such that

$$\mu\tau^{4k+1} = \tau^3 = \tau^{6k+3} = \pm\tau^{6k+3} = 1.$$

In particular, this means  $s = 1$  and  $\tau^3 = 1$ . The condition  $1 = \mu\tau^{4k+1} = \mu\tau^{k+1}$  means that  $\mu = \tau^{-k-1}$ , so the kernel is  $\{(1, \tau^{-k-1}, \tau) : \tau^3 = 1\}$ , which has size 3.  $\square$

The kernel  $\ker(\chi \circ T)$  is then  $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_{12k+6}$ ; recall that  $T$  is 3-to-1, so this is three times the size of  $\ker \chi$ . We now identify which combinations of fixed and unfixed variables are possible for  $\gamma \in \ker \chi$ .

**Lemma 3.9.** *Let  $\gamma = T(s, \mu, \tau) \in \ker \chi$ . The possible combinations of fixed and unfixed variables for  $\gamma$  are given by the table below. We state the conditions on  $(s, \mu, \tau) \in \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_{12k+6}$  such that  $\gamma = T(\pm 1, \mu, \tau)$  fix this combination of variables, and also the number of such  $\gamma$  (remembering that  $T$  is 3-to-1).*

Fixed variables	Number of $\gamma = T(s, \mu, \tau) \in \ker \chi$	$s$	$\mu$	$\tau$
$\{0, 1, 2, 3, 4\}$	1	1	$\tau^{-4(k+1)}$	$\tau^3 = 1$
$\{0\}$	1	1	$\tau^{-4(k+1)}$	$\tau^3 = -1$
$\{1, 2, 3\}$	1	-1	$\tau^{-4k-1}$	$\tau^3 = 1$
$\{2, 3, 4\}$	2	1	$\mu \neq \tau^{-k-1}$	$\tau^3 = 1$
$\{2, 3\}$	2	-1	$\mu \neq \tau^{-k-1}$	$\tau^3 = 1$
$\emptyset$	$6k + 2$	See below		

Three further cases which do not contribute to  $\text{HH}^*$ :  $\{3\}$ ,  $\{4\}$ , and  $\{3, 4\}$

*Proof.* Let  $\gamma = T(s, \mu, \tau)$  with  $s \in \{\pm 1\}$ ,  $\mu \in \boldsymbol{\mu}_3$ ,  $\tau \in \boldsymbol{\mu}_{12k+6}$ .

If  $x_0$  is fixed then  $\tau^{-4(k+1)} = s\mu$ . This means that  $\tau^{24(k+1)} = 1$ , but  $\tau^{12k+6} = 1$ , so  $\tau^6 = 1$ . Therefore  $\tau^{-4(k+1)}$  is a cube root of unity, which means that  $s = 1$ . This means that the other variables transform as  $\mu\tau^{4k+1} = \tau^{-3}$ ,  $\tau^3$ ,  $\tau^{6k+3}$  and  $\tau^{6k+3}$ . There are two possibilities:  $\tau^3 = 1$  (which fixes all variables) or  $\tau^3 = -1$  (which fixes none).

If  $x_1$  is fixed then  $\mu = \tau^{-4k-1}$ , so  $\tau^{-12k-3} = 1$ , but  $\tau^{12k+6} = 1$ , so  $\tau^3 = 1$ . This means that  $x_2$  and  $x_3$  are also fixed. The variable  $x_0$  transforms as  $s\mu^{-1}\tau^{-4k-4} = s\tau^{-3} = s$ , so either  $x_0$  is fixed (as in the previous case) or  $s = -1$ , in which case both  $x_0$  and  $x_4$  are unfixed.

If  $x_2$  is fixed then  $\tau^3 = 1$  so  $\tau^{6k+3} = 1$  and  $x_3$  is also fixed. If  $x_0$  or  $x_1$  is fixed then we are in a previous case; assume they are not. Then  $\mu \neq \tau^{-k-1}$  and  $s$  can take on either value because both  $\mu$  and  $\tau^{-k-1}$  are in  $\boldsymbol{\mu}_3$ , so  $\mu = -\tau^{-k-1}$  is impossible. If  $s = 1$  then  $x_4$  is fixed (yielding fixed variables  $\{2, 3, 4\}$ ); otherwise we get fixed variables  $\{2, 3\}$ .

Finally, if none of  $x_0, x_1, x_2$  are fixed then the remaining variables can be fixed in any combination. We will see in Theorem 3.10 that the only such  $\gamma$  which contribute  $\gamma$ -monomials to  $\text{HH}^*$  are those which fix no variables. There are  $6k + 2$  of these. To see this,

we argue as follows. If  $x_3$  is not fixed then  $\tau^{6k+3} \neq 1$ , so  $\tau^{6k+3} = -1$ . If  $x_4$  is not fixed then  $s\tau^{6k+3} = -1$  means that  $s = 1$ . The remaining conditions become

$$\mu \neq \tau^{-4(k+1)}, \quad \mu \neq \tau^{-4k-1}.$$

The second condition always holds because  $\tau^{6k+3} = -1$ , so  $\tau^{-4k-1}$  is not a cube root of unity ( $(\tau^{-4k-1})^3 = \tau^{-12k-3} = \tau^3 \neq 1$ ). The first condition implies  $1 = (\tau^{-4(k+1)})^3 = \tau^{12k+12} = \tau^6$ , which can hold only if  $\tau^3 = -1$ . Therefore there are  $6k$  roots of  $\tau^{6k+3} = -1$  for which  $\mu$  can take on any value and 3 roots of  $\tau^3 = -1$  for which  $\mu$  can be two out of the three roots of unity. This gives a total of  $3(6k+2)$  combinations  $(1, \mu, \tau)$ , and this triple-counts the available  $\gamma$ s because  $T$  is 3-to-1.  $\square$

We now pick the following monomial bases  $J_\gamma$  for the relevant Jacobian rings:

$$\begin{aligned} \text{Jac}(\mathbf{w}_{|x_1, x_2, x_3, x_4}) &= \text{Jac}(\mathbf{w}_{|x_1, x_2, x_3}) \\ &= \mathbb{C}[x_1, x_2, x_3]/(3x_1^2x_2, x_1^3 + (2k+1)x_2^{2k}x_3, x_2^{2k+1} + 2x_3) \\ &= \mathbb{C}\{1, x_2, x_2^2, \dots, x_2^{4k+1}, x_1, x_1x_2, x_1x_2^2, \dots, x_1x_2^{4k+1}, x_1^2\} \\ \text{Jac}(\mathbf{w}_{|x_2, x_3, x_4}) &= \text{Jac}(\mathbf{w}_{|x_2, x_3}) \\ &= \mathbb{C}[x_2, x_3]/((2k+1)x_2^{2k}x_3, x_2^{2k+1} + 2x_3) \\ &= \mathbb{C} \cdot \{1, x_2, x_2^2, \dots, x_2^{4k}\} \\ \text{Jac}(\mathbf{w}_{|x_3, x_4}) &= \mathbb{C}[x_3, x_4]/(2x_3, 2x_4) = \mathbb{C} \cdot 1 \\ \text{Jac}(\mathbf{w}_{|x_3}) &= \text{Jac}(\mathbf{w}_{|x_4}) = \mathbb{C} \cdot 1 \end{aligned}$$

**Theorem 3.10.** *If  $\mathbf{w} = x_1^3x_2 + x_2^{2k+1}x_3 + x_3^2 + x_4^2$  then  $\text{HH}^*(\mathbb{A}^5, \Gamma_{\mathbf{w}}, \mathbf{w})$  satisfies*

$$\dim \text{HH}^3 = 6k + 5, \quad \dim \text{HH}^d = 1 \text{ for } d \leq 1$$

and  $\dim \text{HH}^d = 0$  for  $d = 2$  and  $d \geq 4$ .

The  $\text{HH}^*$  contributions for these singularities are as follows:

Monomial	Type	Degree in $\text{HH}^*$	Number of contributions
$x_0^\vee x_1^\vee x_2^\vee x_3^\vee x_4^\vee$	C	3	$6k + 2$
$x_0^\vee x_1^\vee x_2^\vee x_3^\vee x_4^\vee$	B	3	1
$x_0^\vee x_1^\vee x_2^{2k} x_4^\vee$	C	3	2
$x_0^{(6k+3)p-1} x_1^\vee x_2^\vee x_3^\vee x_4^\vee$	A	$-4(k+1)p + 2$	$p \geq 1$ even
$x_0^{(6k+3)p+3q} x_2^{2q}$	A	$-4(k+1)p - 2q$	$p \geq 0, 0 \leq q \leq 2k, p = q \pmod{2}$
$x_0^{(6k+3)p+3q+4} x_1 x_2^{2q+3}$	A	$-4(k+1)p - 2q - 2$	$p \geq 0, 0 \leq q \leq 2k - 1, p = q \pmod{2}$
$x_0^{(6k+3)p+6k+4} x_1 x_2$	A	$-4(k+1)(p+1)$	$p \geq 0$ even
$x_0^{(6k+3)p+6k+2} x_1^2$	A	$-4(k+1)p - 4k - 2$	$p \geq 0$ even

Type B contributions in  $\text{HH}^{d+1}$  for each type A monomial contributing to  $\text{HH}^d$ .

*Proof.* Assuming the stated monomials are correct, the patient reader can check that every degree less than or equal to 1 picks up precisely one contribution as  $p$  and  $q$  vary (it suffices to check this over the degree range from 0 to  $-8(k+1)$ ). We will therefore focus on establishing the list of contributing  $\gamma$ -monomials.

We work one set of fixed variables at a time and figure out which  $\gamma$ -monomials can contribute.

$\{0, 1, 2, 3, 4\}$ : The possible  $A$ -type monomials are as follows.

$x_0^{b_0} x_2^{b_2}$  with  $0 \leq b_2 \leq 4k+1$ . For this to contribute, we need  $b_0 = 0 \pmod 6$  and  $3b_2 - 4(k+1)b_0 = (12k+6)u$  for some  $u \in \mathbb{Z}$ . If we write  $b_0 = 6\beta_0$  and  $2\beta_0 = (2k+1)p+q$  with  $p \geq 0$ ,  $q \in \{0, 1, \dots, 2k\}$ ,  $p = q \pmod 2$ , then we get

$$b_2 = (4k+2)(2(k+1)p+u) + 4(k+1)q,$$

so if we reduce modulo  $4k+2$  we get  $b_2 = 2q \pmod{4k+2}$ . Since  $0 \leq b_2 \leq 4k+1$  and  $q \leq 2k$ , this determines  $b_2$ . The result is a contribution  $x_0^{(6k+3)p+3q} x_2^{2q} \in \text{HH}^{-4(k+1)p-2q}$  for all  $p \geq 0$ ,  $q \in \{0, 1, \dots, 2k\}$  with  $p = q \pmod 2$ .

$x_0^{b_0} x_1 x_2^{b_2}$  with  $0 \leq b_2 \leq 4k+1$ . For this to contribute, we need  $b_0 = 1 \pmod 3$ ,  $b_0 = 0 \pmod 2$  (so  $b_0 = 4 \pmod 6$ ) and  $3b_2 + 4k + 1 - 4(k+1)b_0 = (12k+6)u$  for some  $u \in \mathbb{Z}$ . As in the previous case, we write  $b_0 = 6\beta_0 + 4$  and  $2\beta_0 = (2k+1)p+q$  with  $p \geq 0$ ,  $q \in \{0, 1, \dots, 2k\}$ ,  $p = q \pmod 2$ . Arguing as before, we deduce that  $b_2 = 2q + 3 \pmod{4k+2}$ . This means  $b_2 = 2q + 3$  except in the case  $q = 2k$ , when  $b_2 = 1$ . We get contributions  $x_0^{(6k+3)p+3q+4} x_1 x_2^{2q+3} \in \text{HH}^{-4p(k+1)-2q-2}$  for  $p \geq 0$ ,  $q \in \{0, 1, \dots, 2k-1\}$  with  $p = q \pmod 2$  and a contribution  $x_0^{(6k+3)p+6k+4} x_1 x_2 \in \text{HH}^{-4(p+1)(k+1)}$ .

$x_0^{b_0} x_1^2$ . For this to contribute, we need  $b_0 = 0 \pmod 2$ ,  $b_0 = 2 \pmod 3$  (so  $b_0 = 2 \pmod 6$ ) and  $8k+2 - 4(k+1)b_0 = (12k+6)u$  for some  $u \in \mathbb{Z}$ . If we write  $b_0 = 6\beta_0 + 2$  and  $2\beta_0 = (2k+1)p+q$  with  $p \geq 0$ ,  $q \in \{0, 1, \dots, 2k\}$ ,  $p = q \pmod 2$  then we get  $q = 2k \pmod{2k+1}$  and so  $x_0^{(6k+3)p+6k+2} x_1^2 \in \text{HH}^{-4(k+1)p-4k-2}$  (with  $p$  even).

We also get corresponding  $B$ -type monomials by replacing  $x_0^{b_0}$  with  $x_0^{b_0+1} x_0^\vee$ .

$\{0\}$ : Any type  $A$  contribution is  $x_0^{b_0} x_1^\vee \cdots x_4^\vee$ . We need

$$b_0 = -1 \pmod 2, \quad b_0 = -1 \pmod 3 \Rightarrow b_0 = 5 \pmod 6$$

and  $-2(6k+3) - 3 - (4k+1) - 4(k+1)b_0 = (12k+6)u$  for some  $u \in \mathbb{Z}$ . Write  $b_0 = 6\beta_0 - 1$ . Then we get  $-4(k+1)\beta_0 = (2k+1)(u+1)$ . Since  $\gcd(4(k+1), 2k+1) = 1$ , we deduce that  $\beta_0 = (2k+1)p$  and  $u+1 = -4(k+1)p$  for some  $p$ . In other words, we get  $x_0^{(12k+6)p-1} x_1^\vee \cdots x_4^\vee \in \text{HH}^{2-4(k+1)p}$  ( $p \geq 1$ ). There is a corresponding  $B$ -type monomial  $x_0^{(12k+6)p} x_0^\vee x_1^\vee \cdots x_4^\vee \in \text{HH}^{3-4(k+1)p}$  ( $p \geq 0$ ).

$\{1, 2, 3\}$ : The possible  $\gamma$ -monomials are of type  $C$ . They have the form  $x_0^\vee x_1^{b_1} x_2^{b_2} x_4^\vee$  with  $b_1 = 0, 1$ ,  $b_2 = 0, 1, \dots, 4k+1$  or  $b_1 = 2$ ,  $b_2 = 0$ . If this contributes then we have

$b_1 = b_0 = -1 \pmod{3}$ , which leaves the only possibility as  $x_0^\vee x_1^2 x_4^\vee$ . This transforms under the action of  $T(1, 1, \tau)$  as  $\tau^{6k+3}$ , which is not an integer power of  $\tau^{12k+6}$ , so this monomial does not contribute.

$\{2, 3, 4\}$ : The possible  $\gamma$ -monomials are  $x_0^\vee x_1^\vee x_2^{b_2} \in C_\gamma$ , which transform nontrivially under the action of  $T(-1, 1, 1)$  and hence do not contribute to  $\text{HH}^*$ .

$\{2, 3\}$ : There are two  $\gamma$  fixing precisely  $x_2, x_3$ . The only  $\gamma$ -monomials are  $x_0^\vee x_1^\vee x_2^{b_2} x_4^\vee$  with  $b_2 = 0, 1, \dots, 4k$ . These transform according to the character  $\tau^{3b_2-6k}$ , which is an integer power of  $\tau^{12k+6}$  if and only if  $b_2 = 2k$ . This yield two contributions  $(\gamma, x_0^\vee x_1^\vee x_2^{2k} x_4^\vee) \in \text{HH}^3$ .

$\{3, 4\}$ : The only  $\gamma$ -monomial is  $x_0^\vee x_1^\vee x_2^\vee \in C_\gamma$ , which transforms nontrivially under the action of  $T(-1, 1, 1)$  and hence does not contribute to  $\text{HH}^*$ .

$\{3\}$ : The only  $\gamma$ -monomial  $x_0^\vee x_1^\vee x_2^\vee x_4^\vee \in C_\gamma$  which transforms as  $\tau^{6k+3}$  under the action of  $T(1, 1, \tau)$  and hence does not contribute to  $\text{HH}^*$ .

$\{4\}$ : The only  $\gamma$ -monomial  $x_0^\vee x_1^\vee x_2^\vee x_3^\vee \in C_\gamma$  which transforms as  $\tau^{6k+3}$  under the action of  $T(1, 1, \tau)$  and hence does not contribute to  $\text{HH}^*$ .

$\emptyset$ : The  $C$ -type monomial  $x_0^\vee \cdots x_4^\vee \in \text{HH}^3$  contributes whenever  $\gamma$  has no fixed variables; there are precisely  $6k + 2$  such elements  $\gamma$ .  $\square$

### 3.3 More $cA_\ell$ examples

By [23, Theorem 1.1], any  $cA_\ell$  singularity with a small resolution is given by an equation  $x_1^2 + x_2^2 + f(x_3, x_4) = 0$  where germ of the plane curve  $f = 0$  at the origin has  $\ell + 1$  distinct smooth branches, and conversely, any such singularity admits a small resolution (the converse was also proved in [15, p. 676]).

Let

$$\mathbf{w} = \check{\mathbf{w}} = x_1^2 + x_2^2 + x_3 x_4 (x_3^{\ell-1} + x_4^{k(\ell-1)}).$$

The singularity  $\check{\mathbf{w}} = 0$  is of type  $cA_\ell$ : the  $x_3 = x_4$  slice has an  $A_\ell$  singularity at the origin. The curve  $x_3 x_4 (x_3^{\ell-1} + x_4^{k(\ell-1)}) = 0$  has multiplicity  $\ell + 1$  and  $\ell + 1$  distinct branches at the origin:

$$x_3 = 0, \quad x_4 = 0, \quad \text{and} \quad x_3 + \mu x_4^k = 0 \text{ for } \mu^{\ell-1} = -1.$$

Therefore, this singularity admits a small resolution.

**Lemma 3.11.** *There exists a surjective 2-to-1 homomorphism  $T: \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_{2(\ell-1)} \times \mathbb{C}^\times \rightarrow \Gamma_{\mathbf{w}}$  which we will construct in the proof. The composition  $\chi \circ T$  is given by  $(\pm 1, \sigma, \tau) \mapsto \sigma^2 \tau^{2k\ell+2}$ .*

*Proof.* The group  $\Gamma_{\mathbf{w}}$  is defined by the equations

$$t_0 t_1 t_2 t_3 t_4 = t_1^2 = t_2^2 = t_3^\ell t_4 = t_3 t_4^{k(\ell-1)+1},$$

which imply  $t_3^{\ell-1} = t_4^{k(\ell-1)}$ , so  $t_3 = \xi t_4^k$  for some  $\xi$  with  $\xi^{\ell-1} = 1$ . Substituting back, we get

$$t_1^2 = t_2^2 = \xi t_4^{k\ell+1}.$$

Pick a square root  $\sigma$  for  $\xi$  and a square root  $\tau$  for  $t_4$  such that  $t_1 = \sigma\tau^{k\ell+1}$ ; then  $t_2 = \pm\sigma\tau^{k\ell+1}$ ,  $t_3 = \sigma^2\tau^{2k}$ ,  $t_4 = \tau^2$ ,  $t_0 = \pm\sigma^{-2}\tau^{-2(k+1)}$ . This shows that the homomorphism

$$T(\pm 1, \sigma, \tau) = (\pm\sigma^{-2}\tau^{-2(k+1)}, \sigma\tau^{k\ell+1}, \pm\sigma\tau^{k\ell+1}, \sigma^2\tau^{2k}, \tau^2)$$

is surjective. To see that it is 2-to-1, note that its kernel consists of triples  $(1, \sigma, \tau)$  for which  $\tau^2 = 1$  (so  $\tau = \pm 1$ ) and  $\sigma = \tau^{-k\ell-1}$ . This has size 2.  $\square$

The kernel  $\ker(\chi \circ T)$  is the subgroup

$$\{(\pm 1, \sigma, \tau) \in \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_{2(\ell-1)} \times \boldsymbol{\mu}_{2(k\ell+1)(\ell-1)} : \tau^{2(k\ell+1)} = \sigma^{-2}\}.$$

The projection to  $\tau \in \boldsymbol{\mu}_{2(k\ell+1)(\ell-1)}$  is surjective and split by the map  $\tau \mapsto (1, \tau^{-k\ell-1}, \tau)$ ; its kernel consists of triples  $(\pm 1, \pm 1, 1)$ , so there is an isomorphism

$$\begin{aligned} \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_{2(k\ell+1)(\ell-1)} &\rightarrow \ker(\chi \circ T) \\ (s_1, s_2, \tau) &\mapsto (s_1, s_2\tau^{-(k\ell+1)}, \tau). \end{aligned}$$

We will work with elements of this group; since  $T$  is 2-to-1, this will mean that we overcount contributions to  $\text{HH}^*$  by a factor of 2. We now identify which combinations of fixed and unfixed variables are possible for  $\gamma \in \ker \chi$ .

**Lemma 3.12.** *The possible combinations of fixed variables are given in the table below, along with the number of elements  $\gamma \in \ker \chi$  which give rise to these fixed variables.*

Fixed variables	$\#\gamma$
$\emptyset$	$k\ell(\ell-1)$
$\{0\}$	$\ell-2$
$\{0, 3, 4\}$	1
$\{0, 1, 2\}$	$\ell-2$
$\{0, 1, 2, 3, 4\}$	1
<i>The following cases occur, but do not contribute to <math>\text{HH}^*</math>:</i>	
$\{1\}, \{2\}, \{1, 2\}, \{0, 1\}, \{0, 2\}, \{1, 3, 4\}, \{2, 3, 4\}$	

*Proof.* Let  $\gamma = T(s_1, s_2\tau^{-(k\ell+1)}, \tau)$  with  $s_1, s_2 \in \{1, -1\}$  and  $\tau \in \boldsymbol{\mu}_{2(k\ell+1)(\ell-1)}$ .

If  $x_0$  is fixed then  $s_1\tau^{2(k\ell+1)-2k-2} = 1$ , so  $\tau^{2(\ell-1)k} = s_1$ . Since  $\tau^{2(\ell-1)(k\ell+1)} = 1$  this implies  $\tau^{2(\ell-1)} = s_1^\ell$ , and therefore  $s_1^{\ell(k\ell+1)} = 1$ . If  $s_1 = 1$  then this always holds. If  $s_1 = -1$  then this holds if and only if  $\ell(k\ell+1)$  is even. Therefore the element which fix  $x_0$  are those of



the form  $T(1, s_2\tau^{-(k\ell+1)}, \tau)$  with  $\tau^{2(\ell-1)} = 1$  and (if  $\ell(k\ell + 1)$  is even)  $T(-1, s_2\tau^{-(k\ell+1)}, \tau)$  with  $\tau^{2(\ell-1)} = (-1)^\ell$ .

$x_1$  is fixed if and only if  $\sigma\tau^{k\ell+1} = s_2 = 1$ .

$x_2$  is fixed if and only if  $s_1\sigma\tau^{k\ell+1} = s_1s_2 = 1$ , that is  $s_1 = s_2$ .

$x_4$  is fixed if and only if  $\tau^2 = 1$ . That is  $\tau = \pm 1$ .

$x_3$  is fixed if and only if  $\sigma^2\tau^{2k} = \tau^{-2k\ell-2+2k} = \tau^{-2(k(\ell-1)+1)} = 1$ . Note that

$$\gcd(k\ell + 1, k(\ell - 1) + 1) = 1 \quad \text{and} \quad \gcd(n - 1, k(\ell - 1) + 1) = 1,$$

so the only way we can simultaneously solve  $\tau^{2(k\ell+1)(\ell-1)} = 1$  and  $\tau^{-2(k(\ell-1)+1)} = 1$  is if  $\tau^2 = 1$ . This means that  $x_3$  is fixed if and only if  $\tau = \pm 1$  (if and only if  $x_4$  is also fixed).

The following table enumerates the possibilities for combinations of fixed variables and the counts<sup>5</sup> of  $\gamma \in \ker \chi$  which fix this combination of variables (omitting any which turn out not to contribute to  $\text{HH}^*$ ).

$s_1$	$s_2$	$\tau^2$	$\tau^{2(\ell-1)}$	fixed variables	$\#\gamma$
1	1	1	1	0 1 2 3 4	1
		$\neq 1$	1	0 1 2	$\ell - 2$
			$\neq 1$	1 2	
1	-1	1	1	0 3 4	1
		$\neq 1$	1	0	$\ell - 2$
			$\neq 1$		$k\ell(\ell - 1)$
-1	1	1	1	0 1 3 4	
		$\neq 1$	1	0 1	
			$(-1)^\ell$	0 1	
			else	1	
-1	-1	1	1	0 2 3 4	
		$\neq 1$	1	0 2	
			$(-1)^\ell$	0 2	
			else	2	

□

We pick the monomial basis  $x_3^a x_4^b$ ,  $0 \leq a \leq \ell - 1$ ,  $0 \leq b \leq k(\ell - 1) - 1$  for the Jacobian ring of  $\mathbf{w}|_{x_{j_1} = \dots = x_{j_{\ell-k}} = 0}$  when  $x_3$  and  $x_4$  are fixed and the monomial basis 1 when they are not.

<sup>5</sup>Recall that if we count elements of  $\ker(\chi \circ T)$  then we overcount by a factor of 2. We have removed this factor of 2 in the table.

**Theorem 3.13.** *If  $\mathbf{w} = x_1^2 + x_2^2 + x_3x_4(x_3^{\ell-1} + x_4^{k(\ell-1)})$  then  $\mathrm{HH}^*(\mathbb{A}^5, \Gamma_{\mathbf{w}}, \mathbf{w})$  satisfies*

$$\dim \mathrm{HH}^3 = (k\ell + 1)(\ell - 1), \quad \dim \mathrm{HH}^d = \ell \text{ for } d \leq 1$$

and  $\dim \mathrm{HH}^d = 0$  for  $d = 2$  and  $d \geq 4$ .

The  $\mathrm{HH}^*$  contributions for these singularities are given by the following table.

Monomial	Type	Degree in $\mathrm{HH}^*$	Number of contributions
$x_0^\vee x_1^\vee x_2^\vee x_3^\vee x_4^\vee$	$C$	3	$k\ell(\ell - 1)$
$x_0^\vee x_1^\vee x_2^\vee x_3^\vee x_4^\vee$	$B$	3	$\ell - 1$
$x_0^{(k\ell+1)p+q\ell+r} v x_3^r x_4^{q(\ell-1)+r}$	$A$	$-2(k+1)p - 2q$	$\begin{cases} 0 \leq q \leq k-1 \\ 0 \leq r \leq \ell-1 \\ p \geq 0 \end{cases}$
$x_0^{(k\ell+1)p+k\ell} v x_3^{\ell-1}$	$A$	$-2(k+1)p - 2k$	$p \geq 0$
$x_0^{(k\ell+1)p+k\ell} v x_4^{k(\ell-1)}$	$A$	$-2(k+1)p - 2k$	$p \geq 0$
$x_0^{(k\ell+1)p+k\ell} v x_3^\vee x_4^\vee$	$A$	$-2(k+1)p - 2k$	$\ell - 2, p \geq 0$
<i>Type B contributions in <math>\mathrm{HH}^{d+1}</math> for each type A monomial contributing to <math>\mathrm{HH}^d</math></i>			

In this table, we have written

$$v = \begin{cases} 1 & \text{if } b_0 = 0 \pmod{2}, \\ x_1^\vee x_2^\vee & \text{if } b_0 = 1 \pmod{2}. \end{cases}$$

*Proof.* For each  $\gamma$ -monomial  $m$ , let  $b_0, \dots, b_4$  be the total exponents of  $x_0, \dots, x_4$  in  $m$ . This monomial transforms under  $T(\pm 1, \sigma, \tau)$  as

$$(\pm 1)^{b_0+b_2} \sigma^{-2b_0+b_1+b_2+2b_3} \tau^{-2b_0(k+1)+(k\ell+1)(b_1+b_2)+2kb_3+2b_4},$$

which agrees with  $(\chi \circ T)^{\otimes u}(\pm 1, \sigma, \tau)$  if and only if

$$b_0 = b_2 \pmod{2}, \tag{3.1}$$

$$b_1 + b_2 + 2b_3 = 2b_0 + 2u \pmod{2(\ell - 1)}, \tag{3.2}$$

$$(k\ell + 1)(b_1 + b_2) + 2kb_3 + 2b_4 = 2(k\ell + 1)u + 2b_0(k + 1). \tag{3.3}$$

Reducing Equation (3.2) modulo 2 tells us that  $b_1 = b_2 \pmod{2}$ . For  $i = 1, 2$ , the only possibilities for  $b_i$  are 0 (if  $x_i$  is fixed) or  $-1$  (if  $x_i$  is not fixed). Thus, if  $(\gamma, m)$  contributes to  $\mathrm{HH}^*$  then either  $x_1$  and  $x_2$  are both fixed or neither is fixed. This immediately rules out the contributions from  $\gamma$  with fixed variables  $\{1\}$ ,  $\{2\}$ ,  $\{0, 1\}$ ,  $\{0, 2\}$ ,  $\{1, 3, 4\}$ , and  $\{2, 3, 4\}$ . Moreover, if  $x_2$  is fixed then  $b_2 = 0$  so  $b_0 = 0$  so  $x_0$  must also be fixed (or else we would have  $b_0 = -1$ ). This rules out contributions with fixed variables  $\{1, 2\}$ .

We now dispose of the type  $C$  contributions. These come from  $(\gamma, x_0^\vee x_1^\vee x_2^\vee x_3^\vee x_4^\vee) \in \mathrm{HH}^3$  where  $\gamma$  fixes no variables; there are  $k\ell(\ell - 1)$  of these.

Since  $b_1 = b_2 \pmod{2}$ , Equations (3.2) and (3.3) become

$$b_3 = b_0 + u - b_1 \pmod{\ell - 1} \quad (3.4)$$

$$kb_3 + b_4 = k(\ell - 1)(u - b_1) + (k + 1)(b_0 + u - b_1). \quad (3.5)$$

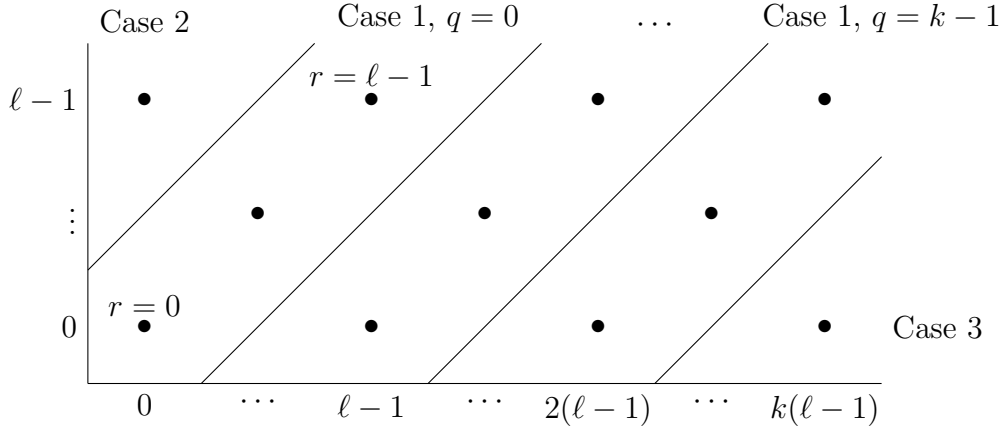
Reducing (3.5) modulo  $\ell - 1$  yields

$$b_4 = u - b_1 + b_0 = b_3 \pmod{\ell - 1}.$$

We distinguish the following cases:

1.  $b_3 = r, b_4 = q(\ell - 1) + r$  with  $q = 0, 1, \dots, k - 1$  and  $r = 0, 1, \dots, \ell - 1$ .
2.  $b_3 = \ell - 1, b_4 = 0$
3.  $b_3 = 0, b_4 = k(\ell - 1)$
4.  $b_3 = b_4 = -1$ .

We illustrate Cases 1–3 in the diagram below for  $\ell = 2, k = 3$ :



In what follows, we let  $d = \gcd(\ell - 1, k + 1) = \gcd(k + 1, k\ell + 1) = \gcd(\ell - 1, k\ell + 1)$  and define  $x, y, z$  by

$$k + 1 = dx, \quad k\ell + 1 = dy, \quad \ell - 1 = dz.$$

We will focus on type *A* contributions (there will be corresponding type *B* contributions obtained by multiplying with  $x_0 x_0^\vee$ ).

In Case 1, Equation (3.5) becomes

$$(\ell - 1)q + (k + 1)r = k(\ell - 1)(u - b_1) + (k + 1)(b_0 + u - b_1),$$

so  $k(u - b_1) = q - sx, b_0 + u - b_1 = r + sz$  for some integer  $s$ . Equation (3.4) tells us that  $r + sz = r \pmod{\ell - 1}$ , so  $s = dP$  for some integer  $P$ . If we write  $P = kp + q$  for some  $p$ , we get

$$u - b_1 = -((k + 1)p + q), \quad b_0 = q\ell + r + (k\ell + 1)p.$$

This yields a contribution of

$$x_0^{(k\ell+1)p+q\ell+r} v x_3^r x_4^{q(\ell-1)+r} \in \mathrm{HH}^{-2((k+1)p+q)} \quad \text{where } v = \begin{cases} 1 & \text{if } b_0 = 0 \pmod{2}, \\ x_1^\vee x_2^\vee & \text{if } b_0 = 1 \pmod{2}. \end{cases}$$

In each case, there is precisely one  $\gamma$  contributing this monomial (according to whether the fixed variables are  $\{0, 1, 2, 3, 4\}$  or  $\{0, 3, 4\}$ ). There are  $\ell$  contributions in each degree (as  $r$  varies) and we get every degree congruent to  $-2q \pmod{2(k+1)}$  for  $q = 0, 1, \dots, k-1$ , that is,  $\mathrm{HH}^d$  has rank  $\ell$  for every even  $d \neq -2k \pmod{2(k+1)}$ ,  $d \leq 0$ . The corresponding type  $B$  contributions give  $\dim \mathrm{HH}^d = \ell$  for every odd  $d \neq 1 - 2k \pmod{2(k+1)}$ ,  $d \leq 1$ .

In Cases 2 and 3,  $kb_3 + b_4 = k(\ell - 1)$ , so

$$k(\ell - 1) = k(\ell - 1)(u - b_1) + (k + 1)(b_0 + u - b_1),$$

which implies

$$k(u - b_1) = k - sx, \quad b_0 + u - b_1 = sz$$

for some  $s$ . As before, Equation (3.4) implies  $s = dP$ , so  $k(u - b_1) = k - P(k + 1)$ . This means  $P = kp$  for some  $p$ , so  $u - b_1 = 1 - (k + 1)p$  and  $b_0 = (k\ell + 1)p - 1$ . Thus we get contributions

$$x_0^{(k\ell+1)p-1} vw \in \mathrm{HH}^{-2(k+1)p+2} \quad \text{where } w \in \{x_3^{\ell-1}, x_4^{k(\ell-1)}\} \quad \text{and } v = \begin{cases} 1 & \text{if } b_0 = 0 \pmod{2}, \\ x_1^\vee x_2^\vee & \text{if } b_0 = 1 \pmod{2}. \end{cases}$$

In both cases there is precisely one  $\gamma$  contributing this monomial (according to whether the fixed variables are  $\{0, 1, 2, 3, 4\}$  or  $\{0, 3, 4\}$ ). This gives two contributions in every even degree  $d = 2 \pmod{2(k+1)}$ ,  $d \leq -2k$ .

Finally, in Case 4 we have  $kb_3 + b_4 = -(k + 1)$ , which yields

$$u - b_1 = -(k + 1)p, \quad b_0 = (k\ell + 1)p - 1,$$

and we get a contribution

$$x_0^{(k\ell+1)p-1} v x_3^\vee x_4^\vee \in \mathrm{HH}^{-2(k+1)p+2} \quad \text{where } v = \begin{cases} 1 & \text{if } b_0 = 0 \pmod{2}, \\ x_1^\vee x_2^\vee & \text{if } b_0 = 1 \pmod{2}. \end{cases}$$

In both cases, there are  $\ell - 2$  elements  $\gamma$  contributing these monomials (according to whether the fixed variables are  $\{0, 1, 2\}$  or  $\{0\}$ ). Together with the contributions from Cases 2 and 3, this yields  $\dim \mathrm{HH}^d = \ell$  for every even  $d = 2 \pmod{2(k+1)}$ ,  $d \leq -2k$ . The corresponding type  $B$  contributions give  $\dim \mathrm{HH}^d = \ell$  in every odd degree  $d = 3 \pmod{2(k+1)}$ ,  $d \leq 3$ .

Altogether, we get  $\dim \mathrm{HH}^d = \ell$  if  $d \leq 0$  and  $\dim \mathrm{HH}^3 = (k\ell + 1)(\ell - 1)$ .  $\square$

## 4 Bigrading

### 4.1 Scale-equivalence of bigradings

In this section, we need to work over  $\mathbb{C}$  (or at least an algebraically closed field of characteristic zero).

**Definition 4.1.** A  $\mathbb{Z} \times \mathbb{C}$ -grading on a vector space  $V$  (or *bigrading* for short) is a decomposition

$$V = \bigoplus_{(p,q) \in \mathbb{Z} \times \mathbb{C}} V^{p,q}.$$

Two  $\mathbb{Z} \times \mathbb{C}$ -graded vector spaces  $V = \bigoplus V^{p,q}$  and  $W = \bigoplus W^{p,q}$  are *scale-equivalent* if there is a nonzero  $c \in \mathbb{C}$  such that  $\dim(V^{p,q}) = \dim(W^{p,cq})$  for all  $p, q$ .

Our contact invariant will be a scale-equivalence class of  $\mathbb{Z} \times \mathbb{C}$ -graded vector spaces (in fact, we will be able to find a representative which takes values in  $\mathbb{Z} \times \mathbb{Z}$ ). We now explain how to construct a  $\mathbb{Z} \times \mathbb{C}$ -graded vector spaces out of a certain class of Gerstenhaber algebras.

### 4.2 Bigradings from Gerstenhaber algebras

Let  $\mathfrak{g}^*$  be a Gerstenhaber algebra over  $\mathbb{C}$ ; in particular, there is a Gerstenhaber bracket  $[\cdot, \cdot]$  on  $\mathfrak{g}^*$  satisfying:

$$[x, y] = (-1)^{|x||y|}[y, x], \quad (-1)^{|x||z|}[[x, y], z] + (-1)^{|y||x|}[[y, z], x] + (-1)^{|z||y|}[[z, x], y] = 0$$

The subset  $\mathfrak{g}^1 \subset \mathfrak{g}^*$  is a complex Lie algebra and the bracket gives a representation  $\rho^d: \mathfrak{g}^1 \rightarrow \mathfrak{gl}(\mathfrak{g}^d)$  for each  $d$ .

Let  $\mathfrak{h} \subset \mathfrak{g}^1$  be a Cartan subalgebra, that is a nilpotent, self-normalising subalgebra. A Cartan subalgebra exists and is unique up to automorphisms of  $\mathfrak{g}^1$ ; for example, you can construct one by taking the generalised 0-eigenspace of a regular element (an element  $\zeta \in \mathfrak{g}^1$  is regular if the generalised 0-eigenspace of  $\text{ad}_\zeta$  has the least possible dimension). If  $\rho: \mathfrak{g}^1 \rightarrow \mathfrak{gl}(V)$  is a finite-dimensional complex representation then we get a weight-space decomposition  $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$  where

$$V^\alpha := \{v \in V : (\rho(H) - \alpha(H))^N v = 0 \text{ for some } N\}.$$

In other words,  $V^\alpha$  is a simultaneous generalised eigenspace for  $\{\rho(H) : H \in \mathfrak{h}\}$ , with eigenvalues  $\alpha(H)$ . The weight-space decomposition  $\mathfrak{g}^1 = \bigoplus_\alpha \mathfrak{g}^{1,\alpha}$  of the adjoint representation has  $\mathfrak{h} = \mathfrak{g}^{1,0}$ .

If  $\mathfrak{h}$  has rank 1 then we have  $\mathfrak{h}^* \cong \mathbb{C}$ . If we pick such an identification then the weight-space decomposition gives us a  $\mathbb{Z} \times \mathbb{C}$ -bigrading  $\mathfrak{g}^* = \bigoplus_{p,q} \mathfrak{g}^{p,q}$ . Changing our identification  $\mathfrak{h}^* \cong \mathbb{C}$  yields a scale-equivalent  $\mathbb{Z} \times \mathbb{C}$ -grading.

**Example 4.2.** Let  $A^*$  be a  $\mathbb{Z}$ -graded associative algebra and suppose that its Hochschild cohomology  $\mathrm{HH}^*(A, A)$  has finite dimension in each degree. The Hochschild cochains can be given an additional  $\mathbb{Z}$ -grading so that a graded multilinear map  $A^{\otimes p} \rightarrow A[-q]$  contributes to  $\mathrm{HH}^{p,q}(A, A) \subset \mathrm{HH}^{p+q}(A, A)$ . This  $\mathbb{Z} \times \mathbb{Z}$ -bigrading fits into our setting above. We write  $\mathrm{HH}^*(A, A) \cong \bigoplus_{p,q} \mathrm{HH}^{p,q}(A, A)$ . There is an element  $\mathbf{eu} \in \mathrm{CC}^{1,0}(A, A)$  defined on the graded piece  $A^q$  by  $\mathbf{eu}(a) = qa$ . This is a Hochschild cocycle and defines a class (which we also write as  $\mathbf{eu}$ ) in  $\mathrm{HH}^{1,0}(A, A)$ . This satisfies  $[\mathbf{eu}, c] = qc$  for  $c \in \mathrm{CC}^{*,q}(A, A)$ . In particular, the generalised<sup>6</sup> 0-eigenspace of  $\mathrm{ad}_{\mathbf{eu}}$  is  $\mathrm{HH}^{1,0}(A, A)$ . If  $\mathbf{eu}$  is a regular element of the Lie algebra  $\mathrm{HH}^1(A, A)$  then  $\mathrm{HH}^{1,0}(A, A)$  is a Cartan subalgebra. In particular, if  $\dim(\mathrm{HH}^{1,0}(A, A)) = 1$  then  $\mathbf{eu}$  is necessarily regular and we can take  $\mathfrak{h} = \mathrm{HH}^{1,0}(A, A)$ . In this case, if we identify  $\mathfrak{h}^*$  with  $\mathbb{C}$  by sending  $\mathbf{eu}^*$  to 1 then the weight decomposition gives us the usual bigrading  $\mathrm{HH}^*(A, A) \cong \bigoplus_{p,q} \mathrm{HH}^{p,q}(A, A)$ .

### 4.3 Bigradings on symplectic cohomology

If  $V$  is a Liouville domain with  $c_1(V) = 0$ , the symplectic cohomology  $\mathrm{SH}^*(V)$  is a Gerstenhaber algebra. We will sketch how the bracket is defined; for more detail, see [40, Section 4] or [1, Section 2.5.1]. The bracket  $[x, y]$  is defined by

$$[x, y] = \bigoplus_z (\#\mathcal{M}(z; x, y, H, J)) z,$$

where  $\mathcal{M}(z; x, y, J)$  is the moduli space of solutions  $u: \Sigma \rightarrow \hat{V}$  to Floer's equation

$$(du + X_H \otimes \beta)^{0,1} = 0$$

where:

- $\hat{V}$  is the symplectic completion of  $V$ ;
- $\Sigma$  is a pair-of-pants  $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ , where we consider  $0, 1$  to be positive punctures and  $\infty$  as a negative puncture;
- we equip  $\Sigma$  with a 1-parameter family of positive/negative cylindrical ends, specified by asymptotic markers which rotate once for each puncture. As the parameter varies from  $0$  to  $2\pi$ , the markers at the positive punctures rotate once clockwise and the marker at the negative puncture rotates once anticlockwise;
- $\beta$  is a subclosed 1-form on  $\Sigma$  compatible with the cylindrical ends;
- $u$  has asymptotes  $x, y, z$  respectively at the punctures  $0, 1, \infty$ .

The bracket has degree  $-1$ , that is

$$|x| + |y| = |z| + 1,$$

---

<sup>6</sup>Since  $\mathrm{ad}_{\mathbf{eu}}$  is semisimple on the level of cochains, it remains semisimple in its action on cohomology, so generalised eigenspaces are actual eigenspaces.

where the degree is related to the Conley-Zehnder index by  $|x| = n - \mu_{CZ}(x)$ . Equivalently,

$$n = \mu_{CZ}(x) + \mu_{CZ}(y) - \mu_{CZ}(z) + 1.$$

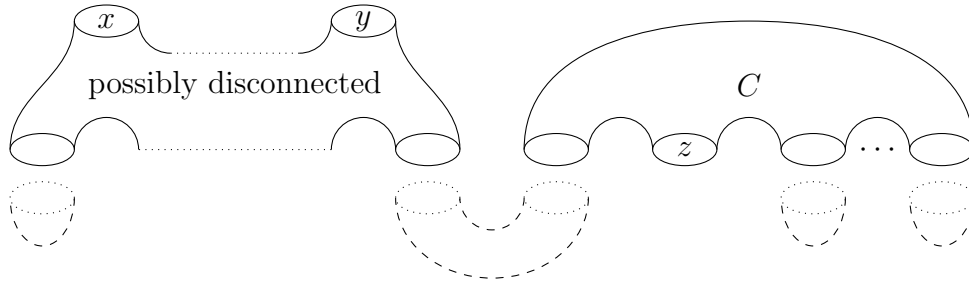
**Lemma 4.3.** *Let  $V$  be a  $2n$ -dimensional Liouville domain with simply-connected boundary and suppose that there is a contact form on  $Y = \partial V$  such that every Reeb orbit  $\gamma$  on  $Y$  satisfies the inequality*

$$\mu_{CZ}(\gamma) \geq \max(5 - n, n - 1).$$

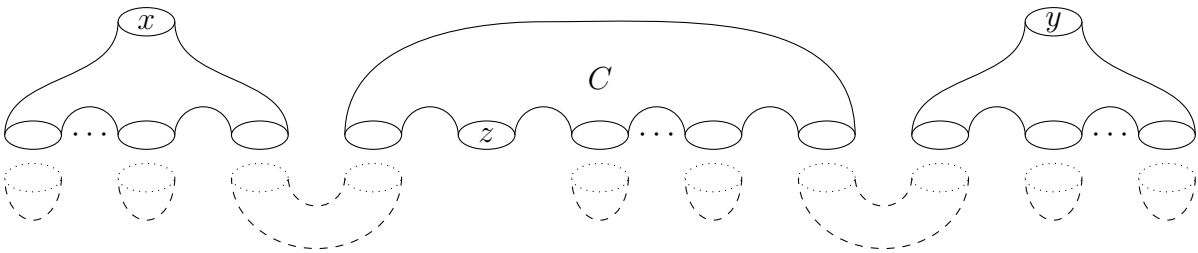
*If  $x, y, z$  are Reeb orbits then there exists a  $J$  such that any  $u \in \mathcal{M}(z; x, y, H, J)$  avoids the interior of  $V$ , that is, every  $u$  stays in the cylindrical end  $\hat{V} \setminus V$ .*

*Proof.* Suppose this is not true. Pick a neck-stretching sequence of almost complex structures  $J_k$  around  $Y$  and a sequence of curves  $u_k \in \mathcal{M}(z; x, y, H, J_k)$  which enter the interior of  $V$ . By the SFT compactness theorem, we can find a convergent subsequence which breaks into levels. There are several cases we need to consider.

Case 1: A break occurs along a separating curve parallel to  $z$  (and possibly other curves).



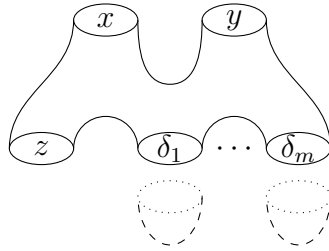
Case 2: Not case 1, but a break occurs along separating curves parallel to  $x$  and to  $y$ .



Case  $3_x$ : Not cases 1–2, but a break occurs along a separating curve parallel to  $x$  (Case  $3_y$  similar).



Case 4: Not cases 1–3, but a break occurs along a contractible loop.



In Cases 1–2, we are left with a component  $C$  which violates the maximum principle (see also the argument from Bourgeois-Oancea [7, Proof of Proposition 5, Step 1] or an alternative argument based on action from Cieliebak-Oancea [12, Proof of Proposition 9.17]).

The argument for Case  $3_x$  is inspired by [12, Appendix A] and [43, Lemma 3.13]. In this case, there are at least two components  $C_1$  and  $C_2$  in the SFT limit, where  $C_1$  has  $x$  as a positive asymptote and  $C_2$  has  $y$  as a positive asymptote. The component  $C_2$  has a negative asymptote at  $z$ , a negative asymptote  $\delta_0$  which connects through lower levels to the component  $C_1$ , and possibly further negative asymptotes  $\delta_1, \dots, \delta_m$ , which are capped off by planes in other levels. The index of  $C_2$  is (we justify this in Remark 4.4 below):

$$\mu_{CZ}(y) - \mu_{CZ}(z) + 1 - \sum_{i=0}^m (\mu_{CZ}(\delta_i) + n - 3). \quad (4.1)$$

We have

$$\mu_{CZ}(y) - \mu_{CZ}(z) + 1 = n - \mu_{CZ}(x) \leq 1$$

because  $\mu_{CZ}(x) \geq n - 1$  by assumption. Moreover  $\mu_{CZ}(\delta_i) + n - 3 \geq 2$  by assumption, so  $\mu_{CZ}(y) - \mu_{CZ}(z) + 1 - \sum_{i=0}^m \mu_{CZ}(\delta_i) \leq 1 - 2 = -1$ , which contradicts the regularity of  $C_2$ .

The argument for Case  $3_y$  is the same as for Case  $3_x$  with the roles of  $x$  and  $y$  interchanged.

Case 4 yields a regular component  $C$  in the SFT limit which has punctures asymptotic to  $x$ ,  $y$  and  $z$  as well as further negative punctures with asymptotes  $\delta_1, \dots, \delta_m$ . The index of  $C$  is equal to the index of the original moduli space minus  $\sum_{i=1}^m (\mu_{CZ}(\delta_i) + n - 3) \geq 2$ , so becomes negative. This is a contradiction.  $\square$



**Remark 4.4.** We now explain the index formula (4.1) from the proof. If we fix the positions of the punctures and all the asymptotic markers, the virtual dimension of this moduli space is (see Schwarz's thesis [37, Theorem 3.3.11]):

$$\mu_{CZ}(y) - \mu_{CZ}(z) - \sum_{i=0}^m \mu_{CZ}(\delta_i) - n(m+1)$$

since  $-m-1$  is the Euler characteristic of the domain. However, the bubbling/breaking which gives rise to the punctures at  $\delta_i$  can happen anywhere, with any asymptotic marker, and the asymptotic markers on  $y$  and  $z$  can move in a 1-parameter family, so we get an additional  $3(m+1)+1$ , which gives Equation (4.1). Note that this is intermediate between the formula in Schwarz's thesis and the formula [5, Corollary 5.4] from Bourgeois's thesis, where all punctures and markers are allowed to move.

**Corollary 4.5.** *Suppose that  $(Y, \xi)$  is a  $(2n-1)$ -dimensional contact manifold which admits a contact form  $\alpha$  for which every closed Reeb orbit  $\gamma$  satisfies*

$$\mu_{CZ}(\gamma) \geq \max(5-n, n-1).$$

*Let  $V_1, V_2$  be Liouville domains with  $c_1(V_i) = 0$  and  $\partial V_i = Y$ . Suppose that  $V_i$  admits a Morse function with no critical points of index 1. Then (a) there is an isomorphism of Lie algebras  $f^1: \text{SH}^1(V_1) \rightarrow \text{SH}^1(V_2)$ , and (b) for each  $d < 0$  there is an isomorphism  $f^d: \text{SH}^d(V_1) \rightarrow \text{SH}^d(V_2)$  which intertwines the representations*

$$\text{ad}: \text{SH}^1(V_i) \rightarrow \mathfrak{gl}(\text{SH}^d(V_i)).$$

*That is, for each  $d < 0$ , we have a commutative diagram:*

$$\begin{array}{ccc} \text{SH}^1(V_1) & \xrightarrow{\text{ad}} & \mathfrak{gl}(\text{SH}^d(V_1)) \\ f^1 \downarrow & & \downarrow f^d \\ \text{SH}^1(V_2) & \xrightarrow{\text{ad}} & \mathfrak{gl}(\text{SH}^d(V_2)) \end{array}$$

*Proof.* Under the assumptions of the corollary, every element of  $\text{SH}^1(V_i)$  or of  $\text{SH}^d(V_i)$  with  $d < 0$  can be represented using Reeb orbits for the contact form  $\alpha$  (rather than critical points of a Morse function on the filling). These Reeb orbits lie in the cylindrical end of the symplectic completion  $\hat{V}_i$  (rather than in the filling), and these cylindrical ends are both symplectomorphic to the half-symplectisation  $[0, \infty) \times Y$ , so in a suitable cochain model of symplectic cohomology, we get identifications  $f^d: SC^d(V_1) \rightarrow SC^d(V_2)$  when  $d = 1$  or  $d < 0$ . Since the contact boundary is index-positive, we know that the differential on these cochain groups is independent of the filling [12, Prop. 9.17], which tells us that  $f^1$  and  $f^d$  induce isomorphisms on cohomology.

By Lemma 4.3, we know there exist almost complex structures for which the Gerstenhaber bracket between these orbits does not involve any contributions from curves entering the filling. This implies that  $f^1$  is an isomorphism of Lie algebras and that  $f^d$  intertwines the adjoint action of  $\mathrm{SH}^1$ .  $\square$

#### 4.4 Bigrading on $\mathrm{HH}^*(\mathrm{mf})$

In all our examples, we calculated  $\mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w}))$  and saw that  $\mathrm{HH}^2 = 0$ . Moreover, we saw in Lemma 2.8 that there is an intrinsically formal algebra  $B$  such that

$$\mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w})) = \mathrm{HH}^*(B).$$

We now compute the usual algebra bigrading on  $\mathrm{HH}^*(B)$  in terms of the  $\gamma$ -monomial contributions from Theorem 2.14.

**Lemma 4.6.** *A  $\gamma$ -monomial  $m$  contributing to  $\mathrm{HH}^d(B)$  contributes to the bigraded piece  $\mathrm{HH}^{d-nb_0, nb_0}(B)$ , where  $b_0$  is the total exponent of  $x_0$  in  $m$ .*

**Remark 4.7.** Recall that  $\mathrm{HH}^d = \bigoplus_q \mathrm{HH}^{d-q, q}$ , so this is really just saying that the bigrading of  $m$  is  $nb_0$ .

*Proof.* Consider the  $\mathbb{G}_m$ -action  $t \cdot (x_0, \dots, x_{n+1}) = (tx_0, \dots, x_{n+1})$ . Since this action leaves  $\mathbf{w}$  invariant, its weights give a second grading on  $\mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$ . Theorem 2.14 comes from an isomorphism between  $\mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$  and a suitable twisted Koszul cohomology group (whose generators are  $\gamma$ -monomials) [3]. This isomorphism respects the  $\mathbb{G}_m$ -action, hence this additional grading is given by the total exponent of  $x_0$  in the  $\gamma$ -monomials contributing to  $\mathrm{HH}$ .

As in Section 2.3, let  $\mathcal{S} = \bigoplus_i S_i$  be the generator of  $\mathcal{F}(V)$  given by a direct sum of vanishing cycles. In [28, Theorem 4.2] it is shown that, we have a generator  $\mathcal{S}$  of  $\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})$  mirror to  $\mathcal{S}$ , where  $\mathcal{S}$  is the pushforward of a generator  $\mathcal{E}$  of  $\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w})$  under the inclusion  $(x_1, \dots, x_{n+1}) \mapsto (0, x_1, \dots, x_{n+1})$ . In particular,  $\mathcal{S}$  is  $\mathbb{G}_m$ -invariant. Using this, [28, Theorem 4.2] shows that the endomorphism  $A_\infty$ -algebra  $\mathcal{B} = \mathrm{end}(\mathcal{S})$  is a formal algebra and the grading on  $B = \mathrm{H}(\mathcal{B})$  is  $n$  times the weight of the  $\mathbb{G}_m$ -action. Therefore, the  $\mathbb{G}_m$ -weight on  $B$  can be understood in terms of the grading of the algebra  $B$ . Indeed, we see it as the weight decomposition for  $\mathrm{ad}_b$  associated to the derivation  $b = n \cdot \mathbf{eu}$ , where  $\mathbf{eu} \in \mathrm{HH}^1$  is the Euler derivation from Example 4.2.  $\square$

In fact, in all of our examples we have  $\dim \mathrm{HH}^{1,0} = 1$ , which means, as in Example 4.2, that the weight decomposition for the representation  $\mathrm{ad}: \mathrm{HH}^1 \rightarrow \bigoplus_d \mathfrak{gl}(\mathrm{HH}^d)$  gives a  $\mathbb{Z} \times \mathbb{C}$ -bigrading which is scale-equivalent to the algebra bigrading, and hence to the bigrading by the total exponent of  $x_0$  by Lemma 4.6.

## 4.5 Proof of Theorem 1.13

Let  $X$  be a cDV singularity and let  $V$  be its Milnor fibre. Let  $\mu$  be the Milnor number of  $X$ . The Milnor fibre admits a Morse function with precisely one minimum and  $\mu$  critical points of index 3; in particular, none of index 1. Since  $X$  is terminal, McLean's theorem [31, Theorem 1.1] tells us that there exists a contact form for which every closed Reeb orbit  $\gamma$  satisfies  $\mu_{CZ}(\gamma) \geq 2 \text{md}(X) = 2$ , where  $\text{md}(X)$  is the minimal discrepancy of  $X$ , which equals 1 by a theorem of Markushevich [30]. Since  $n = 3$ , we have  $\max(5 - n, n - 1) = 2$ , so that all the assumptions of Corollary 4.5 are satisfied.

Consider the Lie algebra  $\text{SH}^1(V)$  and its representation  $\bigoplus_{d < 0} \text{SH}^d(V)$  (where  $\text{SH}^1(V)$  acts by the Gerstenhaber bracket). By Corollary 4.5, this Lie algebra representation is a contact invariant of the link of  $X$ .

We know that Conjecture 2.2 holds for all our Brieskorn-Pham and  $cA_n$  examples, and we are going to assume that it holds for the Laufer examples too. By Theorem 2.6, this tells us that, if  $V$  is the Milnor fibre of  $\check{\mathbf{w}}$ , then

$$\text{SH}^*(V) \cong \text{HH}^*(\text{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$$

as Gerstenhaber algebras. Therefore, the contact invariant Lie algebra representation is equivalent to the representation  $\text{ad}: \text{HH}^1(B) \rightarrow \bigoplus_{d < 0} \mathfrak{gl}(\text{HH}^d(B))$  discussed in Section 4.4. In particular, this gives a  $\mathbb{Z} \times \mathbb{C}$ -grading on  $\bigoplus_{d < 0} \text{SH}^d(V)$  which we can compute in terms of the  $x_0$ -powers of the contributing  $\gamma$ -monomials by Lemma 4.6.

We now show that, for all of our examples, these scale-equivalence classes of  $\mathbb{Z} \times \mathbb{C}$ -gradings distinguish the contact structures.

### 4.5.1 $\ell = 1$

In this case we need to distinguish the contact structures  $\{\alpha_{1,k} : k = 1, 2, 3, \dots\}$  and  $\{\lambda_{1,k} : k = 1, 2, 3, \dots\}$  on  $S^2 \times S^3$ .

The unique contribution to  $\text{HH}^{-2}$  is:

$$\begin{cases} x_0 x_1^\vee x_2^\vee x_3^\vee x_4^\vee & \text{for } \alpha_{1,1}, \\ x_0^2 x_4^2 & \text{for } \alpha_{1,k}, (k \geq 2), \end{cases} \quad x_0^4 x_1 x_2^3 \text{ for } \lambda_{1,k}.$$

To compare the  $\mathbb{Z} \times \mathbb{C}$ -gradings, we rescale to ensure  $\text{SH}^{-2,4} \neq 0$  in all cases. The  $\mathbb{C}$ -bigrading of a monomial  $x_0^{b_0} \cdots x_4^{b_4} \in \text{HH}^d$  is therefore given by:

$$\begin{cases} 4b_0 & \text{for } \alpha_{1,1}, \\ 2b_0 & \text{for } \alpha_{1,k}, (k \geq 2), \end{cases} \quad b_0 \text{ for } \lambda_{1,k}.$$

The unique contribution to  $\text{HH}^{-4}$  is:

$$\begin{cases} x_0^4 \in \text{SH}^{-4,16} & \text{for } \alpha_{1,1}, \\ x_0^3 x_1^\vee x_2^\vee x_3^\vee x_4^\vee \in \text{SH}^{-4,6} & \text{for } \alpha_{1,2}, \\ x_0^4 x_4^4 \in \text{SH}^{-4,8} & \text{for } \alpha_{1,k}, (k \geq 3), \end{cases} \quad x_0^6 x_2^4 \in \text{SH}^{-4,6} \text{ for } \lambda_{1,k}.$$

This already distinguishes  $\alpha_{1,1}$  from everything,  $\alpha_{1,2}$  from the other  $\alpha$ s, and the  $\lambda$ s from the  $\alpha_{1,k}$ ,  $k \neq 2$ .

To distinguish  $\lambda_{1,k}$  from  $\lambda_{1,K}$  with  $k < K$ , observe that the unique contribution to  $\text{SH}^{-4k-2}$  is  $x_0^{6k+2} x_1^2 \in \text{SH}^{-4k-2,6k+2}$  respectively  $x_0^{6k+4} x_1 x_2^{4k+3} \in \text{SH}^{-4k-2,6k+4}$ .

To distinguish  $\alpha_{1,k}$  from  $\alpha_{1,K}$  with  $2 \leq k < K$ , observe that the unique contribution to  $\text{SH}^{-2k}$  is  $x_0^{2k-1} x_1^\vee x_2^\vee x_3^\vee x_4^\vee \in \text{SH}^{-2k,4k-2}$  respectively  $x_0^{2k} x_4^{2k} \in \text{SH}^{-2k,4k}$ .

To distinguish  $\alpha_{1,2}$  from  $\lambda_{1,k}$ ,  $k \geq 2$ , observe that the unique contribution to  $\text{SH}^{-6}$  is  $x_0^4 \in \text{SH}^{-6,8}$  respectively  $x_0^{10} x_1 x_2^2 \in \text{SH}^{-6,10}$ .

To distinguish  $\alpha_{1,2}$  from  $\lambda_{1,1}$ , observe that the unique contribution to  $\text{SH}^{-8}$  is  $x_0^6 x_2^2 \in \text{SH}^{-8,12}$  respectively  $x_0^{10} \in \text{SH}^{-8,10}$ .

## 4.6 $\ell \geq 2$

The contact structures  $\xi_{\ell,k}$  in Theorem 1.13 live on the manifold  $\sharp_\ell(S^2 \times S^3)$ . We can see from the tables in Theorems 3.7, 3.10 and 3.13 that for any of  $\alpha_{\ell,k}$ ,  $\beta_{\ell,k}$ ,  $\delta_{4,k}$ ,  $\lambda_{\ell,k}$ ,  $\epsilon_{6,k}$ ,  $\epsilon_{8,k}$ , the symplectic cohomology  $\text{SH}^d$ ,  $d < 0$ , is supported in a single  $\mathbb{C}$ -bigrading if and only if  $d = -2k$  or  $1 - 2k \pmod{2(k+1)}$ . Therefore, the only possibility for two contact structures  $\xi_{\ell,k}$ ,  $\ell \geq 2$ , to agree is for the indices  $\ell$  and  $k$  to agree.

We also see that, if we bigrade by the total exponent of  $x_0$ ,  $\text{SH}^1$  is supported in bidegrees  $0, 1, 2, \dots, \ell - 1$ . This is enough to fix our  $\mathbb{Z} \times \mathbb{C}$ -grading completely up to scale so that, in all cases, the  $\mathbb{C}$ -bigrading coincides with the total exponent of  $x_0$ .

To distinguish  $\alpha_{\ell,k}$  from  $\beta_{\ell,k}$  when  $k \neq 1$  (the singularities are locally analytically equivalent when  $\ell = 1$ ), observe that the contributions to  $\text{SH}^{-2k}$  have total  $x_0$  exponent  $k\ell$  respectively  $k(\ell + 1) - 1$ . These are different if  $k \neq 1$ .

To distinguish  $\alpha_{4,k}$  and  $\beta_{4,k}$  from  $\delta_{4,k}$ , note that the contributions to  $\text{SH}^{-2k}$  have total  $x_0$  exponents respectively equal to  $5k - 1$ ,  $4k$  and  $6k - 1$ .

To distinguish  $\alpha_{6,k}$  and  $\beta_{6,k}$  from  $\epsilon_{6,k}$ , note that the contributions to  $\text{SH}^{-2k}$  have total  $x_0$  exponents respectively equal to  $7k - 1$ ,  $6k$  and  $12k - 1$ .

To distinguish  $\alpha_{8,k}$  and  $\beta_{8,k}$  from  $\epsilon_{8,k}$ , note that the contributions to  $\text{SH}^{-2k}$  have total  $x_0$  exponents respectively equal to  $9k - 1$ ,  $8k$  and  $30k - 1$ .

## References

- [1] M. Abouzaid. Symplectic cohomology and Viterbo’s theorem. In *Free loop spaces in geometry and topology*, volume 24 of *IRMA Lect. Math. Theor. Phys.*, pages 271–485. Eur. Math. Soc., Zürich, 2015.
- [2] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Volume 1*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2012. Classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous based on a previous translation by Mark Reynolds, Reprint of the 1985 edition.
- [3] M. Ballard, D. Favero, and L. Katzarkov. A category of kernels for equivariant factorizations and its implications for Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 120:1–111, 2014.
- [4] P. Berglund and T. Hübsch. A generalized construction of mirror manifolds. In *Essays on mirror manifolds*, pages 388–407. Int. Press, Hong Kong, 1992.
- [5] F. Bourgeois. *A Morse-Bott approach to contact homology*. ProQuest LLC, Ann Arbor, MI, 2002. Thesis (Ph.D.)–Stanford University.
- [6] F. Bourgeois, T. Ekholm, and Y. Eliashberg. Effect of Legendrian surgery. *Geom. Topol.*, 16(1):301–389, 2012. With an appendix by Sheel Ganatra and Maksim Maydanskiy.
- [7] F. Bourgeois and A. Oancea. An exact sequence for contact- and symplectic homology. *Invent. Math.*, 175(3):611–680, 2009.
- [8] E. Brieskorn. Beispiele zur Differentialtopologie von Singularitäten. *Invent. Math.*, 2:1–14, 1966.
- [9] E. Brieskorn. Die Auflösung der rationalen Singularitäten holomorpher Abbildungen. *Math. Ann.*, 178:255–270, 1968.
- [10] B. Chantraine, G. Dimitroglou Rizell, P. Ghiggini, and R. Golovko. Geometric generation of the wrapped Fukaya category of Weinstein manifolds and sectors. *arXiv:1712.09126*, 2017.
- [11] K. Cieliebak, A. Floer, and H. Hofer. Symplectic homology. II. A general construction. *Math. Z.*, 218(1):103–122, 1995.
- [12] K. Cieliebak and A. Oancea. Symplectic homology and the Eilenberg-Steenrod axioms. *Algebr. Geom. Topol.*, 18(4):1953–2130, 2018. Appendix written jointly with Peter Albers.
- [13] T. Ekholm and Y. Lekili. Duality between Lagrangian and Legendrian invariants. *arXiv:1701.01284*, 2017.

- [14] A. Floer and H. Hofer. Symplectic homology. I. Open sets in  $\mathbf{C}^n$ . *Math. Z.*, 215(1):37–88, 1994.
- [15] R. Friedman. Simultaneous resolution of threefold double points. *Math. Ann.*, 274(4):671–689, 1986.
- [16] M. Futaki and K. Ueda. Homological mirror symmetry for Brieskorn-Pham singularities. In *Proceeding of the 66th Japan Geometry Symposium*, pages 98–107. Saga University, 2009.
- [17] M. Futaki and K. Ueda. Homological mirror symmetry for Brieskorn-Pham singularities. *Selecta Math. (N.S.)*, 17(2):435–452, 2011.
- [18] M. Futaki and K. Ueda. Homological mirror symmetry for singularities of type D. *Math. Z.*, 273(3-4):633–652, 2013.
- [19] B. Gammage. Mirror symmetry for Berglund-Hübsch Milnor fibers. *arXiv:2010.15570*, 2020.
- [20] S. Ganatra. Symplectic Cohomology and Duality for the Wrapped Fukaya Category. *arXiv:1304.7312*, 2013.
- [21] M. Habermann and J. Smith. Homological Berglund-Hübsch mirror symmetry for curve singularities. *arXiv:1903.01351*, to appear in *J. Symp. Geom.*, 2019.
- [22] H. Hofer. Symplectic capacities. In *Geometry of low-dimensional manifolds, 2 (Durham, 1989)*, volume 151 of *London Math. Soc. Lecture Note Ser.*, pages 15–34. Cambridge Univ. Press, Cambridge, 1990.
- [23] S. Katz. Small resolutions of Gorenstein threefold singularities. In *Algebraic geometry: Sundance 1988*, volume 116 of *Contemp. Math.*, pages 61–70. Amer. Math. Soc., Providence, RI, 1991.
- [24] B. Keller. Derived invariance of higher structures on the Hochschild complex. *Preprint, 2003*.
- [25] B. Keller. Deformed Calabi-Yau completions. *J. Reine Angew. Math.*, 654:125–180, 2011. With an appendix by Michel Van den Bergh.
- [26] H. B. Laufer. On  $\mathbf{C}P^1$  as an exceptional set. In *Recent developments in several complex variables (Proc. Conf., Princeton Univ., Princeton, N. J., 1979)*, volume 100 of *Ann. of Math. Stud.*, pages 261–275. Princeton Univ. Press, Princeton, N.J., 1981.
- [27] Y. Lekili and J. Pascaleff. Floer cohomology of  $\mathfrak{g}$ -equivariant Lagrangian branes. *Compos. Math.*, 152(5):1071–1110, 2016.
- [28] Y. Lekili and K. Ueda. Homological mirror symmetry for Milnor fibers via moduli of  $A_\infty$ -structures. *arXiv:1806.04345*, 2018.

- [29] Y. Lekili and K. Ueda. Homological mirror symmetry for Milnor fibers of simple singularities. *arXiv:2004.07374 (to appear in Algebraic Geometry)*, 2020.
- [30] D. Markushevich. Minimal discrepancy for a terminal cDV singularity is 1. *J. Math. Sci. Univ. Tokyo*, 3(2):445–456, 1996.
- [31] M. McLean. Reeb orbits and the minimal discrepancy of an isolated singularity. *Invent. Math.*, 204(2):505–594, 2016.
- [32] S. Mori. On 3-dimensional terminal singularities. *Nagoya Math. J.*, 98:43–66, 1985.
- [33] D. Mumford. The topology of normal singularities of an algebraic surface and a criterion for simplicity. *Inst. Hautes Études Sci. Publ. Math.*, (9):5–22, 1961.
- [34] D. Orlov. Formal completions and idempotent completions of triangulated categories of singularities. *Adv. Math.*, 226(1):206–217, 2011.
- [35] H. C. Pinkham. Factorization of birational maps in dimension 3. In *Singularities, Part 2 (Arcata, Calif., 1981)*, volume 40 of *Proc. Sympos. Pure Math.*, pages 343–371. Amer. Math. Soc., Providence, RI, 1983.
- [36] M. Reid. Minimal models of canonical 3-folds. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, volume 1 of *Adv. Stud. Pure Math.*, pages 131–180. North-Holland, Amsterdam, 1983.
- [37] M. Schwarz. *Cohomology operations from  $S^1$ -cobordisms in Floer homology*. 1995. Thesis (Ph.D.)—ETH Zürich.
- [38] E. Segal. The  $A_\infty$  deformation theory of a point and the derived categories of local Calabi-Yaus. *J. Algebra*, 320(8):3232–3268, 2008.
- [39] P. Seidel. A biased view of symplectic cohomology. In *Current developments in mathematics, 2006*, pages 211–253. Int. Press, Somerville, MA, 2008.
- [40] P. Seidel. Disjoinable Lagrangian spheres and dilations. *Invent. Math.*, 197(2):299–359, 2014.
- [41] G. N. Tjurina. Resolution of singularities of flat deformations of double rational points. *Funkcional. Anal. i Priložen.*, 4(1):77–83, 1970.
- [42] P. Uebele. Symplectic homology of some Brieskorn manifolds. *Math. Z.*, 283(1-2):243–274, 2016.
- [43] P. Uebele. Periodic Reeb flows and products in symplectic homology. *J. Symplectic Geom.*, 17(4):1201–1250, 2019.
- [44] O. van Koert. Contact homology of Brieskorn manifolds. *Forum Math.*, 20(2):317–339, 2008.
- [45] A. N. Varčenko. Contact structures and isolated singularities. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, (2):18–21, 101, 1980.

- [46] C. Viterbo. Functors and computations in Floer homology with applications. I. *Geom. Funct. Anal.*, 9(5):985–1033, 1999.
- [47] C. T. C. Wall. Classification problems in differential topology. VI. Classification of  $(s - 1)$ -connected  $(2s + 1)$ -manifolds. *Topology*, 6:273–296, 1967.

J. D. Evans, UNIVERSITY OF LANCASTER

`j.d.evans@lancaster.ac.uk`

Y. Lekili, IMPERIAL COLLEGE LONDON

`y.lekili@imperial.ac.uk`