Equivariant Fukaya categories at singular values

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Equivariant cohomology

There is a contravariant functor from manifolds M with an S^1 action to graded vector spaces $M \to H^*_{S^1}(M)$ that satisfies the following properties:

i. If the action on M is free, then $H^*_{S^1}(M) = H^*(M/S^1)$.

ii. If $f : M_1 \to M_2$ is an equivariant map inducing a homotopy equivalence, then $f^* : H^*_{S^1}(M_2) \to H^*_{S^1}(M_1)$ is an isomorphism.

iii. If $M = U \cup V$ with U and V open invariant submanifolds of M, then there exists a long exact sequence

$$\to H^{*-1}_{S^1}(U \cap V) \to H^*_{S^1}(M) \to H^*_{S^1}(U) \oplus H^*_{S^1}(V) \to H^*_{S^1}(U \cap V) \to$$

Equivariant cohomology of a point $\bullet$

$$H^*_{S^1}(\bullet) = \mathbb{C}[t], \quad \deg(t) = 2$$

To see this we observe that the circle acts freely on $S^\infty = \{(z_0, z_1, \ldots) \in \mathbb{C}^\infty : |z_0|^2 + |z_1|^2 + \ldots = 1\}$ by $e^{i\theta} \cdot (z_0, z_1, \ldots,) = (e^{i\theta} z_0, e^{i\theta} z_1, \ldots)$.

The infinite sphere is equivariantly contractible to a point, so we get

$$H^*_{S^1}(\bullet) = H^*(S^\infty / S^1) = H^*(\mathbb{C}P^\infty)$$

Every manifold with an S^1 action has an S^1-equivariant map to the *pt*. Hence, $H_{S^1}(M)$ is in fact an $H^*_{S^1}(\bullet)$-module.

The 2-sphere

Consider S^2, we can cover it by $U = S^2 \setminus \{0\}$ and $V = S^2 \setminus \{\infty\}$. The Mayer-Vietoris sequence gives that

$$0 \to H^0_{S^1}(U \cap V) \to H^0_{S^1}(U) \oplus H^0_{S^1}(V) \to H^0_{S^1}(S^2) \to H^1_{S^1}(U \cap V) \to 0$$

and

$$H^i_{S^1}(S^2) \simeq H^i_{S^1}(U) \oplus H^i_{S^1}(V) \text{ for } \; i \geq 2.$$

It follows that

$$H^*_{S^1}(S^2) \simeq \mathbb{C}[x, y]/(xy), \; \deg(x) = \deg(y) = 2$$

The $\mathbb{C}[t]$-action is given by multiplication by $t = x + y$. Setting $t = 0$, we recover $H^*(S^2)$.

Hamiltonian S^1-action

$T^*S^2 = \{xy + z^2 = 1\} \subset \mathbb{C}^3$, exact symplectic manifold.

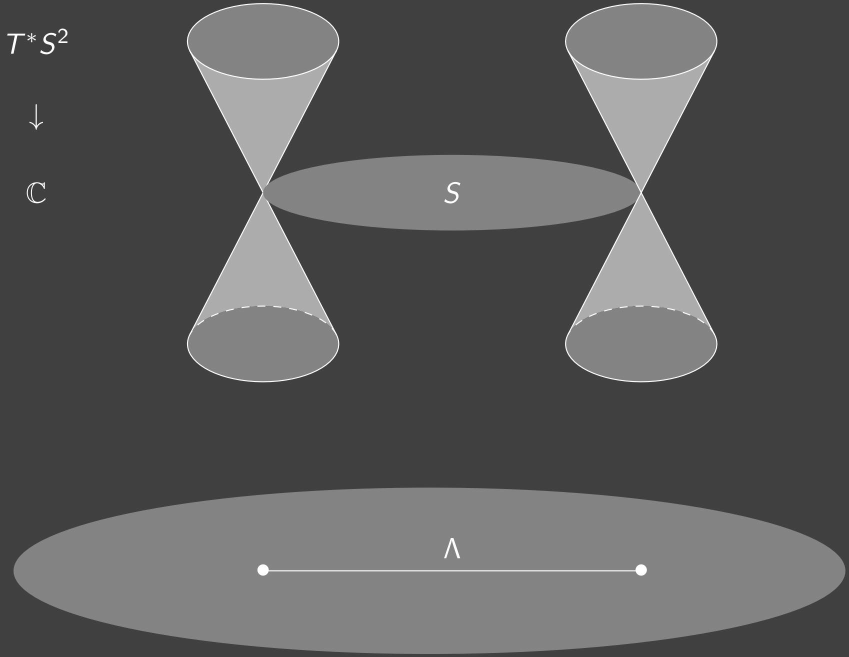
Hamiltonian S^1 action $e^{i\theta} \cdot (x, y, z) \to (e^{i\theta}x, e^{-i\theta}y, z)$

$$\mathcal{W}_{S^1}(T^*S^2)$$

S^1-equivariant wrapped Fukaya category

$S = S^2$ is an exact Lagrangian and fixed by the S^1 action.

$$HF^*_{S^1}(S^2, S^2) = H^*_{S^1}(S^2) = \mathbb{C}[x, y]/(xy), \quad \deg(x) = \deg(y) = 2$$


T^*S^2
$\downarrow$
$\mathbb{C}$
S
Λ

Key observation

Λ is a non-compact Lagrangian in $\mathcal{P} = \mathbb{C} \setminus \{1, -1\}$ pair-of-pants

$$HW^*(\Lambda, \Lambda) = \mathbb{C}[x, y]/xy$$

with $\deg(x) = \deg(y) = 2$ for a certain grading structure on $\mathcal{P}$.

This suggests that there is a quasi-equivalence of $\mathbb{Z}$-graded pre-triangulated categories:

$$\mathcal{W}(\mathcal{P}) \simeq \mathcal{W}_{S^1}(T^* S^2)$$

which we can prove, and we now formulate various generalisations.

Let $Y = \mathbb{C}^n$ or a more general Liouville manifold and $f : Y \to \mathbb{C}$ be a holomorphic map with 0 as a regular value.

Consider the conic fibration $\pi : X \to Y$ is defined on the smooth space

$$X = \{(u, v, \mathbf{w}) : uv = f(\mathbf{w})\}$$

as the restriction of the projection $(u, v, \mathbf{w}) \to \mathbf{w}$. The generic fiber of $\pi : X \to Y$ is isomorphic to a smooth affine conic and it degenerates to a singular conic along the smooth hypersurface $D = \{f(\mathbf{w}) = 0\}$.

The space X admits an Hamiltonian S^1 action given by rotating the fibers: $e^{i\theta} \cdot (u, v, \mathbf{w}) \to (e^{i\theta}u, e^{-i\theta}v, \mathbf{w})$ for $e^{i\theta} \in S^1$.

Conjecture A We have a quasi-equivalence

$$\mathcal{W}_{S^1}(X) \simeq \mathcal{W}(Y \setminus D)$$

More generally, we can consider $f_1, f_2, \dots, f_r : Y \to \mathbb{C}$ holomorphic maps with 0 as a regular value and that the hypersurfaces $\{f_j = 0\}$ intersect in a normal crossing way. Then we can form the smooth space

$$X = \{(u_1, v_1, \dots, u_r, v_r, \mathbf{w}) : u_i v_i = f_i(\mathbf{w}), \text{ for } i = 1, \dots, r\}$$

and the restriction of the projection $(u_1, v_1, \dots, u_r, v_r, \mathbf{w}) \to \mathbf{w}$ defines an iterated conic bundle $\pi : X \to Y$ of rank r whose generic fiber is a product of smooth conics, hence is isomorphic to $(\mathbb{C}^*)^r$.

We also have Hamiltonian action of an r-dimensional torus T on X by rotating the fibers.

Conjecture A We have a quasi-equivalence

$$\mathcal{W}_T(X) \simeq \mathcal{W}(Y \setminus D)$$

Triviality of the deformation

If we delete the divisor $\pi^{-1}D$ from X then what remains is just a principal $(\mathbb{C}^*)^r$ bundle over $Y \setminus D$. So it is not too surprising that there should be a quasi-equivalence:

$$\mathcal{W}_T(X \setminus \pi^{-1}D) \simeq \mathcal{W}(Y \setminus D)$$

Indeed this fits with a more general story about Hamiltonian reduction that we will discuss next.

However, $\mathcal{W}_T(X)$ should be a deformation of $\mathcal{W}_T(X \setminus \pi^{-1}D)$, since including the extra divisor will add terms to the A_∞ structure. From this point of view, Conjecture A is the claim that this deformation is in fact **trivial**.

Seidel's invertible elements

Suppose we have a Hamiltonian S^1 action on a symplectic manifold X. The information of the S^1 action appears in the Fukaya category as an invertible element (due to Seidel):

$$s \in HH^0(\mathcal{W}(X))$$

This is a natural automorphism of the identity functor, so for each object $L \in \mathcal{W}(X)$ it provides an automorphism $s_L : L \xrightarrow{\sim} L$.

Spectral components

The category that we denoted $\mathcal{W}_{S^1}(X)$ in the previous section has objects given by those L such that $s_L = 1_L$.

In fact, for any fixed $\lambda \in \mathbb{C}^*$ one can construct a similar category

$$\mathcal{W}_{S^1}(X)_\lambda$$

by taking objects of $\mathcal{W}(X)$ such that $s_L = \lambda 1_L$.

Teleman refers to these categories as the 'spectral components' of the equivariant Fukaya category.

For example, an S^1-invariant Lagrangian L (which is monotone, has minimal Maslov at least 2, and is equipped with an appropriate spin structure) provides an object of $\mathcal{W}_{S^1}(X)_1$. But if we give L a non-trivial local system, whose monodromy along S^1 orbits is λ, then we have an object of $\mathcal{W}_{S^1}(X)_\lambda$.

Hamiltonian reduction

Consider the Hamiltonian reduction $X /\!\!/_{\alpha} S^1$ at some regular value $\alpha \in \mathbb{R}$ of the moment map. There is a Lagrangian correspondence

$$\Gamma = \{(x, \pi(x)),\ \mu(x) = \alpha\} \subset X^- \times (X /\!\!/_{\alpha} S^1)$$

which induces a functor

$$\mathcal{W}(X) \to \mathcal{W}(X /\!\!/_{\alpha} S^1)$$

Γ is S^1-invariant, so we can use it to define a functor on the equivariant Fukaya category of X. Teleman conjectures that this gives an equivalence

$$\mathcal{W}_{S^1}(X)_{e^\alpha} \cong \mathcal{W}(X /\!\!/_{\alpha} S^1)$$

between the Fukaya category of the Hamiltonian reduction and the corresponding spectral component of the equivariant category. More generally, we can twist the quotient manifold by a B-field β, and this will give the spectral component at $\lambda = e^{\alpha + i\beta}$. A theorem along these lines has been announced by Fukaya.

Let us see what this point-of-view brings to our example, the affine conic $T^*S^2 = \{xy + z^2 = 1\}$.

The moment map is $\mu = |x|^2 - |y|^2$. Any non-zero $\alpha \in \mathbb{R}$ is a regular value of μ, and produces the quotient $\mathbb{C}$. Since $\mathcal{W}(\mathbb{C}) \cong 0$ the spectral component

$$\mathcal{W}_{S^1}(X)_\lambda = 0 \text{ for any } \lambda \textit{ with } |\lambda| \neq 1.$$

However, our observation is about the component at $\lambda = 1$, and this corresponds to the *singular* value $\alpha = 0$ where we cannot do symplectic reduction. But if we simply delete the singularities of the moment map fibre $\mu^{-1}(0)$ then the quotient becomes:

$$\mathcal{P} = \big(\mu^{-1}(0) - (0, 0, \pm 1)\big)/S^1$$

Our observation is that $\mathcal{W}(\mathcal{P})$ is the correct spectral component $\mathcal{W}_{S^1}(X)_1$.

The other components $\lambda = e^{i\beta} \neq 1$ should correspond to a B-field on $\mathcal{P}$ but there are none, and it turns out these categories vanish, as we shall see later.

We can state this melange of our + Teleman's observations as the following conjecture

Conjecture B. Let X be a Hamiltonian S^1-manifold with moment map μ and let α be a singular value in the interior of the moment interval. Under "appropriate hypotheses", there is a quasi-equivalence

$$\mathcal{W}_{S^1}(X)_{e^{\alpha+i\beta}} \cong \mathcal{W}(U/S^1, \beta)$$

where U is the smooth locus in $\mu^{-1}(\alpha)$.

Violation of "appropriate hypothesis" would in general mean that the category on the right hand side should be bulk deformed (as determined by the quantum Kirwan map).

Mirror Symmetry

Given a Hamiltonian S^1 action on X, the Seidel element makes the wrapped Fukaya category $\mathcal{W}(X)$ linear over the ring $\mathbb{C}[s, s^{-1}]$.

Now suppose that X is mirror to an algebraic variety $\check{X}$. Then, since $\mathcal{W}(X) = D^b(\check{X})$, the mirror to s must be an invertible element σ in:

$$HH^0(D^b(\check{X})) = \Gamma(\mathcal{O}_{\check{X}})$$

If we have an S^1 action on the symplectic side then on the mirror we have a function $\sigma : \check{X} \to \mathbb{C}^*$.

More generally X might be mirror to a Landau-Ginzburg model $(\check{X}, \check{W})$. Then $\mathcal{W}(X)$ is equivalent to the category of matrix factorizations $\mathrm{MF}(\check{X}, \check{W})$, but still a function $\sigma : \check{X} \to \mathbb{C}^*$ does provide a natural automorphism of this category, so a possible mirror to the S^1 action on X.

Mirror Symmetry conjecture

Conjecture C. Suppose we have a Hamiltonian S^1 action on a symplectic manifold X. Suppose X has a mirror Landau-Ginzburg model $(\check{X}, \check{W})$, and that the S^1 action is mirror to a function $\sigma : \check{X} \to \mathbb{C}^*$. Then for every $\lambda \in \mathbb{C}^*$ we have an equivalence

$$\mathcal{W}_{S^1}(X)_\lambda \;\cong\; \mathrm{MF}\left(\check{Z}_\lambda, \check{W}|_{\check{Z}_\lambda}\right)$$

where $\check{Z}_\lambda \subset \check{X}$ denotes the hypersurface $\sigma^{-1}(\lambda)$.

This claim will be central to all the mirror symmetry evidence however, it is not really a precise conjecture because we haven't specified what we mean by 'mirror'. In particular it's not enough to just assume that $\mathcal{W}(X) \cong \mathrm{MF}(\check{X}, \check{W})$. It might be better to read it as a *definition* of an 'S^1-equivariant homological mirror'.

A log CY example

Consider

$$X = \mathbb{C}^2 \setminus \{zw = 1\}$$

equipped with the restriction of the standard symplectic form on $\mathbb{C}^2$. This is a log-Calabi Yau surface which is known to be self-mirror. We write

$$\check{X} = \mathbb{C}^2 \setminus \{\check{z}\check{w} = 1\}$$

Hamiltonian S^1 action on X by $e^{i\theta}(z,w) = (e^{i\theta}z, e^{-i\theta}w)$.
On $\check{X}$ this becomes the non-vanishing function:

$$\sigma = 1 - \check{z}\check{w}$$

If $\lambda \in \mathbb{C}^*$ with $\lambda \neq 1$ then $\sigma^{-1}(\lambda) = \mathbb{C}^*$. So the claim is that $\mathcal{W}_{S^1}(X)_\lambda \cong D^b(\mathbb{C}^*)$. But for $\lambda = 1$, we're claiming that $\mathcal{W}_{S^1}(X)_1$ should be equivalent to the derived category of the node $\check{Z}_1 = \{\check{z}\check{w} = 0\}$.

A log CY example

Now what about the symplectic reductions? If we have $\lambda = e^\alpha$ for $\alpha \in \mathbb{R} \setminus 0$ then $\mathcal{W}_{S^1}(X)_\lambda$ should be the wrapped category of the Hamiltonian reduction of X at the moment-map value α. Since this quotient is $\mathbb{C}^*$, and $\mathbb{C}^*$ is self-mirror, everything is consistent.

At the singular value $\alpha = 0$ we can apply our Conjectures A or B, which tell us to delete the singularity from $\mu^{-1}(0)$ before we take the quotient. The result is $\mathbb{C}^\times \setminus \{1\}$ which is the pair-of-pants. This is indeed well known to be the mirror to $\check{Z}_1$.

Recovering the non-equivariant category

Given S^1 action on X we have for each $\lambda \in \mathbb{C}^*$ a spectral component $\mathcal{W}_{S^1}(X)_\lambda$ of the equivariant Fukaya category. These categories have some important extra structure, they are linear over $H^\bullet_{S^1}(pt) = \mathbb{C}[t], \quad \deg t = 2$.

This structure is built into the construction, and all A_∞ structure maps respect it. It is therefore possible to take the fibre of $\mathcal{W}_{S^1}(X)_\lambda$ at $t = 0$. The result is a subcategory

$$\mathcal{W}_{S^1}(X)_\lambda|_{t=0} \subset \mathcal{W}(X)$$

of the ordinary wrapped Fukaya category of X, it is the full subcategory of objects L with $s_L = \lambda 1_L$.

Deformation class

Now suppose we have the set-up of Conjecture A. So we have a rank one conic fibration $\pi : X \to Y$ degenerating over a divisor $D \subset Y$. The conjecture is that:

$$\mathcal{W}_{S^1}(X)_1 \cong \mathcal{W}(Y \setminus D)$$

Since the category on the left is linear over $\mathbb{C}[t]$, the category on the right should be too.
There is an obvious guess for what this extra structure on $\mathcal{W}(Y \setminus D)$ is. Indeed, there is a class $\tau \in SH^2(Y \setminus D)$ corresponding to a simple Reeb orbit going around the divisor D once. It is sometimes called the Borman-Sheridan class. With this choice of τ, the category $\mathcal{W}(Y \setminus D)$ becomes linear over $\mathbb{C}[t]$.

Conjecture D. In the situation of Conjecture A, the action of t on $\mathcal{W}_{S^1}(X)_1$ coincides with the action of τ on $\mathcal{W}(Y \setminus D)$.

relative Fukaya category

On the base space Y we can consider the *relative* wrapped Fukaya category: $\mathcal{W}(Y, D)$. This category is, by construction, linear over a power series ring $\mathbb{C}[[h]]$ where $\deg h = 0$. The fibre at $h = 0$ is a full subcategory of $\mathcal{W}(Y \setminus D)$, containing the Lagrangians that don't have ends at D.
Now consider the space X. The S^1 action makes $\mathcal{W}(X)$ linear over the ring $\mathbb{C}[s^{\pm 1}]$.

Conjecture E. Suppose we have the setup of Conjecture A. Then the relative wrapped Fukaya category $\mathcal{W}(Y, D)$ is equivalent to the completion of $\mathcal{W}(X)$ at $s = 1$.

Suppose X with S^1 action is mirror to $(\check{X}, \check{W})$ with $\sigma : \check{X} \to \mathbb{C}^*$. Then $\mathcal{W}(Y, D)$ will be equivalent to the category of matrix factorizations on the formal scheme obtained by completing $\check{X}$ along the divisor $\check{Z}_1 = \sigma^{-1}(1)$.

A summary

$\pi : X \to Y$ conic fibration with singular fibers over D equipped with an S^1 action.

$(\check{X}, \check{W})$, mirror to X and $\sigma : \check{X} \to \mathbb{C}^*$ mirror to S^1 action.

$$\begin{tikzcd}
\mathrm{MF}(\check{X}, \check{W}) \arrow[r, dashed, "\simeq"] \arrow[d, shift right, "i^*"] & \mathcal{W}(X) \arrow[r, "\pi_*"] & \mathcal{W}(Y, D) \arrow[d, "h=0"] \\
\mathrm{MF}(\sigma^{-1}(1)), \check{W}|_{\sigma^{-1}(1)}) \arrow[r, dashed, "\simeq"] \arrow[u, shift right, "i_*"] & \mathcal{W}_{S^1}(X)_1 \arrow[u, "t=0"] & \mathcal{W}(Y \setminus D) \arrow[l, "\simeq", "\pi^{-1}"']
\end{tikzcd}$$

The End

(...or a beginning?)