ON HOMOLOGICAL MIRROR SYMMETRY FOR THE COMPLEMENT OF A SMOOTH AMPLE DIVISOR IN A K3 SURFACE

YANKI LEKILI AND KAZUSHI UEDA

ABSTRACT. We introduce a conjecture on homological mirror symmetry relating the symplectic topology of the complement of a smooth ample divisor in a K3 surface to algebraic geometry of type III degenerations, and prove it when the degree of the divisor is either 2 or 4.

Let X be a K3 surface and D be a smooth ample divisor. Since the affine variety $U := X \backslash D$ is simply connected and has the trivial canonical bundle, the wrapped Fukaya category $\mathcal{W}(U)$ (and its full subcategory $\mathcal{F}(U)$ consisting of compact Lagrangians) comes with a unique $\mathbb{Z}$-grading. The symplectomorphism type of the pair (X, D) is determined by the degree $d := D \cdot D$ and the index k of primitivity (i.e., the maximal k such that $[D]/k \in H_2(X; \mathbb{Z})$). The degree d can be any positive even integer, and the index k can be any positive integer satisfying $2k^2 | d$.

A *Type III K3 surface* is a surface X obtained from a collection $(X_i, D_i)_{i=1}^n$ of smooth rational surfaces X_i with anticanonical cycles $D_i \in |-K_{X_i}|$ by identifying irreducible components of D_i in such a way that the dual intersection complex is a triangulation of S^2. Each irreducible double curve $C \subset D_i \cap D_j$ must satisfy the *triple point formula*

$$(1) \qquad (C)^2_{X_i} + (C)^2_{X_j} = \begin{cases} 0 & C \text{ is a nodal curve,} \\ -2 & C \text{ is smooth.} \end{cases}$$

We call the number of triangles in the dual intersection complex and the index of primitivity of the monodromy logarithm of the degeneration (which is also determined by the dual intersection complex [FS86, Theorem 6.5]) as the *degree* and the *index* of the Type III K3 surface. Any Type III K3 surfaces with the same degree and index are related by a sequence of Type I and Type II modifications shown in Figures 1 and 2 [FS86, Theorem 0.6.3].
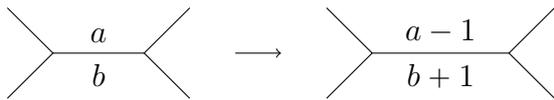
FIGURE 1. Type I modification
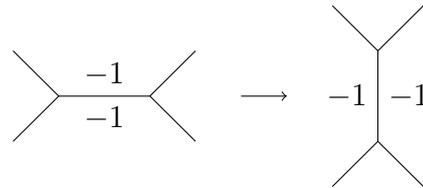
FIGURE 2. Type II modification

Here, the integer on an edge of the intersection complex denotes the self-intersection number of the corresponding irreducible double curve. These modifications are given by the Atiyah flop on the total space of the smoothing, and hence do not change the derived category $\operatorname{coh} X$ of coherent sheaves by [BP10, Theorem 1.1].

We fix the complex structure of a Type III K3 surface by the following two conditions:

(1) For any $i \in \{1, \ldots, n\}$, the complex structure of the log Calabi–Yau surface $X_i \backslash D_i$ is the one appearing in [HK]. This means that a suitable corner blow-up of (X_i, D_i) is obtained from a toric surface by blowing up (possibly infinitely near) points on the toric boundary whose toric coordinates are -1.

(2) Irreducible components of the anticanonical cycles are glued torically. This means that when two irreducible curves on the boundaries are glued together, the centers of the blow-ups are identified.

Conjecture 1. *Let U be the complement of a smooth ample divisor of degree d and index k in a K3 surface. Let further X be a Type III K3 surface of degree d and index k. Then there exist equivalences*

$$\text{(2)} \qquad \qquad \mathcal{W}(U) \simeq \operatorname{coh} X$$

and

$$\text{(3)} \qquad \qquad \mathcal{F}(U) \simeq \operatorname{perf} X.$$

Theorem 2. *Conjecture 1 holds in degree 2 and 4.*

Proof. The Milnor fiber

$$\text{(4)} \qquad \qquad U := \left\{ (x, y, z) \in \mathbb{C}^2 \mid x^2 + y^6 + z^6 = 1 \right\}$$

of the Brieskorn–Pham singularity obtained as the suspension of the Sebastiani–Thom sum of two copies of the A_5-singularity is the complement of a smooth ample divisor of degree 2 in a K3 surface.

Let $\mathbb{P} := [(\mathbb{A}^4 \backslash \mathbf{0})/\Gamma]$ be the quotient stack of the complement of the origin $\mathbf{0}$ in $\mathbb{A}^4 = \operatorname{Spec} \mathbb{C}[x, y, z, w]$ by the diagonal action of

$$\text{(5)} \qquad \qquad \Gamma := \left\{ (\alpha, \beta, \gamma, \delta) \in (\mathbb{G}_m)^4 \mid \alpha^2 = \beta^6 = \gamma^6 = \alpha\beta\gamma\delta \right\},$$

and Z be the hypersurface of $\mathbb{P}$ defined by $x^2 + y^6 + z^6 + xyzw$. It is shown in [LU, Theorem 1.7.(ii)] that

$$\text{(6)} \qquad \qquad \mathcal{W}(U) \simeq \operatorname{coh} Z$$

and

$$\text{(7)} \qquad \qquad \mathcal{F}(U) \simeq \operatorname{perf} Z.$$

Let $\tilde{\mathbb{P}}$ be a crepant resolution of the coarse moduli scheme of $\mathbb{P}$, and X be the hypersurface of $\tilde{\mathbb{P}}$ defined by $x^2 + y^6 + z^6 + xyzw$. One has

$$\text{(8)} \qquad \qquad \operatorname{coh} Z \simeq \operatorname{coh} X$$

and

$$\text{(9)} \qquad \qquad \operatorname{perf} Z \simeq \operatorname{perf} X$$

by (the proof of) [BP10, Theorem 2.1].

Note that Z has a unique singular point $[0 : 0 : 0 : 1]$ contained in the open substack of $\mathbb{P}$ defined by $w \neq 0$ whose coarse moduli scheme V is isomorphic to $\mathbb{A}^3 / \left\langle \frac{1}{2}(1, 0, 1), \frac{1}{6}(0, 1, 5) \right\rangle$. One can choose $\tilde{\mathbb{P}}$ in such a way that the inverse image of V is the toric variety whose fan is the cone over the triangulation given in Figure 3. Then the scheme X consists of the strict transform X_1 of Z and two more irreducible components X_2 and X_3, which are the compact toric divisors in the crepant resolution of V.

An explicit description in terms of toric coordinates shows that the family defined by $x^2 + y^6 + z^6 + tw^6 + xyzw = 0$ inside $\tilde{\mathbb{P}}$ is a semistable degeneration of K3 surfaces whose central fiber is X.

Each of X_2 and X_3 is the toric surface obtained by blowing up $\mathbb{P}^2$ four times as designated by the fan given in Figure 4. The double curve $C = X_2 \cap X_3$ is described on each of X_2 and X_3 as the (-1)-curve obtained as the strict transform $H - E_2 - E_4$ of the line passing through the pair of infinitely near points which are the centers of the successive blow-ups. It follows that the intersection complex of X must be the one shown in Figure 5. The dual intersection complex is a triangulation of S^2 by two triangles.

In order to determine the complex structure of X_1, let Y be the graph of the rational map

$$\text{(10)} \qquad \qquad Z \dashrightarrow \mathbb{P}^2, \quad [x : y : z : w] \mapsto [p : q : r] = [x^2 : y^6 : z^6],$$

i.e., the closure in $Z \times \mathbb{P}^2$ of the graph of the regular map from the complement of $x = y = z = 0$ in Z to $\mathbb{P}^2$. The projection $Y \to \mathbb{P}^2$ is the stacky weighted blow-up of $\mathbb{P}^2$ of weights $(1, 2)$, $(1, 6)$, and $(1, 6)$ at the intersections of the coordinate lines and the line defined by $p + q + r = 0$. The minimal resolution $\tilde{Y}$ of the coarse moduli scheme of Y is a rational surface with an anticanonical cycle whose irreducible components have self-intersection numbers -1, -5, and -5. By blowing down the (-1)-component of the anticanonical cycle, one obtains the rational surface X_1 with an
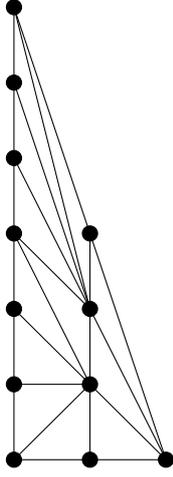
FIGURE 3. A crepant resolution of $\mathbb{A}^3 \big/ \big\langle \frac{1}{2}(1,0,1), \frac{1}{6}(0,1,5) \big\rangle$
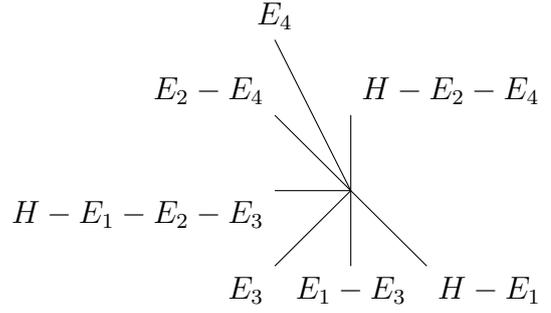
FIGURE 4. A weak toric del Pezzo surface of degree 5
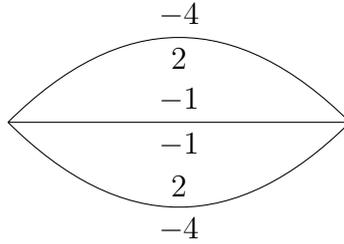
FIGURE 5. The division of S^2 into three bigons

anticanonical cycle consisting of two (-4)-curves. This concludes the proof of Theorem 2 for degree 2.

The story for the degree 4 case is completely parallel. The complement of a smooth ample divisor of degree 4 in a K3 surface is the Milnor fiber of the Brieskorn–Pham singularity obtained as the Sebastiani–Thom sum of three copies of the A_3-singularity. Let $\mathbb{P}$ be the quotient stack of $\mathbb{A}^4 \backslash \mathbf{0}$ by the diagonal action of

(11) $$\Gamma := \left\{ (\alpha, \beta, \gamma, \delta) \in (\mathbb{G}_{\mathrm{m}})^4 \mid \alpha^4 = \beta^4 = \gamma^4 = \alpha\beta\gamma\delta \right\},$$

and Z be the hypersurface of $\mathbb{P}$ defined by $x^4 + y^4 + z^4 + xyzw$. It is shown in [LU, Theorem 1.7.(i)] that

(12) $$\mathcal{W}(U) \simeq \operatorname{coh} Z$$

and

(13) $$\mathcal{F}(U) \simeq \operatorname{perf} Z.$$

A stacky resolution Y of Z is obtained as the graph of the rational map

(14) $$Z \dashrightarrow \mathbb{P}^2, \ [x : y : z : w] \mapsto [x^4 : y^4 : z^4].$$

The projection $Y \to \mathbb{P}^2$ is the weighted blow-up of $\mathbb{P}^2$ of weight $(1, 4)$ at the three intersection points of the coordinate lines and a general line. The minimal resolution of the coarse moduli scheme of Z is a rational surface with an anticanonical cycle consisting of three (-3)-curves. The scheme X consists of this surface and three more irreducible components, which are the compact toric divisors in the crepant resolution of $\mathbb{A}^3 \big/ \big\langle \frac{1}{4}(1,3,0), \frac{1}{4}(0,1,3) \big\rangle$ whose fan is shown in Figure 6. Each of these irreducible components is $\mathbb{P}^2$ blown-up at four points (including infinitely near points) whose fan is shown in Figure 7. The double curves among these two components have self intersection minus one on each irreducible components. The intersection complex of X is the tetrahedron shown in Figure 8. $\quad\square$

3
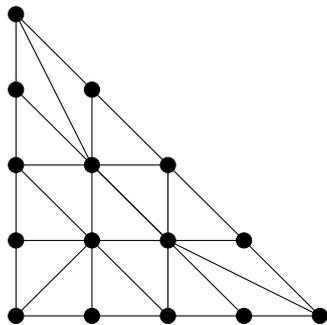
FIGURE 6. A crepant resolution of $\mathbb{A}^3/\left\langle \frac{1}{4}(1,3,0), \frac{1}{4}(0,1,3) \right\rangle$
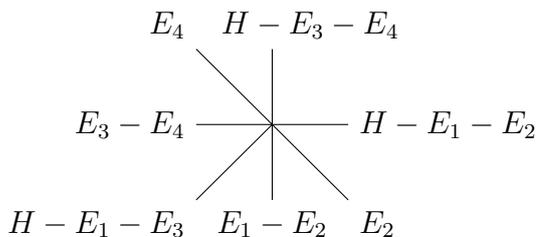
FIGURE 7. A weak del Pezzo surface of degree 5
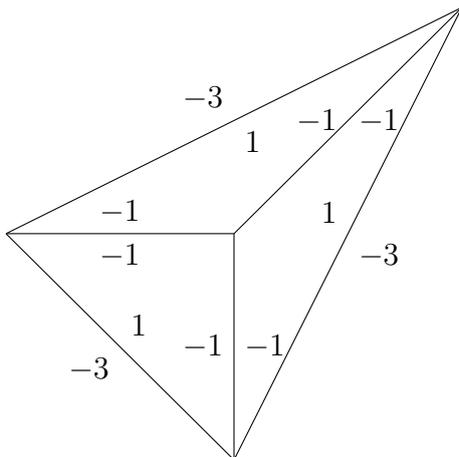
FIGURE 8. The tetrahedron

Theorem 2 combined with [Sei11] proves [LU, Conjecture 1.5] for a smooth ample divisor of degree 2 in a K3 surface.

Corollary 3. *One has*

$$\mathcal{F}(D) \simeq \mathcal{W}(U)/\mathcal{F}(U) \tag{15}$$

when $U = X \backslash D$ is the complement of a smooth ample divisor D in a K3 surface X.

Proof. Let S be the hypersurface of $\operatorname{Spec} \mathbb{C}[x,y,z]/\left\langle \frac{1}{5}(1,1,3) \right\rangle$ defined by

$$x^5 + y^5 + z^5 + xyz = 0. \tag{16}$$

The mirror of the genus two curve is identified in [Sei11] as the total transform H of S in a crepant resolution of $\operatorname{Spec} \mathbb{C}[x,y,z]/\left\langle \frac{1}{5}(1,1,3) \right\rangle$. The surface H has three irreducible components H_i for $i = 1, 2, 3$. The components $H_1 \cong \mathbb{P}^2$ and $H_2 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3))$ are the exceptional divisors of the resolution, and the component H_3 is the strict transform of S. In order to determine the complex structure of H_3, let T be the graph of the rational map

$$S \dashrightarrow \mathbb{P}^2, \quad (x,y,z) \mapsto [p:q:r] = [x:y:z^2]. \tag{17}$$

The projection $T \to \mathbb{P}^2$ is the restriction to an open subset of a blow-up of ten points on the conic $pq = r^2$ defined by $p^5 + q^5 = 0$ followed by a weighted blow-up of weight $(1,2)$ of five points on the coordinate line $r = 0$ defined by $p^5 + q^5 = 0$. It follows that H_3 is an open subscheme of the blow-up of ten points on a conic and five pairs of infinitely near points on a line in $\mathbb{P}^2$.

The compactification of the surface H, whose intersection complex is shown in Figure 9, is related to the type III K3 surface X of degree 2 appearing in the proof of Theorem 2 by deformation (i.e., moving the center of the blow-up to infinitely near points at $p + q = 0$) and a sequence of Type I modifications. Since the deformation does not change the formal neighborhood of the singular locus

4
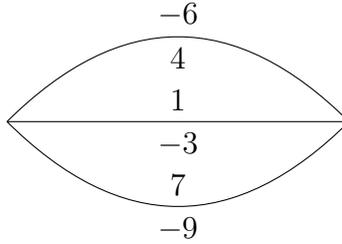
FIGURE 9. Seidel's mirror for the genus 2 curve

and the Type I modification does not change the derived category, the stable derived categories of X and H are equivalent. □

Remark 4. Homological mirror symmetry for 2-dimensional pair of pants (see e.g. [Sei15, She11, Nad, Aur18, LP20] and references therein) implies an equivalence

$$\tag{18} \mathcal{W}(U) \simeq \operatorname{coh} X$$

where

$$\tag{19} U := \left\{ (x, y, z) \in (\mathbb{C}^\times)^3 \mid x + y + z + 1/xyz = 0 \right\}$$

is the complement of a nef divisor of degree 64 and index 4 with 24 nodes in a K3 surface (the quartic mirror) and X is the toric boundary of $\mathbb{P}^3$. Note that X can be turned into a Type III K3 surface by blowing up at 24 points consisting of four points on each of six double curves. It is natural to expect that smoothing a node at the divisor at infinity is mirror to blowing up a point on the double curve.

REFERENCES

[Aur18] Denis Auroux, *Speculations on homological mirror symmetry for hypersurfaces in* $(\mathbb{C}^*)^n$, Surveys in differential geometry 2017. Celebrating the 50th anniversary of the Journal of Differential Geometry, Surv. Differ. Geom., vol. 22, Int. Press, Somerville, MA, 2018, pp. 1–47. MR 3838112 5

[BP10] Vladimir Baranovsky and Jeremy Pecharich, *On equivalences of derived and singular categories*, Cent. Eur. J. Math. **8** (2010), no. 1, 1–14. MR 2593258 1, 2

[FS86] Robert Friedman and Francesco Scattone, *Type* III *degenerations of K3 surfaces*, Invent. Math. **83** (1986), no. 1, 1–39. MR 813580 1

[HK] Paul Hacking and Ailsa Keating, *Homological mirror symmetry for log Calabi-Yau surfaces*, arXiv:2005.05010. 1

[LP20] Yankı Lekili and Alexander Polishchuk, *Homological mirror symmetry for higher-dimensional pairs of pants*, Compos. Math. **156** (2020), no. 7, 1310–1347. MR 4120165 5

[LU] Yankı Lekili and Kazushi Ueda, *Homological mirror symmetry for Milnor fibers via moduli of A_∞-structures*, arXiv:1806.04345. 2, 3, 4

[Nad] David Nadler, *Wrapped microlocal sheaves on pairs of pants*, arXiv:1604.00114. 5

[Sei11] Paul Seidel, *Homological mirror symmetry for the genus two curve*, J. Algebraic Geom. **20** (2011), no. 4, 727–769. MR 2819674 (2012f:53186) 4

[Sei15] _____, *Homological mirror symmetry for the quartic surface*, Mem. Amer. Math. Soc. **236** (2015), no. 1116, vi+129. MR 3364859 5

[She11] Nick Sheridan, *On the homological mirror symmetry conjecture for pairs of pants*, J. Differential Geom. **89** (2011), no. 2, 271–367. MR 2863919 (2012m:53196) 5

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, SOUTH KENSINGTON, LONDON, SW7 2AZ
Email address: y.lekili@imperial.ac.uk

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO, 3-8-1 KOMABA, MEGURO-KU, TOKYO, 153-8914, JAPAN.
Email address: kazushi@ms.u-tokyo.ac.jp