

HOMOLOGICAL MIRROR SYMMETRY FOR MILNOR FIBERS OF SIMPLE SINGULARITIES

YANKI LEKILI AND KAZUSHI UEDA

ABSTRACT. We prove homological mirror symmetry for Milnor fibers of simple singularities, which are among the log Fano cases of [LU, Conjecture 1.5]. The proof is based on a relation between matrix factorizations and Calabi–Yau completions. As an application, we give an explicit computation of the symplectic cohomology group of the Milnor fiber of a simple singularity in all dimensions.

1. INTRODUCTION

A *simple singularity* is an isolated hypersurface singularity of modality zero. Arnold classified such singularities; up to right equivalence, they are given by one of the following:

$$(1.1) \quad \begin{aligned} A_\ell &: x_1^{\ell+1} + x_2^2 + \cdots + x_{n+1}^2 = 0, \\ D_\ell &: x_1^{\ell-1} + x_1 x_2^2 + x_3^2 + \cdots + x_{n+1}^2 = 0, \\ E_6 &: x_1^4 + x_2^3 + x_3^2 + \cdots + x_{n+1}^2 = 0, \\ E_7 &: x_1^3 + x_1 x_2^3 + x_3^2 + \cdots + x_{n+1}^2 = 0, \\ E_8 &: x_1^5 + x_2^3 + x_3^2 + \cdots + x_{n+1}^2 = 0. \end{aligned}$$

In the case $n = 2$, simple surface singularities have many other characterizations, such as Kleinian singularities, rational double points, or canonical singularities, to name a few.

Let $\check{\mathbf{w}}$ be one of these defining polynomials, which we think of as a holomorphic function on \mathbb{C}^{n+1} , and equip $\check{\mathbf{w}}^{-1}(1)$ with the Liouville structure induced from the standard one on \mathbb{C}^{n+1} . This is the Liouville completion of the *Milnor fiber*, which is the Liouville domain obtained by intersecting $\check{\mathbf{w}}^{-1}(1)$ with a ball. Let $\mathcal{W}(\check{\mathbf{w}}^{-1}(1))$ denote the idempotent-complete derived wrapped Fukaya category of $\check{\mathbf{w}}^{-1}(1)$.

For $n \geq 2$, since $\check{\mathbf{w}}^{-1}(1)$ is not a log Calabi–Yau manifold but a log Fano manifold, its mirror is not a manifold but a *Landau–Ginzburg model*, by which we mean a pair of a stack and a section of a line bundle on it. One way to obtain a Landau–Ginzburg mirror of a log Fano manifold is to first remove a divisor to make it log Calabi–Yau, then find its mirror, which is another log Calabi–Yau manifold, and finally add a potential to this mirror [Aur07, Aur09]. This produces a Landau–Ginzburg mirror whose underlying manifold is of the same dimension as the original manifold. When the singularity is toric (i.e., a simple surface singularity of type A), there is a standard choice for the divisor to remove, and the resulting mirror is the Landau–Ginzburg model consisting of a complement of a toric divisor in the minimal resolution of the singularity of the same type and a monomial function on it (see e.g. [AAK16, Section 9.2]). The choice of the divisor is not unique in general, and there are multiple mirrors for a given Milnor fiber.

In this paper, we consider an alternative mirror of the Milnor fiber of a simple singularity based on transposition of invertible polynomials introduced in [BH93, BH95]. A weighted homogeneous polynomial $\mathbf{w} \in \mathbb{C}[x_1, \dots, x_{n+1}]$ with an isolated critical point at the origin is *invertible* if there is an integer matrix $A = (a_{ij})_{i,j=1}^{n+1}$ with non-zero determinant such that

$$(1.2) \quad \mathbf{w} = \sum_{i=1}^{n+1} \prod_{j=1}^{n+1} x_j^{a_{ij}}.$$

The *transpose* of \mathbf{w} is defined as

$$(1.3) \quad \check{\mathbf{w}} = \sum_{i=1}^{n+1} \prod_{j=1}^{n+1} x_j^{a_{ji}},$$

whose exponent matrix \check{A} is the transpose matrix of A . The group

$$(1.4) \quad \Gamma_{\mathbf{w}} := \{(t_0, t_1, \dots, t_{n+1}) \in (\mathbb{G}_m)^{n+2} \mid t_1^{a_{1,1}} \cdots t_{n+1}^{a_{1,n+1}} = \cdots = t_1^{a_{n+1,1}} \cdots t_{n+1}^{a_{n+1,n+1}} = t_0 t_1 \cdots t_{n+1}\}$$

acts naturally on $\mathbb{A}^{n+2} := \text{Spec } \mathbb{C}[x_0, \dots, x_{n+1}]$. Let $\text{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 \cdots x_{n+1})$ denote the idempotent completion of the dg category of $\Gamma_{\mathbf{w}}$ -equivariant coherent matrix factorizations of $\mathbf{w} + x_0 \cdots x_{n+1}$ on \mathbb{A}^{n+2} in the sense of [EP15]. Conjecture 1.1 below is given in [LU, Conjecture 1.5]:

Conjecture 1.1. *For any invertible polynomial \mathbf{w} , one has a quasi-equivalence*

$$(1.5) \quad \text{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 \cdots x_{n+1}) \simeq \mathcal{W}(\check{\mathbf{w}}^{-1}(1)).$$

In other words, the Landau–Ginzburg model $([\mathbb{A}^{n+2}/\Gamma_{\mathbf{w}}], \mathbf{w} + x_0 \cdots x_{n+1})$ is mirror to the Liouville manifold $\check{\mathbf{w}}^{-1}(1)$. The main result of this paper is the following:

Theorem 1.2. *Conjecture 1.1 holds for $n \geq 2$ and $\check{\mathbf{w}}$ one of the defining polynomials of simple singularities appearing in (1.1).*

For $n = 1$, Conjecture 1.1 was recently studied by [Hab] who proved an equivalence of the full subcategories consisting of objects with finite-dimensional morphism spaces.

This paper is organized as follows: In Section 2, we collect basic definitions and results on Calabi–Yau completions and trivial extension algebras. In Section 3, we recall the description of the wrapped Fukaya category of the Milnor fiber of a simple singularity for $n \geq 2$ in terms of the n -Calabi–Yau completion of a Dynkin quiver of the corresponding type. In Section 4, we show that $\text{mf}(\mathbb{A}^{n+2}, \Gamma, \mathbf{w})$ is the n -Calabi–Yau completion of $\text{mf}(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w})$ under mild conditions on Γ and \mathbf{w} . When $\check{\mathbf{w}}$ is one of the defining polynomials of simple singularities appearing in (1.1), $\text{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 \cdots x_{n+1})$ is quasi-equivalent to $\text{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})$. Since $\text{mf}(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w})$ is quasi-equivalent to the derived category of representations of the Dynkin quiver of the corresponding type, Theorem 1.2 is proved. As an application, we explicitly compute the symplectic cohomology of the Milnor fiber of a simple singularity in all dimensions in Section 5.

Acknowledgment: Y. L. is partially supported by the Royal Society (URF) and the NSF grant DMS-1509141. K. U. is partially supported by Grant-in-Aid for Scientific Research (15KT0105, 16K13743, 16H03930).

2. CALABI–YAU COMPLETIONS AND TRIVIAL EXTENSION ALGEBRAS

The n -Calabi–Yau completion (or the *derived n -preprojective algebra*) of a dg algebra \mathcal{A} is defined in [Kel11, Section 4.1] as the tensor algebra

$$(2.1) \quad \Pi_n(\mathcal{A}) := T_{\mathcal{A}}(\theta) := \mathcal{A} \oplus \theta \oplus \theta \otimes_{\mathcal{A}} \theta \oplus \cdots,$$

where the \mathcal{A} -bimodule $\theta := \Theta[n-1]$ is a shift of the inverse dualizing complex $\Theta := \text{hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e)$.

The *Ginzburg dg algebra* \mathcal{G}_Q^n of a quiver Q (without potential) is a model of the n -Calabi–Yau completion $\Pi_n(A_Q)$ of the path algebra A_Q , defined in [Kel11, Section 6.2] after [Gin] as the path algebra of the graded quiver \overline{Q} with same vertices as Q and arrows consisting of

- the original arrows $g \in Q_1$ in degree 1,
- the opposite arrows g^* for each arrow $g \in Q_1$ in degree $1 - n$, and
- loops h_v at each vertex $v \in Q_0$ in degree $1 - n$,

equipped with the differential d given by

$$(2.2) \quad dg = dg^* = 0 \quad \text{and} \quad dh = \sum_{g \in Q_1} g^*g - gg^*$$

where $h = \sum_{v \in Q_0} h_v$.

The *degree n trivial extension algebra* of a finite-dimensional algebra A is defined as $A \oplus A^\vee[-n]$ equipped with the multiplication $(a, f) \cdot (b, g) = (ab, ag + fb)$, where A^\vee is the dual of A as a vector space.

The degree n trivial extension algebra B_Q^n of the path algebra A_Q of a Dynkin quiver Q is the (derived) Koszul dual of \mathcal{G}_Q^n in the sense that

$$(2.3) \quad \mathrm{hom}_{\mathcal{G}_Q^n}(\mathbf{k}_{\mathcal{G}}, \mathbf{k}_{\mathcal{G}}) \simeq B_Q^n, \quad \mathrm{hom}_{(B_Q^n)^{\mathrm{op}}}(\mathbf{k}_B, \mathbf{k}_B) \simeq (\mathcal{G}_Q^n)^{\mathrm{op}},$$

where $\mathbf{k}_{\mathcal{G}} := \bigoplus_{v \in Q_0} S_v$ is the direct sum of simple \mathcal{G}_Q^n -modules S_v associated with vertices $v \in Q_0$, and similarly for \mathbf{k}_B . The Koszul duality implies an isomorphism

$$(2.4) \quad \mathrm{HH}^*(\mathcal{G}_Q^n) \cong \mathrm{HH}^*(B_Q^n)$$

of Hochschild cohomologies (cf. e.g. [Her, Theorem 3.4]).

3. WRAPPED FUKAYA CATEGORY OF THE MILNOR FIBER OF SIMPLE SINGULARITY

Let \check{w} be one of the defining polynomials of a simple singularity and $M^n = \check{w}^{-1}(1)$ be the Milnor fiber, which we view as a Weinstein manifold where the Weinstein structure is induced by restriction from the ambient \mathbb{C}^{n+1} . It is well known that this Weinstein manifold is symplectomorphic (in fact, Weinstein homotopic) to plumbing X_Q of cotangent bundles of spheres T^*S^n according to the Dynkin diagram Q corresponding to the simple singularity. One way to see this is to verify it directly for $n = 1$, and then use the fact that in higher dimensions the Milnor fiber is obtained by stabilization — increasing the dimension corresponds to suspension of the Lefschetz fibration [Sei10]. See also [Abo11] for an explicit construction of a symplectic structure on plumbings. This stabilization point of view also enables one to describe M via Legendrian surgery. Namely M is obtained by attaching critical handles to a Legendrian link Λ_Q^{n-1} on $\partial\mathbb{D}^n$ whose components are unknotted Legendrian spheres S^{n-1} which are clasped together (as in Hopf link) according to the Dynkin diagram Q . The direct sum of co-cores to the critical handles (i.e., cotangent fibers away from the plumbing region) form a generating object of the wrapped Fukaya category by the main theorem in [CRGG], and the surgery formula of [BEE12, Ekh] allows one to explicitly compute the endomorphism algebra of this generator as the Chekanov–Eliashberg algebra $\mathrm{CE}^*(\Lambda_Q^{n-1})$.

This Chekanov–Eliashberg algebra was computed directly in the case $n = 2$ in the paper [EL17] and the resulting dg algebra was shown to be quasi-isomorphic to the derived multiplicative preprojective algebra of the corresponding Dynkin type. Moreover, working over \mathbb{C} , it was shown that the derived multiplicative preprojective algebra of Dynkin type Q is quasi-isomorphic to the Ginzburg algebra \mathcal{G}_Q^2 , also known as the derived (additive) preprojective algebra of Dynkin type Q .

For $n \geq 3$, one can do a direct computation in an analogous way, but we can also deduce this by the Koszul duality result given in [EL, Theorem 58] which shows that $\mathrm{CE}^*(\Lambda_Q^{n-1})$ is the (derived) Koszul dual of the endomorphism algebra of the union of the core spheres of the plumbing. Notice that for $n \geq 3$, \check{w} is suspended at least twice, thus the formality of the endomorphism algebra of vanishing cycles in the compact Fukaya category of $\check{w}^{-1}(1)$ follows automatically by a result of Seidel [Sei10]. Putting it all together, we conclude that $\mathrm{CE}^*(\Lambda_Q^{n-1})$ is Koszul dual to the degree n trivial extension algebra B_Q^n of the path algebra A_Q of a Dynkin quiver of the corresponding type (see also [Li19] for another example).

As a result of these computations, for $n \geq 2$ we have a quasi-isomorphism

$$(3.1) \quad \mathrm{CE}^*(\Lambda_Q^{n-1}) \simeq \mathcal{G}_Q^n$$

over \mathbb{C} , which implies a quasi-equivalence

$$(3.2) \quad \mathcal{W}(\check{w}^{-1}(1)) \simeq \mathrm{perf} \Pi_n(A_Q)$$

between the wrapped Fukaya category of $\check{w}^{-1}(1)$ and the dg derived category of perfect modules over $\Pi_n(A_Q)$.

Remark 3.1. Note from [Sei01, Proposition 3.4] that A_Q is derived equivalent to the Fukaya–Seidel category $\mathcal{F}(\check{w})$ of the LG-model $\check{w}: \mathbb{C}^{n+1} \rightarrow \mathbb{C}$. Thus (3.2) shows that $\mathcal{W}(\check{w}^{-1}(1))$ is the Calabi–Yau completion of $\mathcal{F}(\check{w})$ for $n \geq 2$. Although this relationship between $\mathcal{F}(\check{w})$ and $\mathcal{W}(\check{w}^{-1}(1))$ is not true

in general, we expect it to hold when $\check{\mathbf{w}}$ is a double suspension of an invertible polynomial whose Milnor fiber is a log Fano manifold.

Remark 3.2. The isomorphism (3.1) remains true for $n \geq 3$ over an arbitrary commutative ring but for $n = 2$ we have to require that 2 is invertible for type D_ℓ, E_6, E_7, E_8 , 3 is invertible for type E_6, E_7, E_8 , and 5 is invertible for type E_8 . Otherwise, $\text{CE}^*(\Lambda_Q)$ is quasi-isomorphic to the derived multiplicative preprojective algebra (see [EL19]) which is not quasi-isomorphic to the derived (additive) preprojective algebra $\Pi_n(A_Q)$.

4. MATRIX FACTORIZATIONS AND CALABI–YAU COMPLETIONS

Let Γ be a subgroup of $(\mathbb{G}_m)^{n+1}$ acting diagonally on $\mathbb{A}^{n+1} := \text{Spec } \mathbb{C}[x_1, \dots, x_{n+1}]$. Assume that Γ is a finite extension of the multiplicative group \mathbb{G}_m , so that the group $\text{Char}(\Gamma) := \text{Hom}(\Gamma, \mathbb{G}_m)$ of characters of Γ is a finite extension of \mathbb{Z} . The coordinate ring $\mathbb{C}[x_1, \dots, x_{n+1}]$ has a $\text{Char}(\Gamma)$ -grading coming from the Γ -action on \mathbb{A}^{n+1} , and we set $\chi_i := \deg x_i$ for $i \in \{1, \dots, n+1\}$. Let $\mathbf{w} \in \mathbb{C}[x_1, \dots, x_{n+1}]_\chi := (\mathbb{C}[x_1, \dots, x_{n+1}] \otimes \chi)^\Gamma$ be a homogeneous element of degree $\chi \in \text{Char}(\Gamma)$. Assume that \mathbf{w} has an isolated critical point at the origin, so that the structure sheaf \mathcal{O}_0 of the origin split-generates $\text{mf}(\mathbb{A}^{n+1}, \mathbf{w})$ by [KMVdB11, Proposition A.2] (see also [Orl11, Dyc11]). Let $R \subset \text{Char}(\Gamma)$ be a set of representatives of the group $\text{Char}(\Gamma)/(\chi)$, which we assume to be finite. Then $\mathcal{E} := \bigoplus_{\rho \in R} \mathcal{O}(\rho)$ generates $\text{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w})$, since the autoequivalence $M \mapsto M(\chi)$ of $\text{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w})$ shifting the Γ -weight by χ is isomorphic to the functor $M \mapsto M[2]$ shifting the cohomological grading by 2.

The n -Calabi–Yau completion of the dg Yoneda algebra $\mathcal{A} := \text{hom}(\mathcal{E}, \mathcal{E})$ is given by

$$(4.1) \quad \Pi_n(\mathcal{A}) := \mathcal{A} \oplus \theta \oplus \theta \otimes_{\mathcal{A}} \theta \oplus \dots \simeq \bigoplus_{i=0}^{\infty} \text{hom}(\mathcal{E}, \theta^i(\mathcal{E})),$$

where we abuse notation and use the same symbol for an autoequivalence and its graph bimodule. Note that the autoequivalence θ is isomorphic to a shift of the inverse Serre functor \mathbb{S}^{-1} ;

$$(4.2) \quad \theta \simeq \mathbb{S}^{-1}[n-1].$$

Now, as in [LU, Section 2], we introduce another variable x_0 of degree $\chi_0 := \chi - (\chi_1 + \dots + \chi_{n+1})$, and consider the polynomial ring $\mathbb{C}[x_0, x_1, \dots, x_{n+1}]$ in $n+2$ variables, which naturally contains $\mathbb{C}[x_1, \dots, x_{n+1}]$ as a subring. One can show, e.g., by taking the trivial G, H , and v in [BFK14, Lemma 3.52], that

$$(4.3) \quad \text{mf}(\mathbb{A}^{n+2}, \mathbf{w}) \simeq \text{coh } \mathbb{A}_{x_0}^1 \otimes \text{mf}(\mathbb{A}^{n+1}, \mathbf{w}).$$

As shown in [IT13, Corollary 2.5], graded Auslander–Reiten duality [AR87] implies that

$$(4.4) \quad \mathbb{S} := (\chi_0)[n-1]$$

is a Serre functor on $\text{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w})$. It follows from (4.2) and (4.4) that

$$(4.5) \quad \theta \simeq (-\chi_0).$$

Let \mathcal{F} be the generator of $\text{mf}(\mathbb{A}^{n+2}, \Gamma, \mathbf{w})$ obtained from the tensor product of the generator \mathcal{E} of $\text{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w})$ and the generator $\mathbb{C}[x_0]$ of $\text{coh } \mathbb{A}^1$. If we write both of the forgetful functors $\text{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w}) \rightarrow \text{mf}(\mathbb{A}^{n+1}, \mathbf{w})$ and $\text{mf}(\mathbb{A}^{n+2}, \Gamma, \mathbf{w}) \rightarrow \text{mf}(\mathbb{A}^{n+2}, \mathbf{w})$ as $\overline{(\bullet)}$, then one has

$$(4.6) \quad \text{hom}(\overline{\mathcal{F}}, \overline{\mathcal{F}}) \simeq \text{hom}(\overline{\mathcal{E}}, \overline{\mathcal{E}}) \otimes \mathbb{C}[x_0] \simeq \bigoplus_{\rho \in \text{Char}(\Gamma)} \text{hom}(\mathcal{E}, \mathcal{E}(\rho)) \otimes \mathbb{C}[x_0].$$

Since $\deg(x_0) = \chi_0$, by taking the Γ -invariant part of (4.6) and using (4.5), one obtains

$$(4.7) \quad \text{hom}(\mathcal{F}, \mathcal{F}) \simeq \bigoplus_{i=0}^{\infty} \text{hom}(\mathcal{E}, \mathcal{E}(-i\chi_0)) \simeq \bigoplus_{i=0}^{\infty} \text{hom}(\mathcal{E}, \theta^i(\mathcal{E})) \simeq \Pi_n(\mathcal{A}),$$

which shows that $\text{mf}(\mathbb{A}^{n+2}, \Gamma, \mathbf{w})$ is the n -Calabi–Yau completion of $\text{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w})$.

When $\check{\mathbf{w}}$ is one of the defining polynomials of simple singularities appearing in (1.1), it follows from [Tak, FU11, FU13, HS] and the Knörrer periodicity that there exists a generator \mathcal{G} of $\text{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w})$

whose dg Yoneda algebra is quasi-isomorphic to the path algebra A_Q of a Dynkin quiver of the corresponding type, so that

$$(4.8) \quad \text{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}) \simeq \text{perf } \Pi_n(A_Q).$$

When n is greater than one, the polynomial $\mathbf{w} + x_0 \cdots x_{n+1}$ considered as an element of $\mathbb{C}[[x_0]][[x_1, \dots, x_{n+1}]]$ (i.e., a formal one-parameter deformation of a formal germ of \mathbf{w}) is right equivalent to \mathbf{w} by a formal coordinate change (i.e., there exists $\varphi \in \text{Aut}_{\mathbb{C}[[x_0]]} \mathbb{C}[[x_0]][[x_1, \dots, x_{n+1}]]$ such that $\varphi^*(\mathbf{w} + x_0 \cdots x_{n+1}) = \mathbf{w}$) since the degree of $x_1 \cdots x_{n+1}$ is greater than that of any element in the Jacobi ring

$$(4.9) \quad \text{Jac}_{\mathbf{w}} := \mathbb{C}[x_1, \dots, x_{n+1}] / (\partial_{x_1} \mathbf{w}, \dots, \partial_{x_{n+1}} \mathbf{w})$$

of \mathbf{w} (see e.g. [AGZV85, Section 12.6]). Moreover, one can choose φ to be $\Gamma_{\mathbf{w}}$ -equivariant, so that $\varphi \in \text{Aut}_{\mathbb{C}[[x_0]]} \mathbb{C}[[x_0]][[x_1, \dots, x_{n+1}]]$ (in fact, for any $i \in \{1, \dots, n+1\}$, the coefficient $a_{i, m_1, \dots, m_{n+1}}(x_0)$ of the expansion $\varphi^*(x_i) = \sum_{m_1, \dots, m_{n+1}=0}^{\infty} a_{i, m_1, \dots, m_{n+1}}(x_0) x_1^{m_1} \cdots x_{n+1}^{m_{n+1}}$ is a monomial in x_0). This implies

$$(4.10) \quad \text{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 \cdots x_{n+1}) \simeq \text{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})$$

by [Orl11, Theorem 2.10], and Theorem 1.2 is proved.

5. SYMPLECTIC COHOMOLOGY OF THE MILNOR FIBER OF SIMPLE SINGULARITY

5.1. Symplectic cohomology as Hochschild cohomology. The closed-open map of any Weinstein manifold is an isomorphism because of [Gan12, Theorem 1.1] and the fact, implied by [CRGG, Theorem 1.4] which builds on [Gan12, Gao], that any Weinstein manifold is non-degenerate in the sense of [Gan12, Definition 1.1];

$$(5.1) \quad \text{SH}^*(M) \xrightarrow{\sim} \text{HH}^*(\mathcal{W}(M)).$$

The combination of (3.2), (4.8), (5.1), and the derived Morita invariance of Hochschild cohomology shows that the symplectic cohomology of the Milnor fiber $\check{\mathbf{w}}^{-1}(1)$ of a simple singularity is isomorphic to $\text{HH}^*(\text{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$. As we discuss below, the latter has an explicit description which allows us to compute $\text{SH}^*(\check{\mathbf{w}}^{-1}(1))$ for the Milnor fibers of all simple singularities. Previous partial results in this direction based on different techniques were obtained in [EL17, KvK16, Ueb16], which are in agreement with our computations.

5.2. Hochschild cohomology via matrix factorizations. We use the same notation as in Section 4, and set

$$(5.2) \quad V := \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_{n+1}.$$

Then [Dyc11, CT13, Seg13, BFK14] (cf. also [LU, Theorem 3.1]) shows that $\text{HH}^t(\text{mf}(\mathbb{A}^{n+2}, \Gamma, \mathbf{w}))$ is isomorphic to

$$(5.3) \quad \left(\bigoplus_{\substack{\gamma \in \ker \chi, l \geq 0 \\ t - \dim N_\gamma = 2u}} H^{-2l}(d\mathbf{w}_\gamma) \otimes \chi^{\otimes(u+l)} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right. \\ \left. \oplus \bigoplus_{\substack{\gamma \in \ker \chi, l \geq 0 \\ t - \dim N_\gamma = 2u+1}} H^{-2l-1}(d\mathbf{w}_\gamma) \otimes \chi^{\otimes(u+l+1)} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)^\Gamma.$$

Here $H^i(d\mathbf{w}_\gamma)$ is the i -th cohomology of the Koszul complex

$$(5.4) \quad C^*(d\mathbf{w}_\gamma) := \{ \cdots \rightarrow \Lambda^2 V_\gamma^\vee \otimes \chi^{\otimes(-2)} \otimes S_\gamma \rightarrow V_\gamma^\vee \otimes \chi^\vee \otimes S_\gamma \rightarrow S_\gamma \},$$

where the rightmost term S_γ sits in cohomological degree 0, and the differential is the contraction with

$$(5.5) \quad d\mathbf{w}_\gamma \in (V_\gamma \otimes \chi \otimes S_\gamma)^\Gamma.$$

The vector space V_γ is the subspace of γ -invariant elements in V , S_γ is the symmetric algebra of V_γ , \mathbf{w}_γ is the restriction of \mathbf{w} to $\text{Spec } S_\gamma$, and N_γ is the complement of V_γ in V so that $V \cong V_\gamma \oplus N_\gamma$ as a Γ -module.

If \mathbf{w}_γ has an isolated critical point at the origin, then the cohomology of (5.4) is concentrated in degree 0, so that only the summand

$$(5.6) \quad (\text{Jac}_{\mathbf{w}_\gamma} \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee)^\Gamma$$

with $l = 0$ in (5.3) contributes to $\text{HH}^{2u+\dim N_\gamma}$.

If V_γ contains $\mathbb{C}x_0$, then the Koszul complex $C^*(d\mathbf{w}_\gamma)$ is isomorphic to the tensor product of $C^*(d\mathbf{w}'_\gamma)$ and the complex $\{\mathbf{k}x_0^\vee \otimes \chi^\vee \otimes \mathbf{k}[x_0] \rightarrow \mathbf{k}[x_0]\}$ concentrated in cohomological degree $[-1, 0]$ with the zero differential, where \mathbf{w}'_γ is the restriction of \mathbf{w} to the complement V'_γ of $\mathbb{C}x_0$ in V_γ . If \mathbf{w}'_γ has an isolated critical point at the origin, then $C^*(d\mathbf{w}'_\gamma)$ is quasi-isomorphic to $\text{Jac}_{\mathbf{w}'_\gamma}$ concentrated at the origin, so that only the summands

$$(5.7) \quad (\text{Jac}_{\mathbf{w}'_\gamma} \otimes \mathbb{C}[x_0] \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee)^\Gamma$$

and

$$(5.8) \quad (\mathbb{C}x_0^\vee \otimes \text{Jac}_{\mathbf{w}'_\gamma} \otimes \mathbf{k}[x_0] \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee)^\Gamma$$

with $l = 0$ in (5.3) contribute to $\text{HH}^{2u+\dim N_\gamma}$ and $\text{HH}^{2u+\dim N_\gamma+1}$ respectively.

5.3. Bigrading on Hochschild cohomology. The Hochschild cohomology of a graded algebra B (with no differential) has a bigrading such that

$$(5.9) \quad \text{HH}^{r+s}(B)^s := \text{Ext}_{B^{\text{op}} \otimes B}^r(B, B[s]).$$

When B is the trivial extension algebra B^n of a finite-dimensional algebra A , by introducing a \mathbb{G}_m -action on B^n such that A has weight 0 and $A^\vee[-n]$ has weight n , the s -grading on $\text{HH}^*(B^n)$ can be described as the weight of the induced \mathbb{G}_m -action.

For any positive integer m , the underlying ungraded algebra of the trivial extension algebras B^{mn} is isomorphic to B^n , and only the cohomological gradings are different; that of the former is m times that of the latter. It follows that one has an isomorphism

$$(5.10) \quad \text{HH}^{r+ms}(B^{mn})^{ms} \cong \text{HH}^{r+s}(B^n)^s$$

of vector spaces for any positive integer m such that the parities of n and mn are the same (note that the signs in the Hochschild complex depend on the parity of the cohomological grading).

When Q is a Dynkin quiver, one can transport the \mathbb{G}_m -action on B_Q^n to \mathcal{G}_Q^n through the Koszul duality (2.3), so that g for $g \in Q_1$ has weight 0, g^* for $g \in Q_1$ has weight $-n$, and h_v for $v \in Q_0$ has weight $-n$. This makes the isomorphism (2.4) \mathbb{G}_m -equivariant, so that the \mathbb{G}_m -weights on both sides agree.

Since \mathbf{w} does not depend on x_0 , the \mathbb{G}_m -action on \mathbb{A}^{n+2} such that the weight of x_i is $-n$ for $i = 0$ and 0 for $i \in \{1, \dots, n+1\}$ keeps \mathbf{w} invariant. This induces a \mathbb{G}_m -action on $\text{mf}(\mathbb{A}^{n+2}, \Gamma, \mathbf{w})$, and hence on B_Q^n , whose weight is 0 on A_Q and n on $A_Q^\vee[-n]$ just as in [LU]. This allows us to compute the s -grading on $\text{HH}^*(B_Q^n)$ as the \mathbb{G}_m -weight on (5.3). This \mathbb{G}_m -action is mirror to the one introduced in [SS12] and studied further for type A Milnor fibers in [Sei12].

5.4. Type A_ℓ . Consider the case

$$(5.11) \quad \mathbf{w} = x_1^{\ell+1} + x_2^2 + \dots + x_{n+1}^2 \in \mathbb{C}[x_0, x_1, \dots, x_{n+1}]$$

with

$$(5.12) \quad \Gamma = \Gamma_{\mathbf{w}} := \{ \gamma = (t_0, t_1, \dots, t_{n+1}) \in (\mathbb{G}_m)^{n+2} \mid t_1^{\ell+1} = t_2^2 = \dots = t_{n+1}^2 = t_0 t_1 \dots t_{n+1} \},$$

so that $\chi(\gamma) = t_1^{\ell+1}$, $\ker \chi \cong \boldsymbol{\mu}_{\ell+1} \times (\boldsymbol{\mu}_2)^n$, and $\text{Char}(\Gamma)$ is generated by χ and $\chi_i = \deg x_i$ for $i \in \{0, \dots, n+1\}$ with relations

$$(5.13) \quad \chi = (\ell+1)\chi_1 = 2\chi_2 = \dots = 2\chi_{n+1}.$$

5.4.1. For any $\gamma \in \ker \chi$, one has

$$(5.14) \quad \text{Jac}_{\mathbf{w}_\gamma} = \begin{cases} \mathbb{C}[x_0] \otimes \mathbb{C}[x_1]/(x_1^\ell) & \mathbb{C}x_0 \oplus \mathbb{C}x_1 \subset V_\gamma, \\ \mathbb{C}[x_0] & \mathbb{C}x_0 \subset V_\gamma \text{ and } \mathbb{C}x_1 \not\subset V_\gamma, \\ \mathbb{C}[x_1]/(x_1^\ell) & \mathbb{C}x_0 \not\subset V_\gamma \text{ and } \mathbb{C}x_1 \subset V_\gamma, \\ \mathbb{C} & \text{otherwise.} \end{cases}$$

If we write an element of $\text{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$ as

$$(5.15) \quad x_0^{k_0} x_1^{k_1} \otimes x_{j_1}^\vee \wedge x_{j_2}^\vee \wedge \dots \wedge x_{j_s}^\vee,$$

where $k_0 = 0$ if $\mathbb{C}x_0 \not\subset V_\gamma$ and $k_1 = 0$ if $\mathbb{C}x_1 \not\subset V_\gamma$, then its degree is given by

$$(5.16) \quad k_0\chi_0 + k_1\chi_1 - \chi_{j_1} - \dots - \chi_{j_s},$$

which can be proportional to χ only if V_γ is either V , $\mathbb{C}x_0 \oplus \mathbb{C}x_1$, $\mathbb{C}x_0$, or 0 .

5.4.2. One has $V_\gamma = V$ if and only if γ is the identity element. The degree of $x_0^{k_0} x_1^{k_1} \in \text{Jac}_{\mathbf{w}}$ is

$$(5.17) \quad k_0\chi - (k_0 - k_1)\chi_1 - k_0\chi_2 - \dots - k_0\chi_{n+1},$$

which is proportional to χ if and only if k_0 is even and $\ell + 1$ divides $k_0 - k_1$. Such an element can be written as

$$(5.18) \quad \mathbf{a}_{k,m} := x_0^{k+m(\ell+1)} x_1^k,$$

where $k \in \{0, \dots, \ell - 1\}$ and $m \in \mathbb{N}$ satisfies

- if ℓ is even, then the parities of k and m agree, and
- if ℓ is odd, then k is even.

Since

$$(5.19) \quad \deg \left(x_0^{k+m(\ell+1)} x_1^k \right) = (k + m(\ell + 1))\chi - m\chi - \frac{1}{2}(k + m(\ell + 1))n\chi$$

$$(5.20) \quad = \left((k + m\ell) - \frac{1}{2}(k + m(\ell + 1))n \right) \chi,$$

the element $x_0^{k+m(\ell+1)} x_1^k$ for such (k, m) contributes $\mathbb{C}((k + m(\ell + 1))n)$ to HH^t for $t = 2(k + m\ell) - (k + m(\ell + 1))n$. Similarly, for each such (k, m) , the element

$$(5.21) \quad \mathbf{a}_{k,m} := x_0^\vee \otimes x_0^{k+m(\ell+1)+1} x_1^k \in \mathbb{C}x_0^\vee \otimes \text{Jac}_{\mathbf{w}}$$

contributes $\mathbb{C}((k + m(\ell + 1))n)$ to HH^{t+1} for $t = 2(k + m\ell) - (k + m(\ell + 1))n$.

5.4.3. One has $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1$ if and only if n is even and $\gamma = (1, 1, -1, \dots, -1)$. The degree of

$$(5.22) \quad x_0^{k_0} x_1^{k_1} \otimes x_2^\vee \wedge \dots \wedge x_{n+1}^\vee \in \text{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$$

is given by

$$(5.23) \quad k_0\chi + (k_1 - k_0)\chi_1 - (k_0 + 1)\chi_2 - \dots - (k_0 + 1)\chi_{n+1},$$

which is proportional to χ if and only if k_0 is odd and $\ell + 1$ divides $k_1 - k_0$. Such an element can be written as

$$(5.24) \quad \mathbf{a}_{k,m} := x_0^{k+m(\ell+1)} x_1^k \otimes x_2^\vee \wedge \dots \wedge x_{n+1}^\vee,$$

where $k \in \{0, \dots, \ell - 1\}$ and $m \in \mathbb{N}$ satisfies

- if ℓ is even, then the parities of k and m differ, and
- if ℓ is odd, then k is odd.

Since the degree of this element is

$$(5.25) \quad \left((k + m\ell) - \frac{1}{2}(k + m(\ell + 1) + 1)n \right) \chi,$$

each such (k, m) contributes $\mathbb{C}((k + m(\ell + 1))n)$ to HH^t for

$$(5.26) \quad t = 2 \left((k + m\ell) - \frac{1}{2}(k + m(\ell + 1) + 1)n \right) + \dim N_\gamma$$

$$(5.27) \quad = 2(k + m\ell) - (k + m(\ell + 1))n.$$

Similarly, for each such (k, m) , there is an element $\alpha_{k,m}$ contributing $\mathbb{C}((k + m(\ell + 1))n)$ to HH^{t+1} for $t = 2(k + m\ell) - (k + m(\ell + 1))n$.

5.4.4. One has $V_\gamma = \mathbb{C}x_0$ if and only if both ℓ and n are odd and $\gamma = (1, -1, \dots, -1)$. The degree of

$$(5.28) \quad x_0^{k_0} \otimes x_1^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$$

is given by

$$(5.29) \quad k_0\chi - (k_0 + 1)\chi_1 - (k_0 + 1)\chi_2 - \cdots - (k_0 + 1)\chi_{n+1},$$

which is proportional to χ if and only if $\ell + 1$ divides $k_0 + 1$. Such an element can be written as

$$(5.30) \quad \mathbf{b}_m := x_0^{m(\ell+1)-1} \otimes x_1^\vee \wedge \cdots \wedge x_{n+1}^\vee$$

for $m \in \mathbb{N} \setminus \{0\}$. Since the degree of this element is

$$(5.31) \quad \left((m\ell - 1) - \frac{1}{2}m(\ell + 1)n \right) \chi,$$

each such element contributes $\mathbb{C}((m(\ell + 1) - 1)n)$ to HH^t for

$$(5.32) \quad t = 2 \left((m\ell - 1) - \frac{1}{2}m(\ell + 1)n \right) + \dim N_\gamma$$

$$(5.33) \quad = (2m\ell + 1) - (m(\ell + 1) - 1)n.$$

Similarly, for each $m \in \mathbb{N}$, the element

$$(5.34) \quad \beta_m := x_0^\vee \otimes x_0^{m(\ell+1)} \otimes x_1^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \mathbb{C}x_0^\vee \otimes \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$$

contributes $\mathbb{C}(m(\ell + 1) - 1)n)$ to HH^{t+1} for $t = (2m\ell - 1) - (m(\ell + 1) - 1)n$.

5.4.5. For $\gamma = (t_0, \dots, t_{n+1}) \in \ker \chi$, one has $V_\gamma = 0$ if and only if $t_i \neq 1$ for all $i \in \{0, \dots, n + 1\}$. This is the case if and only if $t_2 = \cdots = t_{n+1} = -1$, $t_1 \in \boldsymbol{\mu}_{\ell+1} \setminus \{1\}$, and

$$(5.35) \quad t_0 = (-1)^n t_1^{-1} \neq 1.$$

If n is odd, then (5.35) holds if and only if $t_1 \neq -1$, so that the number of such γ is ℓ if ℓ is even, and $\ell - 1$ if ℓ is odd. If n is even, then (5.35) always holds, and the number of such γ is ℓ . Each such γ contributes $\mathbb{C}(-n)$ to HH^n .

5.4.6. To sum up, the Hochschild cohomology group has a basis consisting of the following elements:

- $\alpha_{k,m}$ of degree $2(k + m\ell) - (k + m(\ell + 1))n$ and weight $-(k + m(\ell + 1))n$, and $\alpha_{k,m}$ of degree $2(k + m\ell) - (k + m(\ell + 1))n + 1$ and weight $-(k + m(\ell + 1))n$, where (k, m) runs over
 - $\{0, \dots, \ell - 1\} \times \mathbb{N}$ if n is even, and
 - the subset of $\{0, \dots, \ell - 1\} \times \mathbb{N}$ consisting of (k, m) such that
 - * the parities of k and m agree if n is odd and ℓ is even,
 - * k is even if both n and ℓ are odd, and
- if both n and ℓ are odd, then
 - \mathbf{b}_m of degree $2m\ell - 1 - (m(\ell + 1) - 1)n$ and weight $-(m(\ell + 1) - 1)n$ for $m \in \mathbb{N} \setminus \{0\}$, and
 - β_m of degree $2m\ell - (m(\ell + 1) - 1)n$ and weight $-(m(\ell + 1) - 1)n$ for $m \in \mathbb{N}$.
- \mathbf{s}_h of degree n and weight n , where h runs over
 - $\{1, 2, \dots, \ell - 1\}$ if both ℓ and n are odd, and

– $\{1, 2, \dots, \ell\}$ otherwise,

5.4.7. As an example, consider the case $\ell = 1$. The Hochschild cohomology group in this case is spanned by

- $\mathbf{a}_{0,m}$ for $m \in \mathbb{N}$ of degree $-2m(n-1)$ and weight $-2mn$,
- $\boldsymbol{\alpha}_{0,m}$ for $m \in \mathbb{N}$ of degree $-2m(n-1) + 1$ and weight $-2mn$,

and, if n is odd, in addition to the above,

- \mathbf{b}_m for $m \in \mathbb{N} \setminus \{0\}$ of degree $-(2m-1)(n-1)$ and weight $-(2m-1)n$,
- $\boldsymbol{\beta}_m$ for $m \in \mathbb{N}$ of degree $-(2m-1)(n-1) + 1$ and weight $-(2m-1)n$,

and, if n is even, in addition to the above,

- \mathbf{s}_1 of degree n and weight n .

This is consistent with the isomorphism

$$(5.36) \quad \mathrm{SH}^*(T^*S^n) \cong H_{n-*}(\mathcal{L}S^n),$$

which is a special case of the isomorphism between the symplectic cohomology of the cotangent bundle and the homology of the free loop space [Vit, Theorem 3.1] (see e.g. [CJY04, Theorem 2] for the homology of the free loop space of spheres).

Another example is the case when $n = 2$ and ℓ is arbitrary. In this case, $\mathrm{SH}^*(\check{\mathbf{w}}^{-1}(1))$ was computed in [EL17] as a bigraded ring. This is compatible with the computation given here.

5.5. **Type D_ℓ .** Consider the case

$$(5.37) \quad \mathbf{w} = x_1^{2\ell-2} + x_2^2 + \dots + x_{n+1}^2 \in \mathbb{C}[x_0, x_1, \dots, x_{n+1}]$$

with the non-maximal group

$$(5.38) \quad \Gamma = \left\{ \gamma = (t_0, t_1, \dots, t_{n+1}) \in (\mathbb{G}_m)^{n+2} \mid t_1^{\ell-1} t_2 = t_2^2 = \dots = t_{n+1}^2 = t_0 t_1 \dots t_{n+1} \right\},$$

so that $\ker \chi \cong \boldsymbol{\mu}_{2\ell-2} \times (\boldsymbol{\mu}_2)^{n-1}$ and $\mathrm{Char}(\Gamma)$ is generated by χ and $\chi_i = \deg x_i$ for $i \in \{0, \dots, n+1\}$ with relations

$$(5.39) \quad \chi = (\ell-1)\chi_1 + \chi_2 = 2\chi_2 = \dots = 2\chi_{n+1} = \chi_0 + \dots + \chi_{n+1}.$$

By a change of coordinates, this is equivalent to

$$(5.40) \quad \mathbf{w}' = x_1^{\ell-1} x_2 + x_2^2 + \dots + x_{n+1}^2$$

with $\Gamma = \Gamma_{\mathbf{w}'}$, whose Berglund–Hübsch transpose

$$(5.41) \quad \check{\mathbf{w}}' = x_1^{\ell-1} + x_1 x_2^2 + x_3^2 + \dots + x_{n+1}^2$$

defines the D_ℓ -singularity. The relations (5.39) imply

$$(5.42) \quad \chi_2 = \chi - (\ell-1)\chi_1,$$

$$(5.43) \quad \chi = (2\ell-2)\chi_1,$$

$$(5.44) \quad \chi_0 = \chi - \chi_1 - \dots - \chi_n$$

$$(5.45) \quad = (\ell-2)\chi_1 - \chi_3 - \dots - \chi_{n+1}.$$

5.5.1. For any $\gamma \in \ker \chi$, one has

$$(5.46) \quad \mathrm{Jac}_{\mathbf{w}'_\gamma} = \begin{cases} \mathbb{C}[x_0] \otimes \mathbb{C}[x_1]/(x_1^{2\ell-3}) & \mathbb{C}x_0 \oplus \mathbb{C}x_1 \subset V_\gamma, \\ \mathbb{C}[x_0] & \mathbb{C}x_0 \subset V_\gamma \text{ and } \mathbb{C}x_1 \not\subset V_\gamma, \\ \mathbb{C}[x_1]/(x_1^{2\ell-3}) & \mathbb{C}x_0 \not\subset V_\gamma \text{ and } \mathbb{C}x_1 \subset V_\gamma, \\ \mathbb{C} & \text{otherwise.} \end{cases}$$

If we write an element of $\mathrm{Jac}_{\mathbf{w}'_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$ as

$$(5.47) \quad x_0^{k_0} x_1^{k_1} \otimes x_{j_1}^\vee \wedge x_{j_2}^\vee \wedge \dots \wedge x_{j_s}^\vee,$$

where $k_0 = 0$ if $\mathbb{C}x_0 \not\subset V_\gamma$ and $k_1 = 0$ if $\mathbb{C}x_1 \not\subset V_\gamma$, then its degree is given by

$$(5.48) \quad k_0 \chi_0 + k_1 \chi_1 - \chi_{j_1} - \dots - \chi_{j_s},$$

which can be proportional to χ only if

- $V_\gamma \cap (\mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1})$ is either $\mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1}$ or 0.

We will assume this condition for the rest of Section 5.5.

5.5.2. Since $t_0 = (t_1 \cdots t_{n+1})^{-1} = \pm t_1^{-1}$, one has $t_0 = 1$ only if $t_1 = \pm 1$. If $t_1 = 1$, then $t_2 = 1$, and one has $t_0 = 1$ if and only if $(t_3)^{n-1} = 1$. If $t_1 = -1$, then $t_2 = (-1)^{\ell-1}$, and one has $t_0 = 1$ if and only if $(-1)^\ell t_3^{n-1} = 1$. It follows that

- V_γ contains $\mathbb{C}x_0$ if and only if
 - $\gamma = (1, \dots, 1)$, where $V_\gamma = V$,
 - $\gamma = (1, 1, 1, -1, \dots, -1)$ with odd n , where $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$,
 - $\gamma = (1, -1, -1, 1, \dots, 1)$ with even ℓ , where $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1}$,
 - $\gamma = (1, -1, -1, -1, \dots, -1)$ with even ℓ and odd n , where $V_\gamma = \mathbb{C}x_0$,
 - $\gamma = (1, -1, 1, -1, \dots, -1)$ with odd ℓ and even n , where $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_2$.

5.5.3. The smallest positive integer k such that the degree of x_0^k is proportional to χ is $2\ell - 2$. One has

$$(5.49) \quad \deg x_0^{2\ell-2} = (2\ell - 2)(\chi - \chi_1 - \cdots - \chi_{n+1})$$

$$(5.50) \quad = ((2\ell - 3) - (\ell - 1)n)\chi.$$

5.5.4. One has $V_\gamma = V$ if and only if γ is the identity element. The degree of $x_0^{k_0} x_1^{k_1} \in \text{Jac}_{\mathbf{w}}$ is

$$(5.51) \quad k_0\chi - (k_0 - k_1)\chi_1 - k_0\chi_2 - \cdots - k_0\chi_{n+1},$$

which is proportional to χ if and only if k_0 is even and $2\ell - 2$ divides $k_0 - k_1$. Such an element can be written as

$$(5.52) \quad \mathbf{a}_{k,m} := x_0^{2k+(2\ell-2)m} x_1^{2k}$$

for $(k, m) \in \{0, \dots, \ell - 2\} \times \mathbb{N}$ which contributes $\mathbb{C}((2k + (2\ell - 2)m)n)$ to HH^t for $t = 4k + (4\ell - 6)m - (2k + (2\ell - 2)m)n$ since

$$(5.53) \quad \deg x_0^{2k} x_1^{2k} = (2k - kn)\chi.$$

Similarly, for each $(k, m) \in \{0, \dots, \ell - 2\} \times \mathbb{N}$, there is an element $\mathbf{a}_{k,m}$ contributing $\mathbb{C}((2k + (2\ell - 2)m)n)$ to HH^t for $t = 4k + 1 + (4\ell - 6)m - (2k + (2\ell - 2)m)n$.

5.5.5. One has $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$ if and only if $\gamma = (1, 1, 1, -1, \dots, -1)$ and n is odd. The degree of $x_0^{k_0} x_1^{k_1} \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \text{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$ is

$$(5.54) \quad k_0\chi - (k_0 - k_1)\chi_1 - k_0\chi_2 - (k_0 + 1)\chi_3 - \cdots - (k_0 + 1)\chi_{n+1},$$

which is proportional to χ if and only if k_0 is odd and $2\ell - 2$ divides $k_0 - k_1 - (\ell - 1)$. Such an element can be written as

$$(5.55) \quad \mathbf{b}_{k,m} := x_0^{k+\ell-1+(2\ell-2)m} x_1^k \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee$$

for

$$(5.56) \quad (k, m) \in \{(k, m) \in \{0, \dots, 2\ell - 4\} \times \mathbb{Z} \mid k + \ell \text{ is even and } k + \ell - 1 + m(2\ell - 2) \geq 0\}.$$

It contributes $\mathbb{C}((k + \ell - 1 + (2\ell - 2)m)n)$ to HH^t for

$$(5.57) \quad t = 2 \deg \left(x_0^{k+\ell-1+(2\ell-2)m} x_1^k \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \right) + \dim N_\gamma$$

$$(5.58) \quad = 2k + 2\ell - 3 + (4\ell - 6)m - (k + \ell - 1 - (2\ell - 2)m)n,$$

since

$$(5.59) \quad \deg \left(x_0^{k+\ell-1} x_1^k \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \right)$$

$$(5.60) \quad = (k + \ell - 1)\chi - (\ell - 1)\chi_1 - (k + \ell - 1)\chi_2 - (k + \ell)\chi_3 - \cdots - (k + \ell)\chi_{n+1}$$

$$(5.61) \quad = \left(k + \ell - 1 - \frac{1}{2}(k + \ell)n \right) \chi.$$

Similarly, for each

$$(5.62) \quad (k, m) \in \{(k, m) \in \{0, \dots, 2\ell - 4\} \times \mathbb{Z} \mid k + \ell \text{ is even and } k + \ell + m(2\ell - 2) \geq 0\},$$

the element

$$(5.63) \quad \beta_{k,m} := x_0^\vee \otimes x_0^{k+\ell+(2\ell-2)m} x_1^k \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \in ((\mathbb{C}x_0)^\vee \otimes \text{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee)^\Gamma$$

contributes $\mathbb{C}(k + \ell - 1 + (2\ell - 2)m)$ to HH^t for

$$(5.64) \quad t = 2k + 2\ell - 2 + (4\ell - 6)m - (k + \ell - 1 - (2\ell - 2)m)n.$$

5.5.6. One has $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1}$ if and only if $\gamma = (1, -1, -1, 1, \dots, 1)$ with even ℓ . An element of $\text{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$ whose degree is proportional to χ can be written as

$$(5.65) \quad \mathbf{c}_m := x_0^{\ell-2+(2\ell-2)m} \otimes x_1^\vee \wedge x_2^\vee$$

for $m \in \mathbb{N}$, which contributes $\mathbb{C}((\ell - 2 + (2\ell - 2)m)n)$ to HH^t for

$$(5.66) \quad t = 2 \deg(x_0^{\ell-2} \otimes x_1^\vee \wedge x_2^\vee) + \dim N_\gamma$$

$$(5.67) \quad = 2\ell - 4 + (4\ell - 6)m - (\ell - 2 + (2\ell - 2)m)n$$

since

$$(5.68) \quad \deg(x_0^{\ell-2} \otimes x_1^\vee \wedge x_2^\vee) = \left(\ell - 3 - \frac{1}{2}(\ell - 2)n \right) \chi.$$

Similarly, for each $m \in \mathbb{N}$, there is an element γ_m contributing $\mathbb{C}((\ell - 2 + (2\ell - 2)m)n)$ to HH^t for

$$(5.69) \quad t = 2\ell - 3 + (4\ell - 6)m - (\ell - 2 + (2\ell - 2)m)n.$$

5.5.7. One has $V_\gamma = \mathbb{C}x_0$ if and only if ℓ is even, n is odd, and $\gamma = (1, -1, \dots, -1) \in \ker \chi$. The degree of

$$(5.70) \quad x_0^{k_0} \otimes x_1^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \text{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$$

is given by

$$(5.71) \quad k_0\chi - (k_0 + 1)\chi_1 - (k_0 + 1)\chi_2 - \cdots - (k_0 + 1)\chi_{n+1},$$

which is proportional to χ if and only if $2\ell - 2$ divides $k_0 + 1$. Such an element can be written as

$$(5.72) \quad \mathbf{d}_m := x_0^{-1+m(2\ell-2)} \otimes x_1^\vee \wedge \cdots \wedge x_{n+1}^\vee$$

for $m \in \mathbb{N} \setminus \{0\}$. Since

$$(5.73) \quad \deg(x_0^{-1} \otimes x_1^\vee \wedge \cdots \wedge x_{n+1}^\vee) = -\chi,$$

each such element contributes $\mathbb{C}((-1 + (2\ell - 2)m)n)$ to HH^t for

$$(5.74) \quad t = 2 \deg(x_0^{-1+m(2\ell-2)} \otimes x_1^\vee \wedge \cdots \wedge x_{n+1}^\vee) + \dim N_\gamma$$

$$(5.75) \quad = -1 + (4\ell - 6)m - (-1 + (2\ell - 2)m)n.$$

Similarly, for each $m \in \mathbb{N}$, the element

$$(5.76) \quad \delta_m := x_0^\vee \otimes x_0^{m(2\ell-2)} \otimes x_1^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \mathbb{C}x_0^\vee \otimes \text{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$$

contributes $\mathbb{C}((-1 + (2\ell - 2)m)n)$ to HH^t for

$$(5.77) \quad t = (4\ell - 6)m - (-1 + (2\ell - 2)m)n.$$

5.5.8. One has $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_2$ if and only if ℓ is odd, n is even, and $\gamma = (1, -1, 1, -1, \dots, -1)$. The degree of

$$(5.78) \quad x_0^{k_0} \otimes x_1^\vee \wedge x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \text{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$$

is given by

$$(5.79) \quad k_0\chi_0 - \chi_1 - \chi_3 - \cdots - \chi_{n+1} = (k_0(\ell - 2) - 1)\chi_1 - (k_0 + 1)\chi_3 - \cdots - (k_0 + 1)\chi_{n+1},$$

which is proportional to χ if and only if k_0 is odd and $2\ell - 2$ divides $k_0(\ell - 2) - 1$. Such an element can be written as

$$(5.80) \quad \mathbf{e}_m := x_0^{\ell-2+(2\ell-2)m} \otimes x_1^\vee \wedge x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee$$

for $m \in \mathbb{N}$, which contributes $\mathbb{C}((\ell - 2 + (2\ell - 2)m)n)$ to HH^t for

$$(5.81) \quad t = 2\ell - 4 + (4\ell - 6)m - (\ell - 2 + (2\ell - 2)m)n$$

since

$$(5.82) \quad \deg(x_0^{\ell-2} \otimes x_1^\vee \wedge x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee) = \frac{1}{2}(2\ell - 4 - (\ell - 1)n)\chi.$$

Similarly, for each $m \in \mathbb{N}$, there is an element \mathbf{e}_m contributing $\mathbb{C}((\ell - 2 + (2\ell - 2)m)n)$ to HH^t for

$$(5.83) \quad t = 2\ell - 3 + (4\ell - 6)m - (\ell - 2 + (2\ell - 2)m)n.$$

5.5.9. Set $\zeta := \exp(2\pi\sqrt{-1}/(2\ell - 2))$. For a given $\gamma = (t_0, \dots, t_{n+1}) \in \ker \chi$, we write $t_1 = \zeta^p$ for $p \in \{0, \dots, 2\ell - 3\}$. Then one has $t_2 = (-1)^p$, so that

- V_γ contains $\mathbb{C}x_1$ if and only if $p = 0$, and
- V_γ contains $\mathbb{C}x_2$ if and only if p is even.

5.5.10. If $\mathbb{C}x_0 \not\subset V_\gamma$ and $\mathbb{C}x_1 \subset V_\gamma$, then one has $\gamma = (-1, 1, 1, -1, \dots, -1)$ for even n and $V_\gamma = \mathbb{C}x_1 \oplus \mathbb{C}x_2$. The element

$$(5.84) \quad x_1^{\ell-2} \otimes x_0^\vee \wedge x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee$$

has degree

$$(5.85) \quad (\ell - 2)\chi_1 - \chi_0 - \chi_3 - \cdots - \chi_{n+1} = 0,$$

so that it contributes $\mathbb{C}(-n)$ to HH^t for $t = \dim N_\gamma = n$.

5.5.11. An element $\gamma \in \ker \chi$ with $V_\gamma = \mathbb{C}x_2$, $\mathbb{C}x_2 \oplus \cdots \oplus \mathbb{C}x_{n+1}$, or $\mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1}$ does not contribute to HH^* , since $\text{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$ contains a unique element, whose degree is not proportional to χ .

5.5.12. One has $V_\gamma = 0$ if and only if $t_3 = \cdots = t_{n+1} = -1$, $t_1 = \zeta^{2m+1}$ for $m \in \{0, \dots, \ell - 2\}$, and

$$(5.86) \quad t_0 = (-1)^n \zeta^{-2m-1} \neq 1.$$

The number of such γ is $\ell - 2$ if ℓ is even and n is odd, and $\ell - 1$ otherwise. Each such γ contributes $\mathbb{C}(-n)$ to HH^n .

5.5.13. To sum up, the Hochschild cohomology group has a basis consisting of the following elements:

- $\mathbf{a}_{k,m}$ of degree $4k + (4\ell - 6)m - (2k + (2\ell - 2)m)n$ and weight $-(2k + (2\ell - 2)m)n$ for $(k, m) \in \{0, \dots, \ell - 2\} \times \mathbb{N}$,
- $\mathbf{\alpha}_{k,m}$ of degree $4k + 1 + (4\ell - 6)m - (2k + (2\ell - 2)m)n$ and weight $-(2k + (2\ell - 2)m)n$ for $(k, m) \in \{0, \dots, \ell - 2\} \times \mathbb{N}$,
- if n is odd, $\mathbf{b}_{k,m}$ of degree $2k + 2\ell - 3 + (4\ell - 6)m - (k + \ell - 1 - (2\ell - 2)m)n$ and weight $-(k + \ell - 1 + (2\ell - 2)m)n$ for $\{(k, m) \in \{0, \dots, 2\ell - 4\} \times \mathbb{Z} \mid k + \ell \text{ is even and } k + \ell - 1 + m(2\ell - 2) \geq 0\}$,
- if n is odd, $\mathbf{\beta}_{k,m}$ of degree $2k + 2\ell - 2 + (4\ell - 6)m - (k + \ell - 1 - (2\ell - 2)m)n$ and weight $-(k + \ell - 1 - (2\ell - 2)m)n$ for $\{(k, m) \in \{0, \dots, 2\ell - 4\} \times \mathbb{Z} \mid k + \ell \text{ is even and } k + \ell + m(2\ell - 2) \geq 0\}$,
- if ℓ is even, \mathbf{c}_m of degree $2\ell - 4 + (4\ell - 6)m - (\ell - 2 + (2\ell - 2)m)n$ and weight $-(\ell - 2 + (2\ell - 2)m)n$ for $m \in \mathbb{N}$,

- if ℓ is even, γ_m of degree $2\ell - 3 + (4\ell - 6)m - (\ell - 2 + (2\ell - 2)m)n$ and weight $-(\ell - 2 + (2\ell - 2)m)n$ for $m \in \mathbb{N}$,
- if ℓ is even and n is odd, \mathbf{d}_m of degree $-1 + (4\ell - 6)m - (-1 + (2\ell - 2)m)n$ and weight $-(-1 + (2\ell - 2)m)n$ for $m \in \mathbb{N} \setminus \{0\}$,
- if ℓ is even and n is odd, δ_m of degree $(4\ell - 6)m - (-1 + (2\ell - 2)m)n$ and weight $-(-1 + (2\ell - 2)m)n$ for $m \in \mathbb{N}$,
- if ℓ is odd and n is even, \mathbf{e}_m of degree $2\ell - 4 + (4\ell - 6)m - (\ell - 2 + (2\ell - 2)m)n$ and weight $-(\ell - 2 + (2\ell - 2)m)n$ for $m \in \mathbb{N}$,
- if ℓ is odd and n is even, ϵ_m of degree $2\ell - 3 + (4\ell - 6)m - (\ell - 2 + (2\ell - 2)m)n$ and weight $-(\ell - 2 + (2\ell - 2)m)n$ for $m \in \mathbb{N}$, and
- \mathbf{s}_h of degree n and weight n , where h runs over a set consisting of
 - $\ell - 2$ elements if ℓ is even and n is odd,
 - $\ell - 1$ elements if both ℓ and n are odd, and
 - ℓ elements otherwise.

5.6. **Type E_6 .** Consider the case

$$(5.87) \quad \mathbf{w} = x_1^4 + x_2^3 + x_3^2 + \cdots + x_{n+1}^2 \in \mathbb{C}[x_0, x_1, \dots, x_{n+1}]$$

with

$$(5.88) \quad \Gamma = \Gamma_{\mathbf{w}} := \{ \gamma = (t_0, t_1, \dots, t_{n+1}) \in (\mathbb{G}_m)^{n+2} \mid t_1^4 = t_2^3 = t_3^2 = \cdots = t_{n+1}^2 = t_0 t_1 \cdots t_{n+1} \},$$

so that $\ker \chi \cong \mu_4 \times \mu_3 \times (\mu_2)^{n-1}$ and $\text{Char}(\Gamma)$ is generated by χ and $\chi_i = \deg x_i$ for $i \in \{0, \dots, n+1\}$ with relations

$$(5.89) \quad \chi = 4\chi_1 = 3\chi_2 = 2\chi_3 = \cdots = 2\chi_{n+1} = \chi_0 + \cdots + \chi_{n+1}.$$

5.6.1. For any $\gamma \in \ker \chi$, one has

$$(5.90) \quad \text{Jac}_{\mathbf{w}_\gamma} \cong \begin{cases} \mathbb{C}[x_0] & \mathbb{C}x_0 \subset V_\gamma \\ \mathbb{C} & \mathbb{C}x_0 \not\subset V_\gamma \end{cases} \otimes \begin{cases} \mathbb{C}[x_1]/(x_1^3) & \mathbb{C}x_1 \subset V_\gamma \\ \mathbb{C} & \mathbb{C}x_1 \not\subset V_\gamma \end{cases} \otimes \begin{cases} \mathbb{C}[x_2]/(x_2^2) & \mathbb{C}x_2 \subset V_\gamma \\ \mathbb{C} & \mathbb{C}x_2 \not\subset V_\gamma \end{cases}.$$

If we write an element of $\text{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$ as

$$(5.91) \quad x_0^{k_0} x_1^{k_1} x_2^{k_2} \otimes x_{j_1}^\vee \wedge x_{j_2}^\vee \wedge \cdots \wedge x_{j_s}^\vee,$$

where $k_i = 0$ if $\mathbb{C}x_i \not\subset V_\gamma$ for $i = 0, 1, 2$, then its degree is given by

$$(5.92) \quad k_0\chi_0 + k_1\chi_1 + k_2\chi_2 - \chi_{j_1} - \cdots - \chi_{j_s},$$

which can be proportional to χ only if $V_\gamma \cap (\mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1})$ is either $\mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1}$ or 0. We will assume this condition for the rest of Section 5.6.

5.6.2. Since $t_0 = 1$ implies $t_2 = 1$ and $t_1 = \pm 1$, one has the following:

- V_γ contains $\mathbb{C}x_0$ if and only if either
 - $\gamma = (1, \dots, 1)$, where $V_\gamma = V$,
 - $\gamma = (1, 1, 1, -1, \dots, -1)$ with odd n , where $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$,
 - $\gamma = (1, -1, 1, -1, \dots, -1)$ with even n , where $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_2$.

5.6.3. One has $V_\gamma = V$ if and only if γ is the identity element. The degree of $x_0^{k_0} x_1^{k_1} x_2^{k_2} \in \text{Jac}_{\mathbf{w}}$ is

$$(5.93) \quad k_0\chi - (k_0 - k_1)\chi_1 - (k_0 - k_2)\chi_2 - k_0\chi_3 - \cdots - k_0\chi_n,$$

which is proportional to χ if and only if 4 divides $k_0 - k_1$, 3 divides $k_0 - k_2$, if $n = 1$ and, in addition k_0 is even, if $n > 1$.

Thus, for $n = 1$, we must have

$$(5.94) \quad 5k_0 + 3k_1 + 4k_2 = 12m$$

for $m \in \mathbb{N}$, in which case the constant of proportionality is

$$(5.95) \quad t/2 = m$$

For each $m \in \mathbb{N}$ such that $5 \nmid m$, the equation (5.94) has a unique solution with $(k_1, k_2) \in \{0, 1, 2\} \times \{0, 1\}$ and if $5 \mid m$, then there are precisely two contributions with $(k_1, k_2) = (0, 0)$ and $(k_1, k_2) = (2, 1)$ such that $(k_1, k_2, m) \in \{0, 1, 2\} \times \{0, 1\} \times \mathbb{N}$ except if $m = 0$, then only $(k_1, k_2) = (0, 0)$ contributes. Each such (k_1, k_2, m) contributes $\mathbb{C}(k_0 n)$ to HH^t and HH^{t+1} .

For $n > 1$, in addition equation (5.94), we must have that k_0 is even. This forces $k_1 \neq 1$, and the possible (k_0, k_1, k_2) and $t = 2 \deg(x_0^{k_0} x_1^{k_1} x_2^{k_2}) / \chi$ are given by

| (k_1, k_2) | k_0 | t |
|--------------|------------|--------------------------|
| (0, 0) | $12m$ | $22m - 12mn$ |
| (0, 1) | $4 + 12m$ | $8 + 22m - (4 + 12m)n$ |
| (2, 0) | $6 + 12m$ | $12 + 22m - (6 + 12m)n$ |
| (2, 1) | $10 + 12m$ | $20 + 22m - (10 + 12m)n$ |

for $m \in \mathbb{N}$. Each (k_0, k_1, k_2) from (5.96) contributes $\mathbb{C}(k_0 n)$ to HH^t and HH^{t+1} .

5.6.4. One has $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$ if and only if n is odd and $\gamma = (1, 1, 1, -1, \dots, -1) \in \ker \chi$. The degree of

$$(5.97) \quad x_0^{k_0} x_1^{k_1} x_2^{k_2} \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$$

is given by

$$(5.98) \quad k_0 \chi - (k_0 - k_1) \chi_1 - (k_0 - k_2) \chi_2 - (k_0 + 1) \chi_3 - \cdots - (k_0 + 1) \chi_{n+1},$$

which is proportional to χ if and only if k_0 is odd, 4 divides $k_0 - k_1$, and 3 divides $k_0 - k_2$. This forces $k_1 = 1$ and the possible (k_0, k_1, k_2) and

$$(5.99) \quad t = 2 \deg(x_0^{k_0} x_1^{k_1} x_2^{k_2} \otimes x_3^\vee \wedge \cdots \wedge x_n^\vee) / \chi + \dim N_\gamma$$

are given by

| (k_1, k_2) | k_0 | t |
|--------------|-----------|-------------------------|
| (1, 0) | $9 + 12m$ | $17 + 22m - (9 + 12m)n$ |
| (1, 1) | $1 + 12m$ | $1 + 22m - (1 + 12m)n$ |

for $m \in \mathbb{N}$. Each (k_0, k_1, k_2) from (5.100) contributes $\mathbb{C}(k_0 n)$ to HH^t and HH^{t+1} .

5.6.5. One has $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_2$ if and only if n is even and $\gamma = (1, -1, 1, -1, \dots, -1) \in \ker \chi$. The degree of

$$(5.101) \quad x_0^{k_0} x_2^{k_2} \otimes x_1^\vee \wedge x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$$

is given by

$$(5.102) \quad k_0 \chi - (k_0 + 1) \chi_1 - (k_0 - k_2) \chi_2 - (k_0 + 1) \chi_3 - \cdots - (k_0 + 1) \chi_{n+1},$$

which is proportional to χ if and only if 4 divides $k_0 + 1$ and 3 divides $k_0 - k_2$. The possible (k_0, k_2) and

$$(5.103) \quad t = 2 \deg(x_0^{k_0} x_2^{k_2} \otimes x_1^\vee \wedge x_3^\vee \wedge \cdots \wedge x_n^\vee) / \chi + \dim N_\gamma$$

are given by

| k_2 | k_0 | t |
|-------|-----------|-------------------------|
| 0 | $3 + 12m$ | $6 + 22m - (3 + 12m)n$ |
| 1 | $7 + 12m$ | $14 + 22m - (7 + 12m)n$ |

for $m \in \mathbb{N}$. Each (k_0, k_2) from (5.104) contributes $\mathbb{C}(k_0 n)$ to HH^t and HH^{t+1} .

5.6.6. If $V_\gamma = \mathbb{C}x_1$, then one has

$$(5.105) \quad \deg(x_1^{k_1} \otimes x_0^\vee \wedge x_2^\vee \wedge \cdots \wedge x_{n+1}^\vee) = -\chi_0 + k_1\chi_1 - \chi_2 - \cdots - \chi_{n+1}$$

$$(5.106) \quad = -\chi + (k_1 + 1)\chi_1,$$

which is not proportional to χ for any $k_1 \in \{0, 1, 2\}$. Similarly, γ with $\mathbb{C}x_0 \not\subset V_\gamma$ and $V_\gamma \neq 0$ does not contribute to HH^* .

5.6.7. One has $V_\gamma = 0$ if and only if $t_1 \in (\boldsymbol{\mu}_4 \setminus \{1\})$, $t_2 \in (\boldsymbol{\mu}_3 \setminus \{1\})$, and $t_3 = \cdots, t_{n+1} = -1$, since $t_2 \neq 1$ implies $t_0 = (-1)^{n-1}t_1^{-1}t_2^{-1} \neq 1$. There are six such γ , and each of them contributes $\mathbb{C}(-n)$ to HH^n .

5.7. **Type E_7 .** Consider the case

$$(5.107) \quad \mathbf{w} = x_1^3x_2 + x_2^3 + x_3^2 + \cdots + x_{n+1}^2 \in \mathbb{C}[x_0, x_1, \dots, x_{n+1}]$$

with

$$(5.108) \quad \Gamma = \Gamma_{\mathbf{w}} := \{\gamma = (t_0, \dots, t_{n+1}) \in (\mathbb{G}_m)^{n+2} \mid t_1^3t_2 = t_2^3 = t_3^2 = \cdots = t_{n+1}^2 = t_0 \cdots t_{n+1}\},$$

so that $\ker \chi \cong \boldsymbol{\mu}_9 \times (\boldsymbol{\mu}_2)^{n-1}$ and $\mathrm{Char}(\Gamma)$ is generated by χ and $\chi_i = \deg x_i$ for $i \in \{0, \dots, n+1\}$ with relations

$$(5.109) \quad \chi = 3\chi_1 + \chi_2 = 3\chi_2 = 2\chi_3 = \cdots = 2\chi_{n+1} = \chi_0 + \cdots + \chi_{n+1}.$$

These relations imply

$$(5.110) \quad \chi_2 = \chi - 3\chi_1,$$

$$(5.111) \quad 9\chi_1 = 2\chi,$$

$$(5.112) \quad \chi_0 = \chi - \chi_1 - \cdots - \chi_{n+1}$$

$$(5.113) \quad = 2\chi_1 - \chi_3 - \cdots - \chi_{n+1}.$$

5.7.1. For any $\gamma \in \ker \chi$, the intersection $V_\gamma \cap (\mathbb{C}x_1 \oplus \mathbb{C}x_2)$ can be either $\mathbb{C}x_1 \oplus \mathbb{C}x_2$, $\mathbb{C}x_2$, or 0, where $\mathrm{Jac}_{\mathbf{w}_\gamma}$ is isomorphic to $\mathbb{C}[x_1, x_2]/(3x_1^2x_2, x_1^3 + 3x_2^2)$, $\mathbb{C}[x_2]/(3x_2^2)$, or \mathbb{C} respectively. A basis of $\mathbb{C}[x_1, x_2]/(3x_1^2x_2, x_1^3 + 3x_2^2)$ is given by $\{1, x_1, x_1^2, x_1^3, x_1^4, x_2, x_1x_2\}$.

If we write an element of $\mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$ as

$$(5.114) \quad x_0^{k_0} x_1^{k_1} x_2^{k_2} \otimes x_{j_1}^\vee \wedge x_{j_2}^\vee \wedge \cdots \wedge x_{j_s}^\vee,$$

then its degree is given by

$$(5.115) \quad k_0\chi_0 + k_1\chi_1 + k_2\chi_2 - \chi_{j_1} - \cdots - \chi_{j_s},$$

which can be proportional to χ only if $V \cap (\mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1})$ is either $\mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1}$ or 0. We assume this condition for the rest of Section 5.7.

5.7.2. Since $t_0 = 1$ implies $t_1 = t_2 = 1$, one has $\mathbb{C}x_0 \subset V_\gamma$ if and only if either $V_\gamma = V$ or $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$.

5.7.3. One has $V_\gamma = V$ if and only if γ is the identity element. The degree of $x_0^{k_0} x_1^{k_1} x_2^{k_2} \in \mathrm{Jac}_{\mathbf{w}}$ is

$$(5.116) \quad k_0(2\chi_1 - \chi_3 - \cdots - \chi_{n+1}) + k_1\chi_1 + k_2(\chi - 3\chi_1)$$

$$(5.117) \quad = k_2\chi + (2k_0 + k_1 - 3k_2)\chi_1 - k_0\chi_3 - \cdots - k_0\chi_{n+1},$$

which is proportional to χ if and only if, 9 divides $2k_0 + k_1 - 3k_2$ if $n = 1$ and in addition k_0 is even, if $n > 1$. Thus, for $n = 1$, the constant of proportionality is

$$(5.118) \quad t/2 = k_2 + \frac{2}{9}(2k_0 + k_1 - 3k_2)$$

The possible (k_0, k_1, k_2) and $t = 2 \deg(x_0^{k_0} x_1^{k_1} x_2^{k_2})/\chi$ are given by

| (k_1, k_2) | k_0 | t |
|--------------|----------|----------|
| $(0, 0)$ | $9m$ | $8m$ |
| $(1, 0)$ | $4 + 9m$ | $4 + 8m$ |
| $(2, 0)$ | $8 + 9m$ | $8 + 8m$ |
| $(3, 0)$ | $3 + 9m$ | $4 + 8m$ |
| $(4, 0)$ | $7 + 9m$ | $8 + 8m$ |
| $(0, 1)$ | $6 + 9m$ | $5 + 8m$ |
| $(1, 1)$ | $1 + 9m$ | $1 + 8m$ |

for $m \in \mathbb{N}$.

For $n > 1$, the constant of proportionality is

$$(5.120) \quad t/2 = k_2 + \frac{2}{9}(2k_0 + k_1 - 3k_2) - \frac{1}{2}k_0(n-1).$$

The possible (k_0, k_1, k_2) and $t = 2 \deg(x_0^{k_0} x_1^{k_1} x_2^{k_2})/\chi$ are given by

| (k_1, k_2) | k_0 | t |
|--------------|------------|--------------------------|
| $(0, 0)$ | $18m$ | $34m - 18mn$ |
| $(1, 0)$ | $4 + 18m$ | $8 + 34m - (4 + 18m)n$ |
| $(2, 0)$ | $8 + 18m$ | $16 + 34m - (8 + 18m)n$ |
| $(3, 0)$ | $12 + 18m$ | $24 + 34m - (12 + 18m)n$ |
| $(4, 0)$ | $16 + 18m$ | $32 + 34m - (16 + 18m)n$ |
| $(0, 1)$ | $6 + 18m$ | $12 + 34m - (6 + 18m)n$ |
| $(1, 1)$ | $10 + 18m$ | $20 + 34m - (10 + 18m)n$ |

for $m \in \mathbb{N}$. Each (k_0, k_1, k_2) from (5.121) contributes $\mathbb{C}(k_0n)$ to HH^t and HH^{t+1} .

5.7.4. For $n > 1$, in addition, one has $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$ if and only if n is odd and $\gamma = (1, 1, 1, -1, \dots, -1)$. The degree of

$$(5.122) \quad x_0^{k_0} x_1^{k_1} x_2^{k_2} \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$$

is given by

$$(5.123) \quad k_0(2\chi_1 - \chi_3 - \cdots - \chi_n) + k_1\chi_1 + k_2(\chi - 3\chi_1) - \chi_3 - \cdots - \chi_{n+1}$$

$$(5.124) \quad = k_2\chi + (2k_0 + k_1 - 3k_2)\chi_1 - (k_0 + 1)\chi_3 - \cdots - (k_0 + 1)\chi_{n+1},$$

which is proportional to χ if and only if 9 divides $2k_0 + k_1 - 3k_2$ and k_0 is odd, in which case the constant of proportionality is

$$(5.125) \quad k_2 + \frac{2}{9}(2k_0 + k_1 - 3k_2) - \frac{1}{2}(k_0 + 1)(n-1).$$

The possible (k_0, k_1, k_2) and

$$(5.126) \quad t = 2 \deg(x_0^{k_0} x_1^{k_1} x_2^{k_2} \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee) / \chi + \dim N_\gamma$$

are given by

| (k_1, k_2) | k_0 | t |
|--------------|------------|--------------------------|
| $(0, 0)$ | $9 + 18m$ | $18 + 34m - (9 + 18m)n$ |
| $(1, 0)$ | $13 + 18m$ | $26 + 34m - (13 + 18m)n$ |
| $(2, 0)$ | $17 + 18m$ | $24 + 34m - (17 + 18m)n$ |
| $(3, 0)$ | $3 + 18m$ | $6 + 34m - (3 + 18m)n$ |
| $(4, 0)$ | $7 + 18m$ | $14 + 34m - (7 + 18m)n$ |
| $(0, 1)$ | $15 + 18m$ | $30 + 34m - (15 + 18m)n$ |
| $(1, 1)$ | $1 + 18m$ | $2 + 34m - (1 + 18m)n$ |

for $m \in \mathbb{N}$. Each (k_0, k_1, k_2) from (5.127) contributes $\mathbb{C}(k_0n)$ to HH^t and HH^{t+1} .

In addition, for the case $(k_1, k_2) = (2, 0)$, the element $x_0^\vee \otimes x_1^2 \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee$ corresponding to $m = -1$ in (5.127) has degree 0, and contributes $\mathbb{C}(-n)$ to HH^n .

5.7.5. For any $t_1 \in \mu_9$ satisfying $t_2 := t_1^{-3} \neq 1$, the element $\gamma = ((-1)^{n-1}t_1^2, t_1, t_1^{-3}, -1, \dots, -1) \in \ker \chi$ satisfies $V_\gamma = 0$. There are six such elements, each of which contributes $\mathbb{C}(-n)$ to HH^n .

5.8. **Type E_8 .** Consider the case

$$(5.128) \quad \mathbf{w} = x_1^5 + x_2^3 + x_3^2 + \cdots + x_{n+1}^2 \in \mathbb{C}[x_0, x_1, \dots, x_{n+1}]$$

with

$$(5.129) \quad \Gamma = \Gamma_{\mathbf{w}} := \{ \gamma = (t_0, \dots, t_{n+1}) \in (\mathbb{G}_m)^{n+2} \mid t_1^5 = t_2^3 = t_3^2 = \cdots = t_{n+1}^2 = t_0 \cdots t_{n+1} \},$$

so that $\ker \chi \cong \mu_5 \times \mu_3 \times (\mu_2)^{n-1}$ and $\mathrm{Char}(\Gamma)$ is generated by χ and $\chi_i = \deg x_i$ for $i \in \{0, \dots, n+1\}$ with relations

$$(5.130) \quad \chi = 5\chi_1 = 3\chi_2 = 2\chi_3 = \cdots = 2\chi_{n+1} = \chi_0 + \cdots + \chi_{n+1}.$$

5.8.1. If we write an element of $\mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$ as

$$(5.131) \quad x_0^{k_0} x_1^{k_1} x_2^{k_2} \otimes x_{j_1}^\vee \wedge x_{j_2}^\vee \wedge \cdots \wedge x_{j_s}^\vee,$$

then its degree is given by

$$(5.132) \quad k_0\chi_0 + k_1\chi_1 + k_2\chi_2 - \chi_{j_1} - \cdots - \chi_{j_s},$$

which can be proportional to χ only if $V \cap (\mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1})$ is either $\mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{n+1}$ or 0. We assume this condition for the rest of Section 5.8.

5.8.2. Since $t_0 = 1$ implies $t_1 = t_2 = 1$, one has $\mathbb{C}x_0 \subset V_\gamma$ if and only if either $V_\gamma = V$ or $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$.

5.8.3. One has $V_\gamma = V$ if and only if γ is the identity element. The degree of $x_0^{k_0} x_1^{k_1} x_2^{k_2} \in \mathrm{Jac}_{\mathbf{w}}$ is

$$(5.133) \quad k_0\chi - (k_0 - k_1)\chi_1 - (k_0 - k_2)\chi_2 - k_0\chi_3 - \cdots - k_0\chi_{n+1},$$

which is proportional to χ if and only if

$$(5.134) \quad k_0 \equiv k_1 \pmod{5},$$

$$(5.135) \quad k_0 \equiv k_2 \pmod{3},$$

if $n = 1$ and, in addition to the above,

$$(5.136) \quad k_0 \equiv 0 \pmod{2},$$

if $n > 1$.

Thus, for $n = 1$, we must have

$$(5.137) \quad 7k_0 + 3k_1 + 5k_2 = 15m$$

for $m \in \mathbb{N}$, in which case the constant of proportionality is

$$(5.138) \quad t/2 = m$$

For each $m \in \mathbb{N}$ such that $7 \nmid m$, the equation (5.137) has a unique solution with $(k_1, k_2) \in \{0, 1, 2, 3\} \times \{0, 1\}$ and if $7 \mid m$, then there are precisely two contributions with $(k_1, k_2) = (0, 0)$ and $(k_1, k_2) = (3, 1)$ such that $(k_1, k_2, m) \in \{0, 1, 2, 3\} \times \{0, 1\} \times \mathbb{N}$ except if $m = 0$, then only $(k_1, k_2) = (0, 0)$ contributes.

For $n > 1$, we must have

$$(5.139) \quad 7k_0 + 3k_1 + 5k_2 = 15m$$

for $m \in \mathbb{N}$, and in addition k_0 must be in $2\mathbb{N}$. Thus, we can re-write (5.139) as

$$(5.140) \quad k_0 = 6k_1 + 10k_2 + 30m'$$

with $m' = k_0/2 - m$. The constant of proportionality is

$$(5.141) \quad t/2 = \frac{(2-n)k_0}{2} - m' = 6k_1 + 10k_2 + 29m' - (3k_1 + 5k_2 + 15m')n$$

Each $(k_1, k_2, m') \in \{0, 1, 2, 3\} \times \{0, 1\} \times \mathbb{N}$ contributes $\mathbb{C}(k_0n)$ to HH^t and HH^{t+1} .

5.8.4. If $n > 1$, in addition, one has $V_\gamma = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$ if and only if n is odd and $\gamma = (1, 1, 1, -1, \dots, -1)$. The degree of

$$(5.142) \quad x_0^{k_0} x_1^{k_1} x_2^{k_2} \otimes x_3^\vee \wedge \cdots \wedge x_{n+1}^\vee \in \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee$$

is

$$(5.143) \quad k_0\chi - (k_0 - k_1)\chi_1 - (k_0 - k_2)\chi_2 - (k_0 + 1)\chi_3 - \cdots - (k_0 + 1)\chi_{n+1},$$

which is proportional to χ if and only if

$$(5.144) \quad 14k_0 + 6k_1 + 10k_2 - 15(n-1) = 30m$$

for $m \in \mathbb{Z}$ and in addition we must have k_0 odd. Thus, again we can rewrite (5.144) as

$$(5.145) \quad k_0 = 15 + 6k_1 + 10k_2 + 30m'$$

where $m' = (k_0 - n)/2 - m$. The constant of proportionality is

$$(5.146) \quad (t - \dim N_\gamma)/2 = m - \frac{(n-1)k_0}{2} = 15 + 6k_1 + 10k_2 + 29m' - (8 + 3k_1 + 5k_2 + 15m)n$$

Each $(k_1, k_2, m') \in \{0, 1, 2, 3\} \times \{0, 1\} \times \mathbb{Z}$ such that

$$15 + 6k_1 + 10k_2 + 30m' \geq 0$$

contributes $\mathbb{C}(k_0n)$ to HH^t and HH^{t+1} .

5.8.5. For any $(t_1, t_2) \in (\boldsymbol{\mu}_5 \setminus \{1\}) \times (\boldsymbol{\mu}_3 \setminus \{1\})$, the element $\gamma = ((-1)^{n-1}(t_1 t_2)^{-1}, t_1, t_2, -1, \dots, -1)$ satisfies $V_\gamma = 0$. There are eight such elements, each of which contributes $\mathbb{C}(-n)$ to HH^n .

REFERENCES

- [AAK16] Mohammed Abouzaid, Denis Auroux, and Ludmil Katzarkov, *Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces*, Publ. Math. Inst. Hautes Études Sci. **123** (2016), 199–282. MR 3502098 1
- [Abo11] Mohammed Abouzaid, *A topological model for the Fukaya categories of plumbings*, J. Differential Geom. **87** (2011), no. 1, 1–80. MR 2786590 (2012h:53193) 3
- [AGZV85] V. I. Arnol'd, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of differentiable maps. Vol. I*, Monographs in Mathematics, vol. 82, Birkhäuser Boston Inc., Boston, MA, 1985, The classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous and Mark Reynolds. MR 777682 (86f:58018) 5
- [AR87] Maurice Auslander and Idun Reiten, *Almost split sequences for \mathbf{Z} -graded rings*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 232–243. MR 915178 (89b:13031) 4
- [Aur07] Denis Auroux, *Mirror symmetry and T-duality in the complement of an anticanonical divisor*, J. Gökova Geom. Topol. GGT **1** (2007), 51–91. MR 2386535 (2009f:53141) 1
- [Aur09] ———, *Special Lagrangian fibrations, wall-crossing, and mirror symmetry*, Surveys in differential geometry. Vol. XIII. Geometry, analysis, and algebraic geometry: forty years of the Journal of Differential Geometry, Surv. Differ. Geom., vol. 13, Int. Press, Somerville, MA, 2009, pp. 1–47. MR MR2537081 1
- [BEE12] Frédéric Bourgeois, Tobias Ekholm, and Yasha Eliashberg, *Effect of Legendrian surgery*, Geom. Topol. **16** (2012), no. 1, 301–389, With an appendix by Sheel Ganatra and Maksim Maydanskiy. MR 2916289 3
- [BFK14] Matthew Ballard, David Favero, and Ludmil Katzarkov, *A category of kernels for equivariant factorizations and its implications for Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **120** (2014), 1–111. MR 3270588 4, 5
- [BH93] Per Berglund and Tristan Hübsch, *A generalized construction of mirror manifolds*, Nuclear Phys. B **393** (1993), no. 1-2, 377–391. MR MR1214325 (94k:14031) 1
- [BH95] Per Berglund and Måns Henningson, *Landau-Ginzburg orbifolds, mirror symmetry and the elliptic genus*, Nuclear Phys. B **433** (1995), no. 2, 311–332. MR 1310310 (96d:58131) 1
- [CT13] Andrei Căldăraru and Junwu Tu, *Curved A_∞ algebras and Landau-Ginzburg models*, New York J. Math. **19** (2013), 305–342. MR 3084707 5

- [CJY04] Ralph L. Cohen, John D. S. Jones, and Jun Yan, *The loop homology algebra of spheres and projective spaces*, Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001), Progr. Math., vol. 215, Birkhäuser, Basel, 2004, pp. 77–92. MR 2039760 9
- [CRGG] Baptiste Chantraine, Georgios Dimitroglou Rizell, Paolo Ghiggini, and Roman Golovko, *Geometric generation of the wrapped Fukaya category of Weinstein manifolds and sectors*, arXiv:1712.09126. 3, 5
- [Dyc11] Tobias Dyckerhoff, *Compact generators in categories of matrix factorizations*, Duke Math. J. **159** (2011), no. 2, 223–274. MR 2824483 (2012h:18014) 4, 5
- [Ekh] Tobias Ekholm, *Holomorphic curves for Legendrian surgery*, arXiv:1906.07228. 3
- [EL] Tobias Ekholm and Yankı Lekili, *Duality between Lagrangian and Legendrian invariants*, arXiv:1701.01284. 3
- [EL17] Tolga Etgü and Yankı Lekili, *Koszul duality patterns in Floer theory*, Geom. Topol. **21** (2017), no. 6, 3313–3389. MR 3692968 3, 5, 9
- [EL19] Tolga Etgü and Yankı Lekili, *Fukaya categories of plumbings and multiplicative preprojective algebras*, Quantum Topol. **10** (2019), no. 4, 777–813. MR 4033516 4
- [EP15] Alexander I. Efimov and Leonid Positselski, *Coherent analogues of matrix factorizations and relative singularity categories*, Algebra Number Theory **9** (2015), no. 5, 1159–1292. MR 3366002 2
- [FU11] Masahiro Futaki and Kazushi Ueda, *Homological mirror symmetry for Brieskorn-Pham singularities*, Selecta Math. (N.S.) **17** (2011), no. 2, 435–452. MR 2803848 (2012e:14083) 4
- [FU13] ———, *Homological mirror symmetry for singularities of type D*, Math. Z. **273** (2013), no. 3-4, 633–652. MR 3030671 4
- [Gan12] Sheel Ganatra, *Symplectic Cohomology and Duality for the Wrapped Fukaya Category*, ProQuest LLC, Ann Arbor, MI, 2012, Thesis (Ph.D.)—Massachusetts Institute of Technology. MR 3121862 5
- [Gao] Yuan Gao, *Functors of wrapped Fukaya categories from Lagrangian correspondences*, arXiv:1712.00225. 5
- [Gin] Victor Ginzburg, *Calabi–Yau algebras*, unpublished, arXiv:0612139. 2
- [Hab] Matthew Habermann, *Homological mirror symmetry for invertible polynomials in two variables*, preprint arXiv:2003.01106. 2
- [HS] Matthew Habermann and Jack Smith, *Homological Berglund–Hübsch mirror symmetry for curve singularities*, to appear in J. Symplectic Geom. arXiv:1903.01351. 4
- [Her] Estanislao Herscovich, *Hochschild (co)homology of Koszul dual pairs* J. Noncommut. Geom. **13** (2019), no. 1, 59–85. 3
- [IT13] Osamu Iyama and Ryo Takahashi, *Tilting and cluster tilting for quotient singularities*, Math. Ann. **356** (2013), no. 3, 1065–1105. MR 3063907 4
- [Kel11] Bernhard Keller, *Deformed Calabi–Yau completions*, J. Reine Angew. Math. **654** (2011), 125–180, With an appendix by Michel Van den Bergh. MR 2795754 2
- [KMVdB11] Bernhard Keller, Daniel Murfet, and Michel Van den Bergh, *On two examples by Iyama and Yoshino*, Compos. Math. **147** (2011), no. 2, 591–612. MR 2776613 4
- [KvK16] Myeonggi Kwon and Otto van Koert, *Brieskorn manifolds in contact topology*, Bull. Lond. Math. Soc. **48** (2016), no. 2, 173–241. MR 3483060 5
- [LU] Yankı Lekili and Kazushi Ueda, *Homological mirror symmetry for Milnor fibers via moduli of A_∞ -structures*, arXiv:1806.04345. 1, 2, 4, 5, 6
- [Li19] Yin Li, *Koszul duality via suspending Lefschetz fibrations*, J. Topol. **12** (2019), no. 4, 1174–1245. MR 3977875 3
- [Orl11] Dmitri Orlov, *Formal completions and idempotent completions of triangulated categories of singularities*, Adv. Math. **226** (2011), no. 1, 206–217. MR 2735755 4, 5
- [Seg13] Ed Segal, *The closed state space of affine Landau–Ginzburg B-models*, J. Noncommut. Geom. **7** (2013), no. 3, 857–883. MR 3108698 5
- [Sei01] Paul Seidel, *More about vanishing cycles and mutation*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publishing, River Edge, NJ, 2001, pp. 429–465. MR MR1882336 (2003c:53125) 3
- [Sei10] ———, *Suspending Lefschetz fibrations, with an application to local mirror symmetry*, Comm. Math. Phys. **297** (2010), no. 2, 515–528. MR 2651908 3
- [Sei12] ———, *Lagrangian homology spheres in (A_m) Milnor fibres via \mathbb{C}^* -equivariant A_∞ -modules*, Geom. Topol. **16** (2012), no. 4, 2343–2389. MR 3033519 6
- [SS12] Paul Seidel and Jake P. Solomon, *Symplectic cohomology and q-intersection numbers*, Geom. Funct. Anal. **22** (2012), no. 2, 443–477. MR 2929070 6
- [Tak] Atsushi Takahashi, *Matrix factorizations and representations of quivers I*, math.AG/0506347. 4
- [Ueb16] Peter Uebele, *Symplectic homology of some Brieskorn manifolds*, Math. Z. **283** (2016), no. 1-2, 243–274. MR 3489066 5
- [Vit] Claude Viterbo, *Functors and computations in Floer homology with applications Part II*, arXiv:1805.01316. 9

DEPARTMENT OF MATHEMATICS KING'S COLLEGE LONDON STRAND LONDON WC2R 2LS

Email address: `yanki.lekili@kcl.ac.uk`

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO, 3-8-1 KOMABA, MEGURO-KU,
TOKYO, 153-8914, JAPAN.

Email address: `kazushi@ms.u-tokyo.ac.jp`