HOMOLOGICAL MIRROR SYMMETRY FOR MILNOR FIBERS OF SIMPLE SINGULARITIES

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Abstract. We prove homological mirror symmetry for Milnor fibers of simple singularities in dimensions greater than one, which are among the log Fano cases of [LU, Conjecture 1.5]. The proof is based on a relation between matrix factorizations and Calabi–Yau completions. As an application, we give an explicit computation of the Hochschild cohomology group of the derived n-preprojective algebra of a Dynkin quiver for any $n \geq 1$, and the symplectic cohomology group of the Milnor fiber of any simple singularity in any dimension greater than one.

1. Introduction

A *simple singularity* is an isolated hypersurface singularity of modality zero. Arnold classified such singularities; up to right equivalence, they are given by one of the following:

$$\begin{aligned} &A_\ell \colon x_1^{\ell+1} + x_2^2 + \cdots + x_{n+1}^2 = 0, \quad \ell = 1, 2, \dots \\ &D_\ell \colon x_1^{\ell-1} + x_1 x_2^2 + x_3^2 + \cdots + x_{n+1}^2 = 0, \quad \ell = 4, 5, \dots \\ &E_6 \colon x_1^4 + x_2^3 + x_3^2 + \cdots + x_{n+1}^2 = 0, \\ &E_7 \colon x_1^3 + x_1 x_2^3 + x_3^2 + \cdots + x_{n+1}^2 = 0, \\ &E_8 \colon x_1^5 + x_2^3 + x_3^2 + \cdots + x_{n+1}^2 = 0. \end{aligned} \tag{1.1}$$

In the case $n = 2$, simple surface singularities have many other characterizations, such as Kleinian singularities, rational double points, or canonical singularities, to name a few.

Let $\check{\mathbf{w}}$ be one of these defining polynomials, which we think of as a holomorphic function on $\mathbb{C}^{n+1}$, and equip $\check{\mathbf{w}}^{-1}(1)$ with the Liouville structure induced from the standard one on $\mathbb{C}^{n+1}$. This is the Liouville completion of the *Milnor fiber*, which is the Liouville domain obtained by intersecting $\check{\mathbf{w}}^{-1}(1)$ with a ball. Let $\mathcal{W}(\check{\mathbf{w}}^{-1}(1))$ denote the idempotent-complete derived wrapped Fukaya category of $\check{\mathbf{w}}^{-1}(1)$.

For $n \geq 2$, since $\check{\mathbf{w}}^{-1}(1)$ is not a log Calabi–Yau manifold but a log Fano manifold, its mirror is not a manifold but a *Landau–Ginzburg model*, by which we mean a pair of a stack and a section of a line bundle on it. One way to obtain a Landau–Ginzburg mirror of a log Fano manifold is to first remove a divisor to make it log Calabi–Yau, then find its mirror, which is another log Calabi–Yau manifold, and finally add a potential to this mirror [Aur07, Aur09]. This produces a Landau–Ginzburg mirror whose underlying manifold is of the same dimension as the original manifold. When the singularity is toric (i.e., a simple surface singularity of type A), there is a standard choice for the divisor to remove, and the resulting mirror is the Landau–Ginzburg model consisting of a complement of a toric divisor in the minimal resolution of the singularity of the same type and a monomial function on it (see e.g. [AAK16, Section 9.2]). The choice of the divisor is not unique in general, and there are multiple mirrors for a given Milnor fiber.

In this paper, we consider an alternative mirror of the Milnor fiber of a simple singularity based on transposition of invertible polynomials introduced in [BH93, BH95]. A weighted homogeneous polynomial $\mathbf{w} \in \mathbb{C}[x_1, \dots, x_{n+1}]$ with an isolated critical point at the origin is *invertible* if there is an integer matrix $A = (a_{ij})_{i,j=1}^{n+1}$ with non-zero determinant such that

$$\mathbf{w} = \sum_{i=1}^{n+1} \prod_{j=1}^{n+1} x_j^{a_{ij}}. \tag{1.2}$$

The *transpose* of $\mathbf{w}$ is defined as

$$\check{\mathbf{w}} = \sum_{i=1}^{n+1} \prod_{j=1}^{n+1} x_j^{a_{ji}}, \tag{1.3}$$

whose exponent matrix $\check{A}$ is the transpose matrix of A. The group

$$\Gamma_{\mathbf{w}} \coloneqq \left\{ (t_0, t_1, \ldots, t_{n+1}) \in (\mathbb{G}_{\mathrm{m}})^{n+2} \;\middle|\; t_1^{a_{1,1}} \cdots t_{n+1}^{a_{1,n+1}} = \cdots = t_1^{a_{n+1,1}} \cdots t_{n+1}^{a_{n+1,n+1}} = t_0 t_1 \cdots t_{n+1} \right\} \tag{1.4}$$

acts naturally on $\mathbb{A}^{n+2} \coloneqq \operatorname{Spec} \mathbb{C}[x_0, \ldots, x_{n+1}]$. Let $\operatorname{mf}\left(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 \cdots x_{n+1}\right)$ denote the idempotent completion of the dg category of $\Gamma_{\mathbf{w}}$-equivariant coherent matrix factorizations of $\mathbf{w} + x_0 \cdots x_{n+1}$ on $\mathbb{A}^{n+2}$ in the sense of [EP15]. Conjecture 1.1 below is given in [LU, Conjecture 1.5]:

Conjecture 1.1. *For any invertible polynomial* $\mathbf{w}$*, one has a quasi-equivalence*

$$\operatorname{mf}\left(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 \cdots x_{n+1}\right) \simeq \mathcal{W}\left(\check{\mathbf{w}}^{-1}(1)\right). \tag{1.5}$$

In other words, the Landau–Ginzburg model $\left(\left[\mathbb{A}^{n+2}/\Gamma_{\mathbf{w}}\right], \mathbf{w} + x_0 \cdots x_{n+1}\right)$ is mirror to the Liouville manifold $\check{\mathbf{w}}^{-1}(1)$. The main result of this paper is the following:

Theorem 1.2. *Conjecture 1.1 holds for $n \geq 2$ and $\check{\mathbf{w}}$ one of the defining polynomials of simple singularities appearing in* (1.1).

The proof of Theorem 1.2 consists of four steps. The first step is the quasi-equivalence

$$\operatorname{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 \cdots x_{n+1}) \simeq \operatorname{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}), \tag{1.6}$$

which comes from the fact that $\mathbf{w} + x_0 \cdots x_{n+1}$ is right equivalent to $\mathbf{w}$ by a formal change of variables, which holds if $n \geq 2$ and $\mathbf{w}$ defines a simple singularity.

The second step is the quasi-equivalence

$$\operatorname{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}) \simeq \Pi_n\left(\operatorname{mf}(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w})\right), \tag{1.7}$$

where Π_n denotes the n-Calabi–Yau completion in the sense of [Kel11]. This holds for any invertible polynomial $\mathbf{w}$ and any $n \geq 0$.

The third step is the quasi-equivalence

$$\operatorname{mf}(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w}) \simeq \operatorname{perf} A_Q \tag{1.8}$$

with the dg category $\operatorname{perf} A_Q$ of perfect dg modules over the path algebra A_Q of a Dynkin quiver Q (with any orientation) of the corresponding type. For type A, this is proved in [Tak, Theorem 3.1] for $n = 0$, and the $n \geq 1$ case follows either from the $n = 0$ case and the Knörrer periodicity [Knö87], or as a special case of [FU11, Theorem 1.2]. For type D, this follows from [FU13, Section 4]. For type E, this follows either from the combination of [HS, Theorem 1] and [Sei01, Proposition 3.4] or by finding a suitable mutation of a generator appearing in [HS, Theorem 2]. Note that [KST07, Theorem 3.1] gives a result close to (1.8), which is not exactly the same since the grading group is different.

The last step is

$$\mathcal{W}\left(\check{\mathbf{w}}^{-1}(1)\right) \simeq \Pi_n\left(\operatorname{perf} A_Q\right), \tag{1.9}$$

which holds if $n \geq 2$ and $\check{\mathbf{w}}$ defines a simple singularity. As discussed in Section 3, the proof of (1.9) in [EL17] depends on the computation of the Hochschild cohomology of $\Pi_n\left(\operatorname{perf} A_Q\right)$, which was missing for type E cases in [EL17] and is done in Section 5.

For $n = 1$ not covered by Theorem 1.2, a quasi-equivalence of the full subcategory $\mathcal{F}(\check{\mathbf{w}}^{-1}(0))$ of $\mathcal{W}(\check{\mathbf{w}}^{-1}(0))$ consisting of (direct summands of bounded complexes of) compact Lagrangians and a category $\operatorname{perf} Z_{\mathbf{w}}$ equivalent to the full subcategory of $\operatorname{mf}\left(\mathbb{A}^3, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 x_1 x_2\right)$ consisting of homologically finite objects (i.e., those X satisfying $\dim \bigoplus_{i \in \mathbb{Z}} \operatorname{Ext}^i(X, Y) < \infty$ for any object Y) is given in [Hab, Theorem 1.1].

As an application of (1.7), we compute the Hochschild cohomology group of the n-Calabi–Yau completion $\Pi_n(A_Q)$, also known as the *derived n-preprojective algebra*, of the path algebra A_Q of any Dynkin quiver Q for any $n \geq 1$. It is possible to compute the Hochschild homology along the same line.

The zero-th cohomology of the derived 2-preprojective algebra is the *preprojective algebra*. The Hochschild homology and cohomology of the preprojective algebra of the path algebra of a Dynkin quiver is calculated in [ES98b, ES98a, EE07]. Even the calculus structure in the sense of [GfDT89, TT00] (which includes the Batalin–Vilkovisky structure and is known to be derived invariant [AK19]) is calculated in [Eu10], and it is an interesting problem to do the same for the derived n-preprojective algebra. Note that the preprojective algebra and the derived 2-preprojective algebra of a Dynkin quiver are very different. The derived 2-preprojective algebra of a Dynkin quiver is a smooth dg algebra, which has cohomology in every negative cohomological degree, and moreover is not formal. In contrast, the preprojective algebra is always concentrated in cohomological degree 0 by definition, and the global dimension is infinite for a Dynkin quiver.

It follows from [Gan12, Theorem 1.1], combined with [CRGG, Theorem 1.4] which builds on [Gan12, Gao], that the closed-open map of any Weinstein manifold from the symplectic cohomology to the Hochschild cohomology of the wrapped Fukaya category is an isomorphism:

$$\mathrm{SH}^*(M) \xrightarrow{\sim} \mathrm{HH}^*(\mathcal{W}(M)). \tag{1.10}$$

Hence, by Theorem 1.2, we see that the symplectic cohomology of the Milnor fiber $\check{\mathbf{w}}^{-1}(1)$ of a simple singularity for $n \geq 2$ is isomorphic to $\mathrm{HH}^*\left(\Pi_n\left(A_Q\right)\right)$. This enables us to give an explicit computation of the symplectic cohomology of Milnor fibers of all simple singularities in a uniform way. Previous partial results computing symplectic cohomology for Milnor fibers of simple simple singularities appeared in [EL17] for A_ℓ and D_ℓ in complex dimension 2, and in [KvK16, Ueb16], for various versions of symplectic cohomology for certain higher dimensional A_ℓ-Milnor fibers for which an associated Morse–Bott spectral sequence yields computations. Our computation also shows that $\mathrm{HH}^*\left(\Pi_1\left(A_Q\right)\right)$ is not isomorphic to $\mathrm{SH}^*(\check{\mathbf{w}}^{-1}(0))$ given in [Hab, Section 3.3], which is consistent with the failure of (1.9) for $n = 1$.

This paper is organized as follows: In Section 2, we collect basic definitions and results on Calabi–Yau completions and trivial extension algebras. In Section 3, we recall the description of the wrapped Fukaya category of the Milnor fiber of a simple singularity for $n \geq 2$ in terms of the n-Calabi–Yau completion of a Dynkin quiver of the corresponding type. In Section 4, we prove (1.6) and (1.7). The computation of Hochschild cohomologies of the derived preprojective algebras of Dynkin quivers are given in Section 5.

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2. Calabi–Yau completions and trivial extension algebras

The *n-Calabi–Yau completion* (or the *derived n-preprojective algebra*) of a dg category $\mathscr{A}$ is defined in [Kel11, Section 4.1] as the tensor algebra

$$\Pi_n(\mathscr{A}) := T_{\mathscr{A}}(\theta) := \mathscr{A} \oplus \theta \oplus \theta \otimes_{\mathscr{A}} \theta \oplus \cdots, \tag{2.1}$$

where the $\mathscr{A}$-bimodule $\theta := \Theta[n-1]$ is a shift of the inverse dualizing complex $\Theta := \hom_{\mathscr{A}^{\mathrm{e}}}(\mathscr{A}, \mathscr{A}^{\mathrm{e}})$.

A dg algebra is regarded as a dg category with one object. The Morita invariance of the Calabi–Yau completion shown in [Kel11, Proposition 4.2] implies that Calabi–Yau completion commutes with the operation of taking the dg category of perfect dg modules:

$$\Pi_n\left(\operatorname{perf} \mathscr{A}\right) \simeq \operatorname{perf}(\Pi_n \mathscr{A}). \tag{2.2}$$

The *Ginzburg dg algebra* $\mathscr{G}_Q^n$ of a quiver Q (without potential) is a model of the n-Calabi–Yau completion $\Pi_n\left(A_Q\right)$ of the path algebra A_Q, defined in [Kel11, Section 6.2] after [Gin] as the path algebra of the graded quiver $\overline{Q}$ with same vertices as Q and arrows consisting of

- the original arrows $g \in Q_1$ in degree 1,
- the opposite arrows g^* for each arrow $g \in Q_1$ in degree $1-n$, and
- loops h_v at each vertex $v \in Q_0$ in degree $1-n$,

equipped with the differential d given by

$$dg = dg^* = 0 \;\text{ and }\; dh = \sum_{g\in Q_1} g^*g - gg^* \tag{2.3}$$

where $h = \sum_{v\in Q_0} h_v$.

The *degree n trivial extension algebra* of a finite-dimensional algebra A is defined as $A \oplus A^\vee[-n]$ equipped with the multiplication $(a, f)\cdot(b, g) = (ab, ag + fb)$, where $A^\vee$ is the dual of A as a vector space.

The degree n trivial extension algebra B^n_Q of the path algebra A_Q of a Dynkin quiver Q is the (derived) Koszul dual of $\mathscr{G}^n_Q$ in the sense that

$$\hom_{\mathscr{G}^n_Q}(\mathbf{k}_{\mathscr{G}}, \mathbf{k}_{\mathscr{G}}) \simeq B^n_Q, \qquad \hom_{(B^n_Q)^{\mathrm{op}}}(\mathbf{k}_B, \mathbf{k}_B) \simeq \left(\mathscr{G}^n_Q\right)^{\mathrm{op}}, \tag{2.4}$$

where $\mathbf{k}_{\mathscr{G}} := \bigoplus_{v\in Q_0} S_v$ is the direct sum of simple left $\mathscr{G}^n_Q$-modules S_v associated with vertices $v \in Q_0$, and similarly for $\mathbf{k}_B$ (see e.g. [EL17, Theorem 23, Corollary 25]).

This Koszul duality implies an isomorphism

$$\mathrm{HH}^*\left(\mathscr{G}^n_Q\right) \cong \mathrm{HH}^*\left(B^n_Q\right) \tag{2.5}$$

of Hochschild cohomologies (see e.g. [FMT05, Theorem 1] and [Her19, Theorem 3.4]).

3. Wrapped Fukaya category of the Milnor fiber of simple singularity

Let $\check{\mathbf{w}}$ be one of the defining polynomials of a simple singularity and $M^n = \check{\mathbf{w}}^{-1}(1)$ be the Milnor fiber, which we view as a Weinstein manifold where the Weinstein structure is induced by restriction from the ambient $\mathbb{C}^{n+1}$. It is well known that this Weinstein manifold is symplectomorphic (in fact, Weinstein homotopic) to the plumbing X_Q of cotangent bundles of spheres T^*S^n according to the Dynkin diagram Q corresponding to the simple singularity. One way to see this is to verify it directly for $n = 1$, and then use the fact that in higher dimensions the Milnor fiber is obtained by stabilization — increasing the dimension corresponds to suspension of the Lefschetz fibration [Sei10]. See also [Abo11] for an explicit construction of a symplectic structure on plumbings. This stabilization point of view also enables one to describe M via Legendrian surgery. Namely M is obtained by attaching critical handles to a Legendrian link Λ^{n-1}_Q on $\partial\mathbb{D}^n$ whose components are unknotted Legendrian spheres S^{n-1} which are clasped together (as in Hopf link) according to the Dynkin diagram Q. The direct sum of co-cores to the critical handles (i.e., cotangent fibers away from the plumbing region) form a generating object of the wrapped Fukaya category by the main theorem in [CRGG], and the surgery formula of [BEE12, Ekh] allows one to explicitly compute the endomorphism algebra of this generator as the Chekanov–Eliashberg algebra $\mathrm{CE}^*(\Lambda^{n-1}_Q)$.

This Chekanov–Eliashberg algebra was computed directly in the case $n = 2$ in the paper [EL17] and the resulting dg algebra was shown to be quasi-isomorphic to the derived multiplicative preprojective algebra of the corresponding Dynkin type. Moreover, working over $\mathbb{C}$, for $Q = A_\ell$ or D_ℓ, it was shown in [EL17, Theorem 13] that the derived multiplicative preprojective algebra of Dynkin type Q is quasi-isomorphic to the Ginzburg algebra $\mathscr{G}^2_Q$, also known as the derived (additive) preprojective algebra of Dynkin type Q. It was conjectured in op. cit. that the same result holds for $Q = E_6, E_7, E_8$ and this is indeed so. The key ingredient for the proof of [EL17, Theorem 13] to go through that was missing in the case $Q = E_6, E_7, E_8$ was the computation that

$$\mathrm{HH}^2\left(\mathscr{G}^2_Q\right)^s = 0 \text{ for } s < 0, \tag{3.1}$$

but this follows from computations given in Section 5 below.

For $n \geq 3$, one can do a direct computation in an analogous way, but we can also deduce this by the Koszul duality result given in [EL, Theorem 58] which shows that $\mathrm{CE}^*(\Lambda^{n-1}_Q)$ is the (derived) Koszul dual of the endomorphism algebra of the union of the core spheres of the plumbing. Notice that for $n \geq 3$, $\check{\mathbf{w}}$ is suspended at least twice, thus the formality of the endomorphism algebra of vanishing cycles in the compact Fukaya category of $\check{\mathbf{w}}^{-1}(1)$ follows automatically by [Sei10, Proposition 4.4] (the formality of the A_∞-algebra $\mathcal{A}$ and a $\mathcal{A}$-bimodule $\mathcal{B}/\mathcal{A}$ in Seidel's notation is obvious in the case at hand, since Γ is a tree and one can shift the objects to put all morphisms in degree 0). Putting it all together, we conclude that $\mathrm{CE}^*(\Lambda_Q^{n-1})$ is Koszul dual to the degree n trivial extension algebra B_Q^n of the path algebra A_Q of a Dynkin quiver of the corresponding type (see also [Li19] for another example).

As a result of these computations, for $n \geq 2$ we have a quasi-isomorphism

$$\mathrm{CE}^*\left(\Lambda_Q^{n-1}\right) \simeq \mathscr{G}_Q^n \tag{3.2}$$

over $\mathbb{C}$, which implies a quasi-equivalence

$$\mathcal{W}(\check{\mathbf{w}}^{-1}(1)) \simeq \operatorname{perf} \Pi_n(A_Q) \tag{3.3}$$

between the wrapped Fukaya category of $\check{\mathbf{w}}^{-1}(1)$ and the dg category of perfect modules over $\Pi_n(A_Q)$.

Remark 3.1. Note from [Sei01, Proposition 3.4] that A_Q is derived equivalent to the Fukaya–Seidel category $\mathcal{F}(\check{\mathbf{w}})$ of the LG-model $\check{\mathbf{w}}\colon \mathbb{C}^{n+1} \to \mathbb{C}$. Thus (3.3) shows that $\mathcal{W}\left(\check{\mathbf{w}}^{-1}(1)\right)$ is the Calabi–Yau completion of $\mathcal{F}(\check{\mathbf{w}})$ for $n \geq 2$. Although this relationship between $\mathcal{F}(\check{\mathbf{w}})$ and $\mathcal{W}\left(\check{\mathbf{w}}^{-1}(1)\right)$ is not true in general, we expect it to hold when $\check{\mathbf{w}}$ is a double suspension of an invertible polynomial whose Milnor fiber is a log Fano manifold, since one has

$$\begin{aligned} &\mathbf{w}(x_1, \ldots, x_{n-1}) + x_n^2 + x_{n+1}^2 + x_0 \cdots x_{n+1} \\ &\qquad = \mathbf{w}(x_1, \ldots, x_{n-1}) + \left(\sqrt{1 - \frac{1}{4}(x_0 \cdots x_{n-1})^2 x_n}\right)^2 + \left(x_{n+1} + \frac{1}{2} x_0 \cdots x_n\right)^2 \end{aligned} \tag{3.4}$$

in $\mathbf{k}\llbracket x_0, \ldots, x_{n+1} \rrbracket$.

Remark 3.2. The isomorphism (3.2) remains true for $n \geq 3$ over an arbitrary commutative ring, but for $n = 2$ we have to require that 2 is invertible for type D_ℓ, E_6, E_7, E_8, 3 is invertible for type E_6, E_7, E_8, and 5 is invertible for type E_8. Otherwise, $\mathrm{CE}^*(\Lambda_Q)$ is quasi-isomorphic to the derived multiplicative preprojective algebra (see [EL19]) which is not quasi-isomorphic to the derived (additive) preprojective algebra $\Pi_n(A_Q)$.

4. Matrix factorizations and Calabi–Yau completions

Let Γ be a subgroup of $(\mathbb{G}_\mathrm{m})^{n+1}$ acting diagonally on $\mathbb{A}^{n+1} := \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_{n+1}]$. Assume that Γ is a finite extension of the multiplicative group $\mathbb{G}_\mathrm{m}$, so that the group $\operatorname{Char}(\Gamma) := \operatorname{Hom}(\Gamma, \mathbb{G}_\mathrm{m})$ of characters of Γ is an extension of a finite group by $\mathbb{Z}$. The coordinate ring $\mathbb{C}[x_1, \ldots, x_{n+1}]$ has a $\operatorname{Char}(\Gamma)$-grading coming from the Γ-action on $\mathbb{A}^{n+1}$, and we set $\chi_i := \deg x_i$ for $i \in \{1, \ldots, n+1\}$. Let $\mathbf{w} \in \mathbb{C}[x_1, \ldots, x_{n+1}]_\chi$ be a homogeneous element of degree $\chi \in \operatorname{Char}(\Gamma)$. Assume that $\mathbf{w}$ has an isolated critical point at the origin, so that the structure sheaf $\mathcal{O}_0$ of the origin split-generates $\operatorname{mf}(\mathbb{A}^{n+1}, \mathbf{w})$ by [KMVdB11, Proposition A.2] (see also [Orl11, Dyc11]). Let $R \subset \operatorname{Char}(\Gamma)$ be a set of representatives of the group $\operatorname{Char}(\Gamma)/(\chi)$, which we assume to be finite. Then $\mathcal{E} := \bigoplus_{\rho \in R} \mathcal{O}_0(\rho)$ split-generates $\operatorname{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w})$, since the autoequivalence $M \mapsto M(\chi)$ of $\operatorname{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w})$ shifting the Γ-weight by χ is isomorphic to the functor $M \mapsto M[2]$ shifting the cohomological grading by 2.

The n-Calabi–Yau completion of the dg Yoneda algebra $\mathscr{A} := \hom(\mathcal{E}, \mathcal{E})$ is given by

$$\Pi_n(\mathscr{A}) := \mathscr{A} \oplus \theta \oplus \theta \otimes_{\mathscr{A}} \theta \oplus \cdots \simeq \bigoplus_{i=0}^{\infty} \hom(\mathcal{E}, \theta^i(\mathcal{E})) \tag{4.1}$$

where $\theta = \Theta[n-1]$ as in Section 2, and we abuse notation and use the same symbol for an autoequivalence and its graph bimodule. Since Θ is the graph of the inverse Serre functor $\mathbb{S}^{-1}$, we have

$$\theta = \mathbb{S}^{-1}[n-1]. \tag{4.2}$$

Now, as in [LU, Section 2], we introduce another variable x_0 of degree $\chi_0 := \chi - (\chi_1 + \cdots + \chi_{n+1})$, and consider the polynomial ring $\mathbb{C}[x_0, x_1, \ldots, x_{n+1}]$ in $n+2$ variables, which naturally contains $\mathbb{C}[x_1, \ldots, x_{n+1}]$ as a subring. One has

$$\operatorname{mf}(\mathbb{A}^{n+2}, \mathbf{w}) \simeq \operatorname{mf}(\mathbb{A}^1, 0) \otimes \operatorname{mf}(\mathbb{A}^{n+1}, \mathbf{w}) \tag{4.3}$$

e.g., by the ungraded ($G = H = 1$) version of [BFK14, Lemma 3.52] with $v = 0$; note that $\operatorname{mf}(\mathbb{A}^1, 0)$ is obtained from $\operatorname{coh} \mathbb{A}^1$ by collapsing the cohomological grading to $\mathbb{Z}/2\mathbb{Z}$, and the tensor product of split-generators of $\operatorname{mf}(\mathbb{A}^1, 0)$ and $\operatorname{mf}(\mathbb{A}^{n+1}, \mathbf{w})$ gives a split-generator of $\operatorname{mf}(\mathbb{A}^{n+2}, \mathbf{w})$ since the critical locus of $\mathbf{w}$ as a function on $\mathbb{A}^{n+2}$ is the product of $\mathbb{A}^1$ times that as a function on $\mathbb{A}^{n+1}$.

As shown in [IT13, Theorem 2.5] whose proof carries over directly to Γ-graded cases, graded Auslander–Reiten duality [AR87] implies that

$$\mathbb{S} := (\chi_0)[n-1] \tag{4.4}$$

is a Serre functor on $\operatorname{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w})$. It follows from (4.2) and (4.4) that

$$\theta \simeq (-\chi_0). \tag{4.5}$$

Let $\mathcal{F}$ be the generator of $\operatorname{mf}(\mathbb{A}^{n+2}, \Gamma, \mathbf{w})$ obtained from the tensor product of the generator $\mathcal{E}$ of $\operatorname{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w})$ and the generator $\mathbb{C}[x_0]$ of $\operatorname{coh} \mathbb{A}^1$. If we write both of the forgetful functors $\operatorname{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w}) \to \operatorname{mf}(\mathbb{A}^{n+1}, \mathbf{w})$ and $\operatorname{mf}(\mathbb{A}^{n+2}, \Gamma, \mathbf{w}) \to \operatorname{mf}(\mathbb{A}^{n+2}, \mathbf{w})$ as $\overline{(\bullet)}$, then one has

$$\hom\left(\overline{\mathcal{F}}, \overline{\mathcal{F}}\right) \simeq \hom\left(\overline{\mathcal{E}}, \overline{\mathcal{E}}\right) \otimes \mathbb{C}[x_0] \simeq \bigoplus_{\rho \in \operatorname{Char}(\Gamma)} \hom(\mathcal{E}, \mathcal{E}(\rho)) \otimes \mathbb{C}[x_0]. \tag{4.6}$$

Since $\deg(x_0) = \chi_0$, by taking the Γ-invariant part of (4.6) and using (4.5), one obtains

$$\hom(\mathcal{F}, \mathcal{F}) \simeq \bigoplus_{i=0}^{\infty} \hom(\mathcal{E}, \mathcal{E}(-i\chi_0)) \simeq \bigoplus_{i=0}^{\infty} \hom(\mathcal{E}, \theta^i(\mathcal{E})) \simeq \Pi_n(\mathscr{A}), \tag{4.7}$$

which shows the quasi-equivalence (1.7).

If n is greater than one, then the degree of $x_1 \cdots x_{n+1}$ in $\operatorname{Char}(\Gamma_{\mathbf{w}}) \otimes \mathbb{Q} \cong \mathbb{Q}$ is greater than the degree of $\mathbf{w}$, which is turn is greater than the degree of any element of the Jacobi ring

$$\operatorname{Jac}_{\mathbf{w}} := \mathbb{C}[x_1, \ldots, x_{n+1}]/(\partial_{x_1} \mathbf{w}, \ldots, \partial_{x_{n+1}} \mathbf{w}) \tag{4.8}$$

of $\mathbf{w}$, and the proof of [AGZV85, Section 12.6, Theorem] shows that the polynomial $\mathbf{w} + x_0 \cdots x_{n+1}$ considered as an element of $\mathbb{C}[\![x_0]\!][\![x_1, \ldots, x_{n+1}]\!]$ (i.e., a formal one-parameter deformation of a formal germ of $\mathbf{w}$) is right equivalent to $\mathbf{w}$ by a formal coordinate change (i.e., there exists $\varphi \in \operatorname{Aut}_{\mathbb{C}[\![x_0]\!]} \mathbb{C}[\![x_0]\!][\![x_1, \ldots, x_{n+1}]\!]$ such that $\varphi^*(\mathbf{w} + x_0 \cdots x_{n+1}) = \mathbf{w}$). The proof moreover shows that one can choose φ to be $\Gamma_{\mathbf{w}}$-equivariant, which implies that for any $i \in \{1, \ldots, n+1\}$, the coefficient $a_{i,m_1,\ldots,m_{n+1}}(x_0)$ of the expansion $\varphi^*(x_i) = \sum_{m_1,\ldots,m_{n+1}=0}^{\infty} a_{i,m_1,\ldots,m_{n+1}}(x_0) x_1^{m_1} \cdots x_{n+1}^{m_{n+1}}$ is a monomial in x_0, since the degree of x_0 in $\operatorname{Char}(\Gamma_{\mathbf{w}}) \otimes \mathbb{Q}$ is negative. In particular, one has $\varphi \in \operatorname{Aut}_{\mathbb{C}[x_0]} \mathbb{C}[x_0][\![x_1, \ldots, x_{n+1}]\!]$. This means that that the formal completion of $(\mathbb{A}^{n+2}, \mathbf{w} + x_0 \cdots x_{n+1})$ along $x_1 = \cdots = x_{n+1} = 0$ is isomorphic to that of $(\mathbb{A}^{n+2}, \mathbf{w})$ as a pair of a formal scheme and a regular function on it, so that

$$\operatorname{mf}\left(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 \cdots x_{n+1}\right) \simeq \operatorname{mf}\left(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}\right) \tag{4.9}$$

by [Orl11, Theorem 2.10], and the quasi-equivalence (1.6) is proved.

5. Hochschild cohomology of the derived n-preprojective algebra

5.1. **Hochschild cohomology via matrix factorizations.** We use the same notation as in Section 4, and set

$$V := \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_{n+1}. \tag{5.1}$$

For $\gamma \in \Gamma$, let V_γ be the subspace of γ-invariant elements in V, S_γ be the symmetric algebra of V_γ, $\mathbf{w}_\gamma$ be the restriction of $\mathbf{w}$ to $\operatorname{Spec} S_\gamma$, and N_γ be the Γ-stable complement of V_γ in V so that $V \cong V_\gamma \oplus N_\gamma$ as a Γ-module. Then [Dyc11, CT13, Seg13, BFK14] (cf. also [LU, Theorem 3.1]) shows that $\mathrm{HH}^t(\operatorname{mf}(\mathbb{A}^{n+2}, \Gamma, \mathbf{w}))$ is isomorphic to

(5.2)

$$\bigoplus_{\substack{\gamma \in \ker \chi,\ l \geq 0 \\ t - \dim N_\gamma = 2u}} \left(H^{-2l}(d\mathbf{w}_\gamma) \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)_{(u+\ell)\chi} \oplus \bigoplus_{\substack{\gamma \in \ker \chi,\ l \geq 0 \\ t - \dim N_\gamma = 2u+1}} \left(H^{-2l-1}(d\mathbf{w}_\gamma) \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)_{(u+\ell+1)\chi}.$$

Here $H^i(d\mathbf{w}_\gamma)$ is the i-th cohomology of the Koszul complex

$$C^*(d\mathbf{w}_\gamma) := \left\{ \cdots \to \Lambda^2 V_\gamma^\vee \otimes S_\gamma(-2\chi) \to V_\gamma^\vee \otimes S_\gamma(-\chi) \to S_\gamma \right\}, \tag{5.3}$$

where the rightmost term S_γ sits in cohomological degree 0, and the differential is the contraction with

$$d\mathbf{w}_\gamma \in \left(V_\gamma \otimes S_\gamma\right)_\chi. \tag{5.4}$$

If $\mathbf{w}_\gamma$ has an isolated critical point at the origin, then the cohomology of (5.3) is concentrated in degree 0, so that only the summand

$$\left(\mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)_{u\chi} \tag{5.5}$$

with $l = 0$ in (5.2) contributes to $\mathrm{HH}^{2u+\dim N_\gamma}$.

If V_γ contains $\mathbb{C}x_0$, then the Koszul complex $C^*(d\mathbf{w}_\gamma)$ is isomorphic to the tensor product of $C^*(d\mathbf{w}'_\gamma)$ and the complex $\{\mathbb{C}x_0^\vee \otimes \chi^\vee \otimes \mathbb{C}[x_0] \to \mathbb{C}[x_0]\}$ concentrated in cohomological degree $[-1, 0]$ with the zero differential, where $\mathbf{w}'_\gamma$ is the restriction of $\mathbf{w}$ to the complement V'_γ of $\mathbb{C}x_0$ in V_γ. If $\mathbf{w}'_\gamma$ has an isolated critical point at the origin, then $C^*(d\mathbf{w}'_\gamma)$ is quasi-isomorphic to $\mathrm{Jac}_{\mathbf{w}'_\gamma}$ concentrated in cohomological degree 0, so that only the summands

$$\left(\mathrm{Jac}_{\mathbf{w}'_\gamma} \otimes \mathbb{C}[x_0] \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)_{u\chi} \tag{5.6}$$

and

$$\left(\mathbb{C}x_0^\vee \otimes \mathrm{Jac}_{\mathbf{w}'_\gamma} \otimes \mathbb{C}[x_0] \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)_{u\chi} \tag{5.7}$$

with $l = 0$ in (5.2) contribute to $\mathrm{HH}^{2u+\dim N_\gamma}$ and $\mathrm{HH}^{2u+\dim N_\gamma+1}$ respectively.

Remark 5.1. Although (5.2) may not look identical to [BFK14, Theorem 1.2], the proof in [BFK14, Section 5] actually shows (5.2). One way to think about (5.2) is the following: If we set $\mathbf{w} = 0$ and forget Γ, then the Hochschild–Kostant–Rosenberg theorem gives a quasi-isomorphism of the Hochschild cochain complex of $\operatorname{coh} \mathbb{A}^{n+2}$ and

$$S \to V^\vee \otimes S \to \Lambda^2 V^\vee \otimes S \to \cdots \tag{5.8}$$

as complexes of $\mathbb{C}$-vector spaces (which lifts to a quasi-isomorphism of L_∞-algebras by the Kontsevich formality). If we introduce the potential $\mathbf{w}$, then the complex (5.8) acquires an additional differential $\Lambda^i V^\vee \otimes S \to \Lambda^{i-1} V^\vee \otimes S$ defined as the contraction with $d\mathbf{w} \in V \otimes S$, which decreases the cohomological grading by one so that the cohomological grading is collapsed to $\mathbb{Z}/2\mathbb{Z}$. The introduction of Γ lifts the grading to $\mathbb{Z}$ again and produces 'twisted sectors' from the orbifold HKR theorem, leading to (5.2); recall the isomorphism $(\chi) \simeq [2]$ of endofunctors of $\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma, \mathbf{w})$ and the orbifold HKR theorem

$$\mathrm{HH}^*([X/G]) = \left(\bigoplus_{g \in G} \bigoplus_{p+q=*} H^{p-\operatorname{codim} X^g}\left(X^g, \Lambda^q T_{X^g} \otimes \Lambda^{\operatorname{codim} X^g} N_{X^g/X}\right) \right)_G \tag{5.9}$$

for global quotients appearing, e.g., in [ACH19, Corollary 1.17].

Remark 5.2. The Hochschild cohomology of a graded algebra B (with no differential) has a bigrading such that

$$\mathrm{HH}^{r+s}(B)^s := \mathrm{Ext}^r_{B^{\mathrm{op}} \otimes B}(B, B[s]). \tag{5.10}$$

When B is the trivial extension algebra B^n of a finite-dimensional algebra A, by introducing a $\mathbb{G}_\mathrm{m}$-action on B^n such that A has weight 0 and $A^\vee[-n]$ has weight n, the s-grading on $\mathrm{HH}^*(B^n)$ can be described as the weight of the induced $\mathbb{G}_\mathrm{m}$-action.

For any positive integer m, the underlying ungraded algebra of the trivial extension algebras B^{mn} is isomorphic to B^n, and only the cohomological gradings are different; that of the former is m times that of the latter. It follows that one has an isomorphism

$$\mathrm{HH}^{r+ms}(B^{mn})^{ms} \cong \mathrm{HH}^{r+s}(B^n)^s \tag{5.11}$$

of vector spaces for any positive integer m such that the parities of n and mn are the same (note that the signs in the Hochschild complex depend on the parity of the cohomological grading).

When Q is a Dynkin quiver, one can transport the $\mathbb{G}_\mathrm{m}$-action on B_Q^n to $\mathscr{G}_Q^n$ through the Koszul duality (2.4), so that g for $g \in Q_1$ has weight 0, g^* for $g \in Q_1$ has weight $-n$, and h_v for $v \in Q_0$ has weight $-n$. This makes the isomorphism (2.5) $\mathbb{G}_\mathrm{m}$-equivariant, so that the $\mathbb{G}_\mathrm{m}$-weights on both sides agree.

Since $\mathbf{w}$ does not depend on x_0, the $\mathbb{G}_\mathrm{m}$-action on $\mathbb{A}^{n+2}$ such that the weight of x_i is $-n$ for $i = 0$ and 0 for $i \in \{1, \dots, n+1\}$ keeps $\mathbf{w}$ invariant. This induces a $\mathbb{G}_\mathrm{m}$-action on $\operatorname{mf}(\mathbb{A}^{n+2}, \Gamma, \mathbf{w})$, and hence on B_Q^n, whose weight is 0 on A_Q and n on $A_Q^\vee[-n]$ just as in [LU]. This allows us to compute the s-grading on $\mathrm{HH}^*\left(B_Q^n\right)$ as the $\mathbb{G}_\mathrm{m}$-weight on (5.2). This $\mathbb{G}_\mathrm{m}$-action is mirror to the one introduced in [SS12] and studied further for type A Milnor fibers in [Sei12].

5.2. **Type A_ℓ.** Consider the case

$$\mathbf{w} = x_1^{\ell+1} + x_2^2 + \cdots + x_{n+1}^2 \in \mathbb{C}[x_0, x_1, \dots, x_{n+1}] \tag{5.12}$$

with

$$\Gamma = \Gamma_\mathbf{w} := \left\{\gamma = (t_0, t_1, \dots, t_{n+1}) \in (\mathbb{G}_\mathrm{m})^{n+2} \;\middle|\; t_1^{\ell+1} = t_2^2 = \cdots = t_{n+1}^2 = t_0 t_1 \cdots t_{n+1}\right\}, \tag{5.13}$$

so that $\ker\chi \cong \boldsymbol{\mu}_{\ell+1} \times (\boldsymbol{\mu}_2)^n$ and $\operatorname{Char}(\Gamma)$ is generated by χ and $\chi_i = \deg x_i$ for $i \in \{0, \dots, n+1\}$ with relations

$$\chi = (\ell+1)\chi_1 = 2\chi_2 = \cdots = 2\chi_{n+1} = \chi_0 + \cdots + \chi_{n+1}. \tag{5.14}$$

5.2.1. For any $\gamma \in \ker\chi$, one has

$$\operatorname{Jac}_{\mathbf{w}_\gamma} = \begin{cases} \mathbb{C}[x_0] \otimes \mathbb{C}[x_1]/(x_1^\ell) & \mathbb{C}x_0 \oplus \mathbb{C}x_1 \subset V_\gamma, \\ \mathbb{C}[x_0] & \mathbb{C}x_0 \subset V_\gamma \text{ and } \mathbb{C}x_1 \not\subset V_\gamma, \\ \mathbb{C}[x_1]/(x_1^\ell) & \mathbb{C}x_0 \not\subset V_\gamma \text{ and } \mathbb{C}x_1 \subset V_\gamma, \\ \mathbb{C} & \text{otherwise.} \end{cases} \tag{5.15}$$

If we write an element of $\operatorname{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$ as

$$x_0^{k_0} x_1^{k_1} \otimes x_{j_1}^\vee \wedge x_{j_2}^\vee \wedge \ldots \wedge x_{j_s}^\vee, \tag{5.16}$$

where $k_0 = 0$ if $\mathbb{C}x_0 \not\subset V_\gamma$ and $k_1 = 0$ if $\mathbb{C}x_1 \not\subset V_\gamma$, then its degree is given by

$$k_0\chi_0 + k_1\chi_1 - \chi_{j_1} - \cdots - \chi_{j_s}, \tag{5.17}$$

which can be proportional to χ only if V_γ is either V, $\mathbb{C}x_0 \oplus \mathbb{C}x_1$, $\mathbb{C}x_0$, or 0. We now deal with each of these cases in turn.

5.2.2. One has $V_\gamma = V$ if and only if γ is the identity element. The degree of $x_0^{k_0} x_1^{k_1} \in \operatorname{Jac}_\mathbf{w}$ is

$$k_0\chi - (k_0 - k_1)\chi_1 - k_0\chi_2 - \cdots - k_0\chi_{n+1}, \tag{5.18}$$

which is proportional to χ if and only if k_0 is even and $\ell + 1$ divides $k_0 - k_1$. Such an element can be written as

$$\boldsymbol{a}_{k,m} := x_0^{k+m(\ell+1)} x_1^k, \tag{5.19}$$

where $k \in \{0, \dots, \ell-1\}$ and $m \in \mathbb{N}$ satisfies

- if ℓ is even, then the parities of k and m agree, and
- if ℓ is odd, then k is even.

Since

$$\deg\left(x_0^{k+m(\ell+1)} x_1^k\right) = (k + m(\ell+1))\chi - m\chi - \frac{1}{2}(k + m(\ell+1))n\chi \tag{5.20}$$

$$= \left((k + m\ell) - \frac{1}{2}(k + m(\ell+1))n\right)\chi, \tag{5.21}$$

the element $x_0^{k+m(\ell+1)}x_1^k$ for such (k,m) contributes $\mathbb{C}((k+m(\ell+1))n)$ to HH^t for $t = 2(k+m\ell) - (k+m(\ell+1))n$ by (5.6). Similarly, for each such (k,m), the element

$$\boldsymbol{\alpha}_{k,m} := x_0^\vee \otimes x_0^{k+m(\ell+1)+1}x_1^k \in \mathbb{C}x_0^\vee \otimes \mathrm{Jac}_{\mathbf{w}} \tag{5.22}$$

contributes $\mathbb{C}((k+m(\ell+1))n)$ to HH^{t+1} for $t = 2(k+m\ell) - (k+m(\ell+1))n$ by (5.7).

5.2.3. One has $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1$ if and and only if n is even and $\gamma = (1,1,-1,\dots,-1)$. The degree of

$$x_0^{k_0}x_1^{k_1} \otimes x_2^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \tag{5.23}$$

is given by

$$k_0\chi + (k_1 - k_0)\chi_1 - (k_0+1)\chi_2 - \cdots - (k_0+1)\chi_{n+1}, \tag{5.24}$$

which is proportional to χ if and only if k_0 is odd and $\ell+1$ divides $k_1 - k_0$. Such an element can be written as

$$\boldsymbol{a}_{k,m} := x_0^{k+m(\ell+1)}x_1^k \otimes x_2^\vee \wedge \cdots \wedge x_{n+1}^\vee, \tag{5.25}$$

where $k \in \{0,\dots,\ell-1\}$ and $m \in \mathbb{N}$ satisfies

- if ℓ is even, then the parities of k and m differ, and
- if ℓ is odd, then k is odd.

Since the degree of this element is

$$\left((k+m\ell) - \frac{1}{2}(k+m(\ell+1)+1)n\right)\chi, \tag{5.26}$$

each such (k,m) contributes $\mathbb{C}((k+m(\ell+1))n)$ to HH^t for

$$t = 2\left((k+m\ell) - \frac{1}{2}(k+m(\ell+1)+1)n\right) + \dim N_\gamma \tag{5.27}$$

$$= 2(k+m\ell) - (k+m(\ell+1))n. \tag{5.28}$$

Similarly, for each such (k,m), there is an element $\boldsymbol{\alpha}_{k,m}$ contributing $\mathbb{C}((k+m(\ell+1))n)$ to HH^{t+1} for $t = 2(k+m\ell) - (k+m(\ell+1))n$.

5.2.4. One has $V_\gamma = \mathbb{C}x_0$ if and only if both ℓ and n are odd and $\gamma = (1,-1,\dots,-1)$. The degree of

$$x_0^{k_0} \otimes x_1^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \tag{5.29}$$

is given by

$$k_0\chi - (k_0+1)\chi_1 - (k_0+1)\chi_2 - \cdots - (k_0+1)\chi_{n+1}, \tag{5.30}$$

which is proportional to χ if and only if $\ell+1$ divides k_0+1. Such an element can be written as

$$\boldsymbol{b}_m := x_0^{m(\ell+1)-1} \otimes x_1^\vee \wedge \cdots \wedge x_{n+1}^\vee \tag{5.31}$$

for $m \in \mathbb{N} \setminus \{0\}$. Since the degree of this element is

$$\left((m\ell-1) - \frac{1}{2}m(\ell+1)n\right)\chi, \tag{5.32}$$

each such element contributes $\mathbb{C}((m(\ell+1)-1)n)$ to HH^t for

$$t = 2\left((m\ell-1) - \frac{1}{2}m(\ell+1)n\right) + \dim N_\gamma \tag{5.33}$$

$$= (2m\ell-1) - (m(\ell+1)-1)n. \tag{5.34}$$

Similarly, for each $m \in \mathbb{N}$, the element

$$\boldsymbol{\beta}_m := x_0^\vee \otimes x_0^{m(\ell+1)} \otimes x_1^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \mathbb{C}x_0^\vee \otimes \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \tag{5.35}$$

contributes $\mathbb{C}(m(\ell+1)-1)n)$ to HH^{t+1} for $t = (2m\ell-1) - (m(\ell+1)-1)n$.

5.2.5. For $\gamma = (t_0, \dots, t_{n+1}) \in \ker \chi$, one has $V_\gamma = 0$ if and only if $t_i \neq 1$ for all $i \in \{0, \dots, n+1\}$. This is the case if and only if $t_2 = \cdots = t_{n+1} = -1$, $t_1 \in \boldsymbol{\mu}_{\ell+1} \setminus \{1\}$, and

$$t_0 = (-1)^n t_1^{-1} \neq 1. \tag{5.36}$$

If n is odd, then (5.36) holds if and only if $t_1 \neq -1$, so that the number of such γ is ℓ if ℓ is even, and $\ell - 1$ if ℓ is odd. If n is even, then (5.36) always holds, and the number of such γ is ℓ. Each such γ contributes $\mathbb{C}(-n)$ to HH^n.

5.2.6. To sum up, the Hochschild cohomology group has a basis consisting of the following elements:

- $\boldsymbol{a}_{k,m}$ of degree $2(k + m\ell) - (k + m(\ell+1))n$ and weight $-(k + m(\ell+1))n$, and $\boldsymbol{\alpha}_{k,m}$ of degree $2(k + m\ell) - (k + m(\ell+1))n + 1$ and weight $-(k + m(\ell+1))n$, where
  - if n is even, then (k, m) runs over $\{0, \dots, \ell-1\} \times \mathbb{N}$, and
  - if n is odd, then (k, m) runs over those pairs in $\{0, \dots, \ell-1\} \times \mathbb{N}$ for which
    - the parities of k and m agree, if ℓ is even,
    - k is even, if ℓ is odd,
- if both n and ℓ are odd, then
  - $\boldsymbol{b}_m$ of degree $2m\ell - 1 - (m(\ell+1) - 1)n$ and weight $-(m(\ell+1) - 1)n$ for $m \in \mathbb{N} \setminus \{0\}$, and
  - $\boldsymbol{\beta}_m$ of degree $2m\ell - (m(\ell+1) - 1)n$ and weight $-(m(\ell+1) - 1)n$ for $m \in \mathbb{N}$, and
- $\boldsymbol{s}_h$ of degree n and weight n, where h runs over
  - $\{1, 2, \dots, \ell-1\}$ if both ℓ and n are odd, and
  - $\{1, 2, \dots, \ell\}$ otherwise.

5.2.7. As an example, consider the case $\ell = 1$. Note that the A_1-Milnor fiber is symplectomorphic to the cotangent bundle T^*S^n. The Hochschild cohomology group in this case is spanned by

- $\boldsymbol{a}_{0,m}$ for $m \in \mathbb{N}$ of degree $-2m(n-1)$ and weight $-2mn$,
- $\boldsymbol{\alpha}_{0,m}$ for $m \in \mathbb{N}$ of degree $-2m(n-1) + 1$ and weight $-2mn$,

and, if n is odd, in addition to the above,

- $\boldsymbol{b}_m$ for $m \in \mathbb{N} \setminus \{0\}$ of degree $-(2m-1)(n-1)$ and weight $-(2m-1)n$,
- $\boldsymbol{\beta}_m$ for $m \in \mathbb{N}$ of degree $-(2m-1)(n-1) + 1$ and weight $-(2m-1)n$,

and, if n is even, in addition to the above,

- $\boldsymbol{s}_1$ of degree n and weight n.

This is consistent with the isomorphism

$$\mathrm{SH}^*(T^*S^n) \cong H_{n-*}(\mathscr{L}S^n), \tag{5.37}$$

which is a special case of the isomorphism between the symplectic cohomology of the cotangent bundle and the homology of the free loop space [Vit, Theorem 3.1] (see e.g. [CJY04, Theorem 2] for the homology of the free loop space of spheres).

Another example is the case when $n = 2$ and ℓ is arbitrary. In this case, $\mathrm{SH}^*(\check{\mathbf{w}}^{-1}(1))$ was computed in [EL17] as a bigraded ring. This is compatible with the computation given here.

5.3. **Type D_ℓ.** The Berglund–Hübsch transform of the invertible polynomial

$$\check{\mathbf{v}} = y_1^{\ell-1} + y_1 y_2^2 + y_3^2 + \cdots + y_{n+1}^2 \tag{5.38}$$

defining the D_ℓ-singularity is given by

$$\mathbf{v} = y_1^{\ell-1} y_2 + y_2^2 + \cdots + y_{n+1}^2, \tag{5.39}$$

and one has

$$\Gamma_{\mathbf{v}} = \left\{ \gamma = (t_1, \dots, t_{n+1}) \in (\mathbb{G}_\mathrm{m})^{n+1} \;\middle|\; t_1^{\ell-1} t_2 = t_2^2 = \cdots = t_{n+1}^2 \right\}. \tag{5.40}$$

By completing the square and rescaling, one has

$$\mathbf{v}(y) = \mathbf{w}(x(y)) \tag{5.41}$$

where

$$x_1 = (-1/4)^{1/(2n-2)} y_1,\ x_2 = y_2 + \frac{1}{2} y_1^{\ell-1},\ x_3 = y_3,\ \dots,\ x_{n+1} = y_{n+1} \tag{5.42}$$

and

$$\mathbf{w} = x_1^{2\ell-2} + x_2^2 + \cdots + x_{n+1}^2. \tag{5.43}$$

Although the change of variables (5.42) is neither linear nor diagonal, the induced action of $\Gamma_{\mathbf{v}}$ on $\operatorname{Spec} \mathbb{C}[x_1, \dots, x_{n+1}]$ remains linear and diagonal, so that one can identify $\Gamma_{\mathbf{v}}$ with a proper subgroup of $\Gamma_{\mathbf{w}}$.

Therefore, we will work with

$$\mathbf{w} = x_1^{2\ell-2} + x_2^2 + \cdots + x_{n+1}^2 \in \mathbb{C}[x_0, x_1, \dots, x_{n+1}] \tag{5.44}$$

with the non-maximal group

$$\Gamma = \left\{ \gamma = (t_0, t_1, \dots, t_{n+1}) \in (\mathbb{G}_{\mathrm{m}})^{n+2} \;\middle|\; t_1^{\ell-1} t_2 = t_2^2 = \cdots = t_{n+1}^2 = t_0 t_1 \cdots t_{n+1} \right\}. \tag{5.45}$$

One has $\ker \chi \cong \boldsymbol{\mu}_{2\ell-2} \times (\boldsymbol{\mu}_2)^{n-1}$ and $\operatorname{Char}(\Gamma)$ is generated by χ and $\chi_i = \deg x_i$ for $i \in \{0, \dots, n+1\}$ with relations

$$\chi = (\ell-1)\chi_1 + \chi_2 = 2\chi_2 = \cdots = 2\chi_{n+1} = \chi_0 + \cdots + \chi_{n+1}. \tag{5.46}$$

The relations (5.46) imply

$$\begin{aligned} \chi_2 &= \chi - (\ell-1)\chi_1, && (5.47)\\ \chi &= (2\ell-2)\chi_1, && (5.48)\\ \chi_0 &= \chi - \chi_1 - \cdots - \chi_n && (5.49)\\ &= (\ell-2)\chi_1 - \chi_3 - \cdots - \chi_{n+1}. && (5.50) \end{aligned}$$

5.3.1. For any $\gamma = (t_0, \dots, t_{n+1}) \in \ker \chi$, one has

$$\operatorname{Jac}_{\mathbf{w}_\gamma} = \begin{cases} \mathbb{C}[x_0] \otimes \mathbb{C}[x_1]/(x_1^{2\ell-3}) & \mathbb{C}x_0 \oplus \mathbb{C}x_1 \subset V_\gamma, \\ \mathbb{C}[x_0] & \mathbb{C}x_0 \subset V_\gamma \text{ and } \mathbb{C}x_1 \not\subset V_\gamma, \\ \mathbb{C}[x_1]/(x_1^{2\ell-3}) & \mathbb{C}x_0 \not\subset V_\gamma \text{ and } \mathbb{C}x_1 \subset V_\gamma, \\ \mathbb{C} & \text{otherwise.} \end{cases} \tag{5.51}$$

If we write an element of $\operatorname{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$ as

$$x_0^{k_0} x_1^{k_1} \otimes x_{j_1}^\vee \wedge x_{j_2}^\vee \wedge \ldots \wedge x_{j_s}^\vee, \tag{5.52}$$

where $k_0 = 0$ if $\mathbb{C}x_0 \not\subset V_\gamma$ and $k_1 = 0$ if $\mathbb{C}x_1 \not\subset V_\gamma$, then its degree is given by

$$k_0\chi_0 + k_1\chi_1 - \chi_{j_1} - \cdots - \chi_{j_s}, \tag{5.53}$$

which can be proportional to χ only if

$$V_\gamma \cap (\mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1}) \text{ is either } \mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1} \text{ or } 0, \tag{5.54}$$

that is,

$$t_3 = \cdots = t_{n+1} = \pm 1. \tag{5.55}$$

We will assume this condition for the rest of Section 5.3.

5.3.2. One has $\gamma = (t_0, \dots, t_{n+1}) \in \ker \chi$ if and only if

$$t_1^{\ell-1} t_2 = t_2^2 = \cdots = t_{n+1}^2 = t_0 t_1 \cdots t_{n+1} = 1. \tag{5.56}$$

If $t_0 = 1$, then one has $t_2^2 = \cdots = t_{n+1}^2 = t_1 \cdots t_{n+1} = 1$, so that $t_1 = (t_2 \cdots t_{n+1})^{-1} = \pm 1$.

If $t_1 = 1$, then one has $t_2 = t_2^2$, so that $t_2 = 1$ and $t_3^2 = \cdots = t_{n+1}^2 = t_0 t_3 \cdots t_{n+1} = 1$. Under the assumption (5.55), one has $t_0 = 1$ if and only if $(t_3)^{n-1} = 1$, that is, $t_3 = 1$ or n is even.

If $t_1 = -1$, then $t_2 = (-1)^{\ell-1}$, and one has $t_0 = 1$ if and only if $(-1)^\ell t_3^{n-1} = 1$. It follows that

- V_γ contains $\mathbb{C}x_0$ if and only if
  - $\gamma = (1, \dots, 1)$, where $V_\gamma = V$,
  - $\gamma = (1, 1, 1, -1, \dots, -1)$ with odd n, where $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$,
  - $\gamma = (1, -1, -1, 1, \dots, 1)$ with even ℓ, where $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1}$,
  - $\gamma = (1, -1, -1, -1, \dots, -1)$ with even ℓ and odd n, where $V_\gamma = \mathbb{C}x_0$,
  - $\gamma = (1, -1, 1, -1, \dots, -1)$ with odd ℓ and even n, where $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_2$.

5.3.3. Note for later use that the smallest positive integer k such that the degree of x_0^k is proportional to χ is $2\ell - 2$. One has

$$\deg x_0^{2\ell-2} = (2\ell - 2)(\chi - \chi_1 - \cdots - \chi_{n+1}) \tag{5.57}$$

$$= ((2\ell - 3) - (\ell - 1)n)\, \chi. \tag{5.58}$$

5.3.4. One has $V_\gamma = V$ if and only if γ is the identity element. The degree of $x_0^{k_0} x_1^{k_1} \in \operatorname{Jac}_{\mathbf{w}}$ is

$$k_0 \chi - (k_0 - k_1)\chi_1 - k_0 \chi_2 - \cdots - k_0 \chi_{n+1}, \tag{5.59}$$

which is proportional to χ if and only if k_0 is even and $2\ell - 2$ divides $k_0 - k_1$. Such an element can be written as

$$\boldsymbol{a}_{k,m} := x_0^{2k+(2\ell-2)m} x_1^{2k} \tag{5.60}$$

for $(k, m) \in \{0, \dots, \ell - 2\} \times \mathbb{N}$ which contributes $\mathbb{C}((2k + (2\ell - 2)m)n)$ to HH^t for $t = 4k + (4\ell - 6)m - (2k + (2\ell - 2)m)n$ since

$$\deg x_0^{2k} x_1^{2k} = (2k - kn)\chi. \tag{5.61}$$

Similarly, for each $(k, m) \in \{0, \dots, \ell - 2\} \times \mathbb{N}$, there is an element $\boldsymbol{\alpha}_{k,m}$ contributing $\mathbb{C}((2k + (2\ell - 2)m)n)$ to HH^t for $t = 4k + 1 + (4\ell - 6)m - (2k + (2\ell - 2)m)n$.

5.3.5. One has $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$ if and only if $\gamma = (1, 1, 1, -1, \dots, -1)$ and n is odd. The degree of $x_0^{k_0} x_1^{k_1} \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \operatorname{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$ is

$$k_0 \chi - (k_0 - k_1)\chi_1 - k_0 \chi_2 - (k_0 + 1)\chi_3 - \cdots - (k_0 + 1)\chi_{n+1}, \tag{5.62}$$

which is proportional to χ if and only if k_0 is odd and $2\ell - 2$ divides $k_0 - k_1 - (\ell - 1)$. Such an element can be written as

$$\boldsymbol{b}_{k,m} := x_0^{k+\ell-1+(2\ell-2)m} x_1^k \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \tag{5.63}$$

for

$$(k, m) \in \{(k, m) \in \{0, \dots, 2\ell - 4\} \times \mathbb{Z} \mid k + \ell \text{ is even and } k + \ell - 1 + m(2\ell - 2) \geq 0\}. \tag{5.64}$$

It contributes $\mathbb{C}((k + \ell - 1 + (2\ell - 2)m)n)$ to HH^t for

$$t = 2 \deg \left(x_0^{k+\ell-1+(2\ell-2)m} x_1^k \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \right) / \chi + \dim N_\gamma \tag{5.65}$$

$$= 2k + 2\ell - 3 + (4\ell - 6)m - (k + \ell - 1 - (2\ell - 2)m)n, \tag{5.66}$$

since

$$\deg \left(x_0^{k+\ell-1} x_1^k \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \right) \tag{5.67}$$

$$= (k + \ell - 1)\chi - (\ell - 1)\chi_1 - (k + \ell - 1)\chi_2 - (k + \ell)\chi_3 - \cdots - (k + \ell)\chi_{n+1} \tag{5.68}$$

$$= \left(k + \ell - 1 - \frac{1}{2}(k + \ell)n \right) \chi. \tag{5.69}$$

Similarly, for each

$$(k,m) \in \{(k,m) \in \{0,\dots,2\ell-4\} \times \mathbb{Z} \mid k+\ell \text{ is even and } k+\ell+m(2\ell-2) \geq 0\}, \tag{5.70}$$

the element

$$\boldsymbol{\beta}_{k,m} := x_0^\vee \otimes x_0^{k+\ell+(2\ell-2)m} x_1^k \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \left((\mathbb{C}x_0)^\vee \otimes \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)^\Gamma \tag{5.71}$$

contributes $\mathbb{C}(k+\ell-1+(2\ell-2)m)$ to HH^t for

$$t = 2k+2\ell-2+(4\ell-6)m-(k+\ell-1-(2\ell-2)m)n. \tag{5.72}$$

5.3.6. One has $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1}$ if and only if $\gamma = (1,-1,-1,1,\dots,1)$ with even ℓ. An element of $\mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$ whose degree is proportional to χ can be written as

$$\boldsymbol{c}_m := x_0^{\ell-2+(2\ell-2)m} \otimes x_1^\vee \wedge x_2^\vee \tag{5.73}$$

for $m \in \mathbb{N}$, which contributes $\mathbb{C}((\ell-2+(2\ell-2)m)n)$ to HH^t for

$$t = 2\deg\left(x_0^{\ell-2} \otimes x_1^\vee \wedge x_2^\vee\right)/\chi + \dim N_\gamma \tag{5.74}$$

$$= 2\ell-4+(4\ell-6)m-(\ell-2+(2\ell-2)m)n \tag{5.75}$$

since

$$\deg\left(x_0^{\ell-2} \otimes x_1^\vee \wedge x_2^\vee\right) = \left(\ell-3-\frac{1}{2}(\ell-2)n\right)\chi. \tag{5.76}$$

Similarly, for each $m \in \mathbb{N}$, there is an element $\boldsymbol{\gamma}_m$ contributing $\mathbb{C}((\ell-2+(2\ell-2)m)n)$ to HH^t for

$$t = 2\ell-3+(4\ell-6)m-(\ell-2+(2\ell-2)m)n. \tag{5.77}$$

5.3.7. One has $V_\gamma = \mathbb{C}x_0$ if and only if ℓ is even, n is odd, and $\gamma = (1,-1,\dots,-1) \in \ker\chi$. The degree of

$$x_0^{k_0} \otimes x_1^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \tag{5.78}$$

is given by

$$k_0\chi-(k_0+1)\chi_1-(k_0+1)\chi_2-\cdots-(k_0+1)\chi_{n+1}, \tag{5.79}$$

which is proportional to χ if and only if $2\ell-2$ divides k_0+1. Such an element can be written as

$$\boldsymbol{d}_m := x_0^{-1+m(2\ell-2)} \otimes x_1^\vee \wedge \cdots \wedge x_{n+1}^\vee \tag{5.80}$$

for $m \in \mathbb{N} \setminus \{0\}$. Since

$$\deg\left(x_0^{-1} \otimes x_1^\vee \wedge \cdots \wedge x_{n+1}^\vee\right) = -\chi, \tag{5.81}$$

each such element contributes $\mathbb{C}((-1+(2\ell-2)m)n)$ to HH^t for

$$t = 2\deg\left(x_0^{-1+(2\ell-2)m} \otimes x_1^\vee \wedge \cdots \wedge x_{n+1}^\vee\right)/\chi + \dim N_\gamma \tag{5.82}$$

$$= -1+(4\ell-6)m-(-1+(2\ell-2)m)n. \tag{5.83}$$

Similarly, for each $m \in \mathbb{N}$, the element

$$\boldsymbol{\delta}_m := x_0^\vee \otimes x_0^{m(2\ell-2)} \otimes x_1^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \mathbb{C}x_0^\vee \otimes \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \tag{5.84}$$

contributes $\mathbb{C}((-1+(2\ell-2)m)n)$ to HH^t for

$$t = (4\ell-6)m-(-1+(2\ell-2)m)n. \tag{5.85}$$

5.3.8. One has $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_2$ if and only if ℓ is odd, n is even, and $\gamma = (1, -1, 1, -1, \dots, -1)$. The degree of

$$x_0^{k_0} \otimes x_1^\vee \wedge x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \operatorname{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \tag{5.86}$$

is given by

$$k_0\chi_0 - \chi_1 - \chi_3 - \cdots - \chi_{n+1} = (k_0(\ell-2)-1)\chi_1 - (k_0+1)\chi_3 - \cdots - (k_0+1)\chi_{n+1}, \tag{5.87}$$

which is proportional to χ if and only if k_0 is odd and $2\ell-2$ divides $k_0(\ell-2)-1$. Such an element can be written as

$$\boldsymbol{e}_m := x_0^{\ell-2+(2\ell-2)m} \otimes x_1^\vee \wedge x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \tag{5.88}$$

for $m \in \mathbb{N}$, which contributes $\mathbb{C}((\ell-2+(2\ell-2)m)n)$ to HH^t for

$$t = 2\ell - 4 + (4\ell-6)m - (\ell-2+(2\ell-2)m)n \tag{5.89}$$

since

$$\deg\left(x_0^{\ell-2} \otimes x_1^\vee \wedge x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee\right) = \frac{1}{2}\left(2\ell - 4 - (\ell-1)n\right)\chi. \tag{5.90}$$

Similarly, for each $m \in \mathbb{N}$, there is an element $\boldsymbol{\epsilon}_m$ contributing $\mathbb{C}((\ell-2+(2\ell-2)m)n)$ to HH^t for

$$t = 2\ell - 3 + (4\ell-6)m - (\ell-2+(2\ell-2)m)n. \tag{5.91}$$

5.3.9. Now we move on to the case when $\mathbb{C}x_0 \not\subset V_\gamma$. We divide it into three cases:

- $\mathbb{C}x_1 \subset V_\gamma$.
- $\mathbb{C}x_1 \not\subset V_\gamma$ and $V_\gamma \neq 0$.
- $V_\gamma = 0$.

5.3.10. Set $\zeta := \exp\left(2\pi\sqrt{-1}/(2\ell-2)\right)$. For a given $\gamma = (t_0, \dots, t_{n+1}) \in \ker\chi$, we write $t_1 = \zeta^p$ for $p \in \{0, \dots, 2\ell-3\}$. Then one has $t_2 = (-1)^p$, so that

- V_γ contains $\mathbb{C}x_1$ if and only if $p = 0$, and
- V_γ contains $\mathbb{C}x_2$ if and only if p is even.

5.3.11. If $\mathbb{C}x_0 \not\subset V_\gamma$ and $\mathbb{C}x_1 \subset V_\gamma$, then one has that $\gamma = (-1, 1, 1, -1, \dots, -1)$, n is even, and $V_\gamma = \mathbb{C}x_1 \oplus \mathbb{C}x_2$. The element

$$x_1^{\ell-2} \otimes x_0^\vee \wedge x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \tag{5.92}$$

has degree

$$(\ell-2)\chi_1 - \chi_0 - \chi_3 - \cdots - \chi_{n+1} = 0, \tag{5.93}$$

so that it contributes $\mathbb{C}(-n)$ to HH^t for $t = \dim N_\gamma = n$, and this is the only contribution.

5.3.12. If $\mathbb{C}x_0 \not\subset V_\gamma$, $\mathbb{C}x_1 \not\subset V_\gamma$, and $V_\gamma \neq 0$, then V_γ is either $V_\gamma = \mathbb{C}x_2$, $\mathbb{C}x_2 \oplus \cdots \oplus \mathbb{C}x_{n+1}$, or $\mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1}$. No such γ does not contribute to HH^*, since $\operatorname{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$ is spanned by a single element, whose degree is not proportional to χ.

5.3.13. One has $V_\gamma = 0$ if and only if $t_3 = \cdots = t_{n+1} = -1$, $t_1 = \zeta^{2m+1}$ for $m \in \{0, \dots, \ell-2\}$, and

$$t_0 = (-1)^n \zeta^{-2m-1} \neq 1. \tag{5.94}$$

The number of such γ is $\ell - 2$ if ℓ is even and n is odd, and $\ell - 1$ otherwise. Each such γ contributes $\mathbb{C}(-n)$ to HH^n.

5.3.14. To sum up, the Hochschild cohomology group has a basis consisting of the following elements:

- $\boldsymbol{a}_{k,m}$ of degree $4k + (4\ell - 6)m - (2k + (2\ell - 2)m)n$ and weight $-(2k + (2\ell - 2)m)n$ for $(k, m) \in \{0, \dots, \ell - 2\} \times \mathbb{N}$,
- $\boldsymbol{\alpha}_{k,m}$ of degree $4k + 1 + (4\ell - 6)m - (2k + (2\ell - 2)m)n$ and weight $-(2k + (2\ell - 2)m)n$ for $(k, m) \in \{0, \dots, \ell - 2\} \times \mathbb{N}$,
- if n is odd, $\boldsymbol{b}_{k,m}$ of degree $2k+2\ell-3+(4\ell-6)m-(k+\ell-1-(2\ell-2)m)n$ and weight $-(k+\ell-1+(2\ell-2)m)n$ for $\{(k, m) \in \{0, \dots, 2\ell - 4\} \times \mathbb{Z} \mid k + \ell \text{ is even and } k + \ell - 1 + m(2\ell - 2) \geq 0\}$,
- if n is odd, $\boldsymbol{\beta}_{k,m}$ of degree $2k+2\ell-2+(4\ell-6)m-(k+\ell-1-(2\ell-2)m)n$ and weight $-(k+\ell-1-(2\ell-2)m)n$ for $\{(k, m) \in \{0, \dots, 2\ell - 4\} \times \mathbb{Z} \mid k + \ell \text{ is even and } k + \ell + m(2\ell - 2) \geq 0\}$,
- if ℓ is even, $\boldsymbol{c}_m$ of degree $2\ell-4+(4\ell-6)m-(\ell-2+(2\ell-2)m)n$ and weight $-(\ell-2+(2\ell-2)m)n$ for $m \in \mathbb{N}$,
- if ℓ is even, $\boldsymbol{\gamma}_m$ of degree $2\ell-3+(4\ell-6)m-(\ell-2+(2\ell-2)m)n$ and weight $-(\ell-2+(2\ell-2)m)n$ for $m \in \mathbb{N}$,
- if ℓ is even and n is odd, $\boldsymbol{d}_m$ of degree $-1 + (4\ell - 6)m - (-1 + (2\ell - 2)m)n$ and weight $-(-1 + (2\ell - 2)m)n$ for $m \in \mathbb{N} \setminus \{0\}$,
- if ℓ is even and n is odd, $\boldsymbol{\delta}_m$ of degree $(4\ell - 6)m - (-1 + (2\ell - 2)m)n$ and weight $-(-1 + (2\ell - 2)m)n$ for $m \in \mathbb{N}$,
- if ℓ is odd and n is even, $\boldsymbol{e}_m$ of degree $2\ell - 4 + (4\ell - 6)m - (\ell - 2 + (2\ell - 2)m)n$ and weight $-(\ell - 2 + (2\ell - 2)m)n$ for $m \in \mathbb{N}$,
- if ℓ is odd and n is even, $\boldsymbol{\epsilon}_m$ of degree $2\ell - 3 + (4\ell - 6)m - (\ell - 2 + (2\ell - 2)m)n$ and weight $-(\ell - 2 + (2\ell - 2)m)n$ for $m \in \mathbb{N}$, and
- $\boldsymbol{s}_h$ of degree n and weight n, where h runs over a set consisting of
  - $\ell - 2$ elements if ℓ is even and n is odd,
  - $\ell - 1$ elements if both ℓ and n are odd, and
  - ℓ elements otherwise.

5.4. **Type E_6.** Consider the case

$$\mathbf{w} = x_1^4 + x_2^3 + x_3^2 + \cdots + x_{n+1}^2 \in \mathbb{C}[x_0, x_1, \dots, x_{n+1}] \tag{5.95}$$

with

$$\Gamma = \Gamma_{\mathbf{w}} := \left\{\gamma = (t_0, t_1, \dots, t_{n+1}) \in (\mathbb{G}_{\mathrm{m}})^{n+2} \;\middle|\; t_1^4 = t_2^3 = t_3^2 = \cdots = t_{n+1}^2 = t_0 t_1 \cdots t_{n+1}\right\}, \tag{5.96}$$

so that $\ker\chi \cong \boldsymbol{\mu}_4 \times \boldsymbol{\mu}_3 \times (\boldsymbol{\mu}_2)^{n-1}$ and $\operatorname{Char}(\Gamma)$ is generated by χ and $\chi_i = \deg x_i$ for $i \in \{0, \dots, n+1\}$ with relations

$$\chi = 4\chi_1 = 3\chi_2 = 2\chi_3 = \cdots = 2\chi_{n+1} = \chi_0 + \cdots + \chi_{n+1}. \tag{5.97}$$

5.4.1. For any $\gamma \in \ker\chi$, one has

$$\operatorname{Jac}_{\mathbf{w}_\gamma} \cong \begin{cases} \mathbb{C}[x_0] & \mathbb{C}x_0 \subset V_\gamma \\ \mathbb{C} & \mathbb{C}x_0 \not\subset V_\gamma \end{cases} \otimes \begin{cases} \mathbb{C}[x_1]/(x_1^3) & \mathbb{C}x_1 \subset V_\gamma \\ \mathbb{C} & \mathbb{C}x_1 \not\subset V_\gamma \end{cases} \otimes \begin{cases} \mathbb{C}[x_2]/(x_2^2) & \mathbb{C}x_2 \subset V_\gamma \\ \mathbb{C} & \mathbb{C}x_2 \not\subset V_\gamma. \end{cases} \tag{5.98}$$

If we write an element of $\operatorname{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$ as

$$x_0^{k_0} x_1^{k_1} x_2^{k_2} \otimes x_{j_1}^\vee \wedge x_{j_2}^\vee \wedge \ldots \wedge x_{j_s}^\vee, \tag{5.99}$$

where $k_i = 0$ if $\mathbb{C}x_i \not\subset V_\gamma$ for $i = 0, 1, 2$, then its degree is given by

$$k_0\chi_0 + k_1\chi_1 + k_2\chi_2 - \chi_{j_1} - \cdots - \chi_{j_s}, \tag{5.100}$$

which can be proportional to χ only if $V_\gamma \cap (\mathbb{C}x_3 \oplus \cdots \mathbb{C}x_{n+1})$ is either $\mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1}$ or 0. We will assume this condition for the rest of Section 5.4, and divide the analysis into the following three cases:

- $\mathbb{C}x_0 \subset V_\gamma$.
- $\mathbb{C}x_0 \not\subset V_\gamma$ and $V_\gamma \neq 0$.
- $V_\gamma = 0$.

5.4.2. Since $t_0 = 1$ implies $t_2 = 1$ and $t_1 = \pm 1$, one has the following:

- V_γ contains $\mathbb{C}x_0$ if and only if either
  - $\gamma = (1, \dots, 1)$, where $V_\gamma = V$,
  - $\gamma = (1, 1, 1, -1, \dots, -1)$ with odd $n \geq 3$, where $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$,
  - $\gamma = (1, -1, 1, -1, \dots, -1)$ with even n, where $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_2$.

5.4.3. One has $V_\gamma = V$ if and only if γ is the identity element. The degree of $x_0^{k_0} x_1^{k_1} x_2^{k_2} \in \operatorname{Jac}_{\mathbf{w}}$ is

$$k_0\chi - (k_0 - k_1)\chi_1 - (k_0 - k_2)\chi_2 - k_0\chi_3 - \cdots - k_0\chi_{n+1}, \tag{5.101}$$

which is proportional to χ if and only if

- 4 divides $k_0 - k_1$ and 3 divides $k_0 - k_2$ if $n = 1$, and
- 4 divides $k_0 - k_1$, 3 divides $k_0 - k_2$, and k_0 is even if $n > 1$.

Thus, for $n = 1$, we must have

$$5k_0 + 3k_1 + 4k_2 = 12m \tag{5.102}$$

for $m \in \mathbb{N}$, in which case one has

$$\deg x_0^{k_0} x_1^{k_1} x_2^{k_2} = m\chi. \tag{5.103}$$

For each $m \in \mathbb{N}$ such that $5 \nmid m$, the equation (5.102) has a unique solution with $(k_1, k_2) \in \{0, 1, 2\} \times \{0, 1\}$ and if $5 \mid m$, then there are precisely two contributions with $(k_1, k_2) = (0, 0)$ and $(k_1, k_2) = (2, 1)$ such that $(k_1, k_2, m) \in \{0, 1, 2\} \times \{0, 1\} \times \mathbb{N}$ except if $m = 0$, then only $(k_1, k_2) = (0, 0)$ contributes. Each such (k_1, k_2, m) contributes $\mathbb{C}(k_0 n)$ to HH^t and HH^{t+1} for $t = 2m$.

For $n > 1$, the condition that k_0 is even forces $k_1 \neq 1$, and the possible (k_0, k_1, k_2) and $t = 2\deg(x_0^{k_0} x_1^{k_1} x_2^{k_2})/\chi$ are given by

(5.104)

(k_1, k_2)	k_0	t
$(0, 0)$	$12m$	$22m - 12mn$
$(0, 1)$	$4 + 12m$	$8 + 22m - (4 + 12m)n$
$(2, 0)$	$6 + 12m$	$12 + 22m - (6 + 12m)n$
$(2, 1)$	$10 + 12m$	$20 + 22m - (10 + 12m)n$

for $m \in \mathbb{N}$. Each (k_0, k_1, k_2) from (5.104) contributes $\mathbb{C}(k_0 n)$ to HH^t and HH^{t+1}.

5.4.4. One has $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2 \subsetneq V$ for $\gamma \in \ker \chi$ if and only if n is an odd integer greater than or equal to 3 and $\gamma = (1, 1, 1, -1, \dots, -1)$. The degree of

$$x_0^{k_0} x_1^{k_1} x_2^{k_2} \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \operatorname{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \tag{5.105}$$

is given by

$$k_0\chi - (k_0 - k_1)\chi_1 - (k_0 - k_2)\chi_2 - (k_0 + 1)\chi_3 - \cdots - (k_0 + 1)\chi_{n+1}, \tag{5.106}$$

which is proportional to χ if and only if k_0 is odd, 4 divides $k_0 - k_1$, and 3 divides $k_0 - k_2$. This forces $k_1 = 1$ and the possible (k_0, k_1, k_2) and

$$t = 2\deg(x_0^{k_0} x_1^{k_1} x_2^{k_2} \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee)/\chi + \dim N_\gamma \tag{5.107}$$

are given by

(5.108)

(k_1, k_2)	k_0	t
$(1, 0)$	$9 + 12m$	$17 + 22m - (9 + 12m)n$
$(1, 1)$	$1 + 12m$	$3 + 22m - (1 + 12m)n$

for $m \in \mathbb{N}$. Each (k_0, k_1, k_2) from (5.108) contributes $\mathbb{C}(k_0 n)$ to HH^t and HH^{t+1}.

5.4.5. One has $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_2$ if and only if n is even and $\gamma = (1,-1,1,-1,\dots,-1) \in \ker\chi$. The degree of

$$x_0^{k_0} x_2^{k_2} \otimes x_1^\vee \wedge x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \operatorname{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \tag{5.109}$$

is given by

$$k_0\chi - (k_0+1)\chi_1 - (k_0-k_2)\chi_2 - (k_0+1)\chi_3 - \cdots - (k_0+1)\chi_{n+1}, \tag{5.110}$$

which is proportional to χ if and only if 4 divides k_0+1 and 3 divides k_0-k_2. The possible (k_0,k_2) and

$$t = 2\deg(x_0^{k_0} x_2^{k_2} \otimes x_1^\vee \wedge x_3^\vee \wedge \cdots \wedge x_n^\vee)/\chi + \dim N_\gamma \tag{5.111}$$

are given by

(5.112)

k_2	k_0	t
0	$3+12m$	$6+22m-(3+12m)n$
1	$7+12m$	$14+22m-(7+12m)n$

for $m \in \mathbb{N}$. Each (k_0,k_2) from (5.112) contributes $\mathbb{C}(k_0 n)$ to HH^t and HH^{t+1}.

5.4.6. If $V_\gamma = \mathbb{C}x_1$, then one has

$$\begin{aligned} \deg\left(x_1^{k_1} \otimes x_0^\vee \wedge x_2^\vee \wedge \cdots \wedge x_{n+1}^\vee\right) &= -\chi_0 + k_1\chi_1 - \chi_2 - \cdots - \chi_{n+1} && (5.113)\\ &= -\chi + (k_1+1)\chi_1, && (5.114)\end{aligned}$$

which is not proportional to χ for any $k_1 \in \{0,1,2\}$. Similarly, γ with $\mathbb{C}x_0 \not\subset V_\gamma$ and $V_\gamma \neq 0$ does not contribute to HH^*.

5.4.7. One has $V_\gamma = 0$ if and only if $t_1 \in (\boldsymbol{\mu}_4 \setminus \{1\})$, $t_2 \in (\boldsymbol{\mu}_3 \setminus \{1\})$, and $t_3 = \cdots, t_{n+1} = -1$, since $t_2 \neq 1$ implies $t_0 = (-1)^{n-1} t_1^{-1} t_2^{-1} \neq 1$. There are six such γ, and each of them contributes $\mathbb{C}(-n)$ to HH^n.

5.5. **Type E_7.** Consider the case

$$\mathbf{w} = x_1^3 x_2 + x_2^3 + x_3^2 + \cdots + x_{n+1}^2 \in \mathbb{C}[x_0, x_1, \dots, x_{n+1}] \tag{5.115}$$

with

$$\Gamma = \Gamma_{\mathbf{w}} := \left\{ \gamma = (t_0, \dots, t_{n+1}) \in (\mathbb{G}_{\mathrm{m}})^{n+2} \;\middle|\; t_1^3 t_2 = t_2^3 = t_3^2 = \cdots = t_{n+1}^2 = t_0 \cdots t_{n+1} \right\}, \tag{5.116}$$

so that $\ker\chi \cong \boldsymbol{\mu}_9 \times (\boldsymbol{\mu}_2)^{n-1}$ and $\mathrm{Char}(\Gamma)$ is generated by χ and $\chi_i = \deg x_i$ for $i \in \{0,\dots,n+1\}$ with relations

$$\chi = 3\chi_1 + \chi_2 = 3\chi_2 = 2\chi_3 = \cdots = 2\chi_{n+1} = \chi_0 + \cdots + \chi_{n+1}. \tag{5.117}$$

These relations imply

$$\begin{aligned} \chi_2 &= \chi - 3\chi_1, && (5.118)\\ 9\chi_1 &= 2\chi, && (5.119)\\ \chi_0 &= \chi - \chi_1 - \cdots - \chi_{n+1} && (5.120)\\ &= 2\chi_1 - \chi_3 - \cdots - \chi_{n+1}. && (5.121)\end{aligned}$$

5.5.1. For any $\gamma \in \ker \chi$, the intersection $V_\gamma \cap (\mathbb{C}x_1 \oplus \mathbb{C}x_2)$ can be either $\mathbb{C}x_1 \oplus \mathbb{C}x_2$, $\mathbb{C}x_2$, or 0, where $\mathrm{Jac}_{\mathbf{w}'_\gamma}$ is isomorphic to $\mathbb{C}[x_1, x_2]/(3x_1^2x_2, x_1^3 + 3x_2^2)$, $\mathbb{C}[x_2]/(3x_2^2)$, or $\mathbb{C}$ respectively. A basis of $\mathbb{C}[x_1, x_2]/(3x_1^2x_2, x_1^3 + 3x_2^2)$ is given by $\{1, x_1, x_1^2, x_1^3, x_1^4, x_2, x_1x_2\}$.

If we write an element of $\mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$ as

$$x_0^{k_0} x_1^{k_1} x_2^{k_2} \otimes x_{j_1}^\vee \wedge x_{j_2}^\vee \wedge \ldots \wedge x_{j_s}^\vee, \tag{5.122}$$

then its degree is given by

$$k_0\chi_0 + k_1\chi_1 + k_2\chi_2 - \chi_{j_1} - \cdots - \chi_{j_s}, \tag{5.123}$$

which can be proportional to χ only if $V \cap (\mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1})$ is either $\mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1}$ or 0. We assume this condition for the rest of Section 5.5.

5.5.2. For $\gamma = (t_0, \ldots, t_{n+1}) \in \ker \chi$, one has $t_1^2 = t_0 t_3 \cdots t_{n+1} = \pm t_0$ and $t_2^2 = t_0 t_1 t_3 \cdots t_{n+1} = \pm t_0 t_1$, so that the condition $t_0 = 1$ implies $t_1^2 = \pm 1$ and $t_2^2 = \pm t_1$, which together with $t_2^3 = 1$ imply $t_1 = t_2 = 1$. Hence one has $\mathbb{C}x_0 \subset V_\gamma$ if and only if either $V_\gamma = V$ or $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$.

5.5.3. One has $V_\gamma = V$ if and only if γ is the identity element. The degree of $x_0^{k_0} x_1^{k_1} x_2^{k_2} \in \mathrm{Jac}_{\mathbf{w}}$ is

$$k_0(2\chi_1 - \chi_3 - \cdots - \chi_{n+1}) + k_1\chi_1 + k_2(\chi - 3\chi_1) \tag{5.124}$$
$$= k_2\chi + (2k_0 + k_1 - 3k_2)\chi_1 - k_0\chi_3 - \cdots - k_0\chi_{n+1}, \tag{5.125}$$

which is proportional to χ if and only if

- 9 divides $2k_0 + k_1 - 3k_2$ if $n = 1$, and
- 9 divides $2k_0 + k_1 - 3k_2$ and k_0 is even if $n > 1$.

For $n = 1$, one has

$$t := 2\deg(x_0^{k_0} x_1^{k_1} x_2^{k_2})/\chi \tag{5.126}$$
$$= 2k_2 + \frac{4}{9}(2k_0 + k_1 - 3k_2). \tag{5.127}$$

The possible (k_0, k_1, k_2) and t are given by

(5.128)

(k_1, k_2)	k_0	t
$(0, 0)$	$9m$	$8m$
$(1, 0)$	$4 + 9m$	$4 + 8m$
$(2, 0)$	$8 + 9m$	$8 + 8m$
$(3, 0)$	$3 + 9m$	$4 + 8m$
$(4, 0)$	$7 + 9m$	$8 + 8m$
$(0, 1)$	$6 + 9m$	$6 + 8m$
$(1, 1)$	$1 + 9m$	$2 + 8m$

for $m \in \mathbb{N}$. Each (k_0, k_1, k_2) from (5.128) contributes $\mathbb{C}(k_0 n)$ to HH^t and HH^{t+1}.

In addition, for the case $(k_1, k_2) = (2, 0)$, the element $x_0^\vee \otimes x_1^2$ corresponding to $m = -1$ in (5.128) has degree 0, and contributes $\mathbb{C}(-1)$ to HH^1.

For $n > 1$, one has

$$t := 2\deg(x_0^{k_0} x_1^{k_1} x_2^{k_2})/\chi \tag{5.129}$$
$$= 2k_2 + \frac{4}{9}(2k_0 + k_1 - 3k_2) - k_0(n - 1). \tag{5.130}$$

The possible (k_0, k_1, k_2) and t are given by

(5.131)

(k_1, k_2)	k_0	t
$(0,0)$	$18m$	$34m - 18mn$
$(1,0)$	$4+18m$	$8+34m-(4+18m)n$
$(2,0)$	$8+18m$	$16+34m-(8+18m)n$
$(3,0)$	$12+18m$	$24+34m-(12+18m)n$
$(4,0)$	$16+18m$	$32+34m-(16+18m)n$
$(0,1)$	$6+18m$	$12+34m-(6+18m)n$
$(1,1)$	$10+18m$	$20+34m-(10+18m)n$

for $m \in \mathbb{N}$. Each (k_0, k_1, k_2) from (5.131) contributes $\mathbb{C}(k_0 n)$ to HH^t and HH^{t+1}.

5.5.4. For $n > 1$, in addition, one has $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$ if and only if n is odd and $\gamma = (1, 1, 1, -1, \ldots, -1)$. The degree of

$$x_0^{k_0} x_1^{k_1} x_2^{k_2} \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \tag{5.132}$$

is given by

$$k_0(2\chi_1 - \chi_3 - \cdots - \chi_{n+1}) + k_1\chi_1 + k_2(\chi - 3\chi_1) - \chi_3 - \cdots - \chi_{n+1} \tag{5.133}$$

$$= k_2\chi + (2k_0 + k_1 - 3k_2)\chi_1 - (k_0 + 1)\chi_3 - \cdots - (k_0 + 1)\chi_{n+1}, \tag{5.134}$$

which is proportional to χ if and only if 9 divides $2k_0 + k_1 - 3k_2$ and k_0 is odd. The possible (k_0, k_1, k_2) and

$$t \coloneqq 2 \deg \left(x_0^{k_0} x_1^{k_1} x_2^{k_2} \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \right) / \chi + \dim N_\gamma \tag{5.135}$$

are given by

(5.136)

(k_1, k_2)	k_0	t
$(0,0)$	$9+18m$	$17+34m-(9+18m)n$
$(1,0)$	$13+18m$	$25+34m-(13+18m)n$
$(2,0)$	$17+18m$	$33+34m-(17+18m)n$
$(3,0)$	$3+18m$	$7+34m-(3+18m)n$
$(4,0)$	$7+18m$	$15+34m-(7+18m)n$
$(0,1)$	$15+18m$	$29+34m-(15+18m)n$
$(1,1)$	$1+18m$	$3+34m-(1+18m)n$

for $m \in \mathbb{N}$. Each (k_0, k_1, k_2) from (5.136) contributes $\mathbb{C}(k_0 n)$ to HH^t and HH^{t+1}.

In addition, for the case $(k_1, k_2) = (2, 0)$, the element $x_0^\vee \otimes x_1^2 \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee$ corresponding to $m = -1$ in (5.136) has degree 0, and contributes $\mathbb{C}(-n)$ to HH^n.

5.5.5. One has $V_\gamma = 0$ for $\gamma = (t_0, \ldots, t_{n+1}) \in \ker \chi$ if and only if $t_1 \in \boldsymbol{\mu}_9 \setminus \{1\}$, $t_2 \coloneqq t_1^{-3} \neq 1$, and $t_3 = \cdots = t_{n+1} = -1$, in which case one has $t_0 = (-1)^{n-1} t_1^2 \neq 1$. The set $\{t_1 \in \boldsymbol{\mu}_9 \mid t_1^3 \neq 1\}$ consists of six elements, each of which contributes $\mathbb{C}(-n)$ to HH^n.

5.6. **Type E_8.** Consider the case

$$\mathbf{w} = x_1^5 + x_2^3 + x_3^2 + \cdots + x_{n+1}^2 \in \mathbb{C}[x_0, x_1, \ldots, x_{n+1}] \tag{5.137}$$

with

$$\Gamma = \Gamma_{\mathbf{w}} \coloneqq \left\{ \gamma = (t_0, \ldots, t_{n+1}) \in (\mathbb{G}_\mathrm{m})^{n+2} \;\middle|\; t_1^5 = t_2^3 = t_3^2 = \cdots = t_{n+1}^2 = t_0 \cdots t_{n+1} \right\}, \tag{5.138}$$

so that $\ker \chi \cong \boldsymbol{\mu}_5 \times \boldsymbol{\mu}_3 \times (\boldsymbol{\mu}_2)^{n-1}$ and $\mathrm{Char}(\Gamma)$ is generated by χ and $\chi_i = \deg x_i$ for $i \in \{0, \ldots, n+1\}$ with relations

$$\chi = 5\chi_1 = 3\chi_2 = 2\chi_3 = \cdots = 2\chi_{n+1} = \chi_0 + \cdots + \chi_{n+1}. \tag{5.139}$$

5.6.1. If we write an element of $\mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$ as

$$x_0^{k_0} x_1^{k_1} x_2^{k_2} \otimes x_{j_1}^\vee \wedge x_{j_2}^\vee \wedge \ldots \wedge x_{j_s}^\vee, \tag{5.140}$$

then its degree is given by

$$k_0\chi_0 + k_1\chi_1 + k_2\chi_2 - \chi_{j_1} - \cdots - \chi_{j_s}, \tag{5.141}$$

which can be proportional to χ only if $V \cap (\mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1})$ is either $\mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1}$ or 0. We assume this condition for the rest of Section 5.6.

5.6.2. Since $t_0 = 1$ implies $t_1 = t_2 = 1$, one has $\mathbb{C}x_0 \subset V_\gamma$ if and only if either $V_\gamma = V$ or $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$.

5.6.3. One has $V_\gamma = V$ if and only if γ is the identity element. The degree of $x_0^{k_0} x_1^{k_1} x_2^{k_2} \in \mathrm{Jac}_{\mathbf{w}}$ is

$$k_0\chi - (k_0 - k_1)\chi_1 - (k_0 - k_2)\chi_2 - k_0\chi_3 - \cdots - k_0\chi_{n+1}, \tag{5.142}$$

which is proportional to χ if and only if

- 5 divides $k_0 - k_1$ and 3 divides $k_0 - k_2$ if $n = 1$, and
- 5 divides $k_0 - k_1$, 3 divides $k_0 - k_2$, and k_0 is even if $n > 1$.

For $n = 1$, we must have

$$7k_0 + 3k_1 + 5k_2 = 15m \tag{5.143}$$

for $m \in \mathbb{N}$, in which case one has

$$t := 2\deg\left(x_0^{k_0} x_1^{k_1} x_2^{k_2}\right)/\chi = 2m. \tag{5.144}$$

For each $m \in \mathbb{N}$ such that $7 \nmid m$, the equation (5.143) has a unique solution with $(k_1, k_2) \in \{0,1,2,3\} \times \{0,1\}$ and if $7 \mid m$, then there are precisely two contributions with $(k_1, k_2) = (0,0)$ and $(k_1, k_2) = (3,1)$ such that $(k_1, k_2, m) \in \{0,1,2,3\} \times \{0,1\} \times \mathbb{N}$ except if $m = 0$, then only $(k_1, k_2) = (0,0)$ contributes.

For $n > 1$, we must have

$$7k_0 + 3k_1 + 5k_2 = 15m \tag{5.145}$$

for $m \in \mathbb{N}$, and in addition k_0 must be in $2\mathbb{N}$. Thus, we can re-write (5.145) as

$$k_0 = 6k_1 + 10k_2 + 30m' \tag{5.146}$$

with $m' = k_0/2 - m$. One has

$$\begin{aligned} t &:= 2\deg\left(x_0^{k_0} x_1^{k_1} x_2^{k_2}\right)/\chi & (5.147)\\ &= 12k_1 + 20k_2 + 58m' - (6k_1 + 10k_2 + 30m')n. & (5.148)\end{aligned}$$

Each $(k_1, k_2, m') \in \{0,1,2,3\} \times \{0,1\} \times \mathbb{N}$ contributes $\mathbb{C}(k_0 n)$ to HH^t and HH^{t+1}.

5.6.4. If $n > 1$, in addition, one has $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$ if and only if n is odd and $\gamma = (1,1,1,-1,\ldots,-1)$. The degree of

$$x_0^{k_0} x_1^{k_1} x_2^{k_2} \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \tag{5.149}$$

is

$$k_0\chi - (k_0 - k_1)\chi_1 - (k_0 - k_2)\chi_2 - (k_0 + 1)\chi_3 - \cdots - (k_0 + 1)\chi_{n+1}, \tag{5.150}$$

which is proportional to χ if and only if

$$14k_0 + 6k_1 + 10k_2 = 30m \tag{5.151}$$

for $m \in \mathbb{Z}$ and in addition we must have k_0 odd. Thus, again we can rewrite (5.151) as

$$k_0 = 15 + 6k_1 + 10k_2 + 30m' \tag{5.152}$$

where $m' = (k_0 - 1)/2 - m$. One has

$$t \coloneqq 2\deg\left(x_0^{k_0}x_1^{k_1}x_2^{k_2} \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee\right)/\chi + \dim N_\gamma \tag{5.153}$$

$$= 2\left(k_0 - \frac{1}{5}(k_0 - k_1) - \frac{1}{3}(k_0 - k_2) - \frac{1}{2}(k_0 + 1)(n-1)\right) + (n-1) \tag{5.154}$$

$$= 29 + 12k_1 + 20k_2 + 58m' - (15 + 6k_1 + 10k_2 + 30m')n \tag{5.155}$$

Each $(k_1, k_2, m') \in \{0,1,2,3\} \times \{0,1\} \times \mathbb{Z}$ such that

$$15 + 6k_1 + 10k_2 + 30m' \geq 0 \tag{5.156}$$

contributes $\mathbb{C}(k_0 n)$ to HH^t and HH^{t+1}.

5.6.5. An element $\gamma = (t_0, \ldots, t_{n+1}) \in \ker\chi$ satisfies $V_\gamma = 0$ if and only if $t_1 \in \boldsymbol{\mu}_5 \setminus \{1\}$, $t_2 \in \boldsymbol{\mu}_3 \setminus \{1\}$, $t_3 = \cdots = t_{n+1} = -1$, and $t_0 = (-1)^{n-1}(t_1 t_2)^{-1}$. There are eight such elements, each of which contributes $\mathbb{C}(-n)$ to HH^n.

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