

KODAIRA FIBRES AND WRAPPED FLOER COHOMOLOGY

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ABSTRACT. Let F be a singular fibre of a relatively minimal complex elliptic fibration with smooth total space, and let Ω be a nonvanishing holomorphic two-form near F . We show that a small neighbourhood of F is a Weinstein domain for $\text{Re } \Omega$ whose completion is a Legendrian surgery, with cocores obtained by completing holomorphic disks transverse to the components of F . For every coefficient field and every multiplicative bulk class, we compute the wrapped Floer cohomology of these cocores and prove that it is concentrated in degree zero. The cocores generate, so the bulk-deformed wrapped Fukaya category is equivalent to the category of perfect modules over an explicit algebra: a multiplicative preprojective algebra of affine type for the normal crossing fibres, and a quiver algebra with relations for types *II*, *III* and *IV*. Applications include formality of the affine plumbing dg algebras, mirror equivalences with resolved affine surfaces at the trivial bulk class, and with quotient stacks of algebraic tori at root-of-unity bulk classes for the four star-shaped fibres.

1. INTRODUCTION

Let $\pi : X \rightarrow \Delta$ be a relatively minimal complex elliptic fibration with smooth total space, proper over a disk, and with just one singular fibre,

$$F = \sum_{i=1}^r m_i C_i,$$

and let Ω be a nonvanishing holomorphic two-form near F . A holomorphic disk transverse to C_i at a smooth point of F_{red} is Lagrangian for $\text{Re } \Omega$, and its projection to the base has degree m_i . Our goal is to determine the algebra of bulk-deformed wrapped morphisms between the exact planes obtained by completing these disks.

Let $W = \{|\pi| \leq \epsilon\}$ be a small neighbourhood of F . We show in Section 2 that W is a Weinstein domain for $\text{Re } \Omega$ whose completion $M = M_F$ is a Legendrian surgery determined by the Kodaira type of F , and that these planes are the cocores L_i of its 2-handles (Theorem 2.1). For the normal crossing types the attaching link is the link of the plumbing of two-spheres according to the dual graph of F ; for *II*, *III* and *IV* it is the rainbow closure of σ_1^3 , σ_1^4 and $(\sigma_1 \sigma_2)^3$.

Fix a coefficient field $\mathbf{k}$. A multiplicative bulk class is a character

$$\mathbf{b} : H_2(M; \mathbb{Z}) \longrightarrow \mathbf{k}^\times,$$

equivalently a class in $H^2(M; \mathbf{k}^\times)$. It modifies Floer operations by assigning multiplicative weights to holomorphic-curve counts, as described in Section 5.1. Since M retracts onto F , the component classes $[C_i]$ form a basis of $H_2(M; \mathbb{Z})$, so $\mathbf{b}$ is specified by the independent parameters

$$q_i = \mathbf{b}([C_i]), \quad \mathbf{q} = (q_i) \in (\mathbf{k}^\times)^r.$$

We write $\mathcal{W}_{\mathbf{b}}(M)$ for the split-closed bulk-deformed wrapped Fukaya category. Its branes carry trivialisations of the bulk class along their Lagrangians; the contractible planes L_i therefore define objects for every $\mathbf{b}$. When all $q_i = 1$, this is the ordinary wrapped Fukaya category.

Write $\Lambda^{\mathbf{q}}(Q)$ for the multiplicative preprojective algebra of Crawley-Boevey and Shaw [CS06] with vertex parameters q_i , recalled in Equation (3.2), and let $A_F(\mathbf{q})$ be the algebra in Table 1; the algebras A_{III} and A_{IV} are the quotients of the path algebras of the quivers in Figure 9 by the relations (3.18) and (3.19). Let $e_i \in A_F(\mathbf{q})$ be the vertex idempotent corresponding to C_i and its cocore L_i . We give the cocores their holomorphic gradings, which restrict to grading zero on the transverse holomorphic disks (Section 4.1).

Fibre	$A_F(\mathbf{q})$
$I_n, n \geq 1$	$\Lambda^{\mathbf{q}}(\tilde{A}_{n-1})$
$I_n^*, n \geq 0$	$\Lambda^{\mathbf{q}}(\tilde{D}_{n+4})$
IV^*, III^*, II^*	$\Lambda^{\mathbf{q}}(\tilde{E}_6), \Lambda^{\mathbf{q}}(\tilde{E}_7), \Lambda^{\mathbf{q}}(\tilde{E}_8)$
II	$\mathbf{k}\langle x, y, z \rangle / (q + x + z + zyx, 1 + x + z + xyz)$
III, IV	$A_{III}(q_1, q_2), A_{IV}(q_1, q_2, q_3)$

TABLE 1. The algebras associated to Kodaira fibres; see Equations (3.2) and (3.17) to (3.19).

Theorem 1.1. *Let F be a singular fibre, M the completion of a neighbourhood of F , and $L_1, \dots, L_r$ the cocores obtained by completing holomorphic disks transverse to the components $C_1, \dots, C_r$ at smooth points of F_{red} , with their holomorphic gradings. For every field $\mathbf{k}$ and every $\mathbf{b} \in H^2(M; \mathbf{k}^\times)$, the Lagrangians L_i generate $\mathcal{W}_{\mathbf{b}}(M)$ and*

$$\text{HW}_{\mathbf{b}}^*(L_i, L_j; \mathbf{k}) = \begin{cases} e_j A_F(\mathbf{q}) e_i, & * = 0, \\ 0, & * \neq 0. \end{cases}$$

Their endomorphism A_∞ algebra has minimal model $A_F(\mathbf{q})$. Consequently,

$$\mathcal{W}_{\mathbf{b}}(M) \simeq \text{Perf } A_F(\mathbf{q}).$$

We first realise M as a Legendrian surgery whose cocores are the completions of the transverse holomorphic disks, and compute the Chekanov–Eliashberg dg algebra of its attaching link with bulk parameters. A uniform estimate for the relative periods of the elliptic fibres shows that convex wrapping produces only degree-zero chords, independently of the bulk class. The surgery formula and cocore generation then identify $\mathcal{W}_{\mathbf{b}}(M)$ with $\text{Perf } A_F(\mathbf{q})$.

Corollary 1.2. *For every affine diagram Q occurring in Table 1, every field $\mathbf{k}$, and every $\mathbf{q} \in (\mathbf{k}^\times)^{Q_0}$, the canonical projection*

$$\mathcal{B}_Q^{\mathbf{q}} \longrightarrow H^0(\mathcal{B}_Q^{\mathbf{q}}) \cong \Lambda^{\mathbf{q}}(Q)$$

is a quasi-isomorphism. Here $\mathcal{B}_Q^{\mathbf{q}}$ is the plumbing dg algebra defined in Definition 3.4.

Proof. The surgery comparison identifies its cohomology with the wrapped endomorphisms in Theorem 1.1; see Proposition 5.3. These are concentrated in degree zero, where the projection is the identity on $\Lambda^{\mathbf{q}}(Q)$. $\square$

Remark 1.3. Kaplan and Schedler conjecture that the dg multiplicative preprojective algebra of every connected non-Dynkin quiver is quasi-isomorphic to its degree-zero cohomology [KS23, Conjecture 1.3]. They prove this for quivers containing a cycle [KS23, Theorem 3.7 and Proposition 4.4]. Corollary 1.2 establishes the conjecture for every extended Dynkin quiver, over every field and at every parameter tuple; the new cases are the affine D and E diagrams. The comparison with the plumbing dg algebra is explained in Section 3.2. Together with Kaplan–Schedler’s self-duality theorem, this also proves the 2-Calabi–Yau assertion of their Conjecture 1.1 [KS23] for affine D and E quivers, over every field and at every nonzero parameter tuple; see Lemma 6.7.

For $F = I_n$ ($n \geq 1$), I_n^* ($n \geq 0$), IV^* , III^* and II^* , let $Q = \tilde{A}_{n-1}$, $\tilde{D}_{n+4}$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$, respectively, and write M_Q for the completion of a neighbourhood of F , which by Theorem 2.1 is a plumbing of cotangent bundles of two-spheres according to Q . Put

$$S_Q = \text{Spec } Z(\Lambda^1(Q)).$$

Explicitly, the equations of S_Q are given in the following table.

$$\begin{array}{l|l} \tilde{A}_{n-1} & xyz + xy + z^n = 0 \\ \tilde{D}_{n+4} & xyz + xy^2 - p_{n-1}x^2y + p_nxz - z^2 = 0 \\ \tilde{E}_6 & xyz + z^2 + x^2z + y^3 = 0 \\ \tilde{E}_7 & xyz + z^2 + y^3 + x^3y = 0 \\ \tilde{E}_8 & xyz + z^2 + y^3 + x^5 = 0. \end{array}$$

Here $p_{-1} = -1$, $p_0 = 0$, and $p_{j+1} = x(p_j + p_{j-1})$; see [KS23, Corollary 6.7] for type A and Section 6.3 for types D and E .

Corollary 1.4. *For the affine D and E types, assume that $\text{char } \mathbf{k}$ is different from 2 in type $\tilde{D}_{n+4}$, from 2, 3 in types $\tilde{E}_6, \tilde{E}_7$, and from 2, 3, 5 in type $\tilde{E}_8$. The origin of S_Q is a $\mathbf{k}$ -rational du Val singularity of type D_{n+4} , E_6 , E_7 or E_8 , respectively. There is one additional $\mathbf{k}$ -rational split A_1 point precisely in type $\tilde{D}_{n+4}$ with $n \geq 1$ and $\text{char } \mathbf{k} = p > 2$ satisfying $p \mid 2n^n - (1 - n)^{n-1}$ and $p \nmid 2n - 1$; otherwise the origin is the only singular point. If $\tilde{S}_Q \rightarrow S_Q$ is the minimal resolution, then*

$$\mathcal{W}(M_Q) \simeq \text{Perf}(\Lambda^1(Q)) \simeq \text{Perf}(\tilde{S}_Q).$$

For $F = II, III, IV$, over every field let S_F be the affine surface with equation

$$\begin{array}{l|l} II & xyz + x + z + 1 = 0, \\ III & xyz + x + y - z + 2 = 0, \\ IV & xyz + x^2 + 3x + y - z + 3 = 0. \end{array}$$

Then $S_F = \text{Spec } Z(A_F(\mathbf{1}))$. The surface S_{II} is smooth; S_{III} and S_{IV} each have a unique singular point $(-1, -1, 1)$, of type A_1 and A_2 , respectively. Writing $\tilde{S}_F$ for the minimal resolution, with $\tilde{S}_{II} = S_{II}$, we have

$$\mathcal{W}(M_F) \simeq \text{Perf } A_F(\mathbf{1}) \simeq \text{Perf}(\tilde{S}_F).$$

Remark 1.5. For types II, III, IV and I_0^* , the categorical mirror descriptions over $\mathbb{C}$ at the trivial bulk class overlap with Beimler, Hu, Olsen and Shende [Bei+26, Theorem 1.1]. These overlaps arise from Hitchin systems. The proofs are different, via microlocal sheaves and toric mirror symmetry there, and via the algebras $\Lambda^1(Q)$ and $A_F(\mathbf{1})$ here.

Corollary 1.4 should also be obtainable, for every Q , from the general homological mirror symmetry theorems of Hacking–Keating [HK22] and Varolgüneş [Var26] for affine log Calabi–Yau surfaces, once M_Q is identified with the Liouville completion of such a surface and their mirror with $\tilde{S}_Q$; some additional work is required.

Remark 1.6. For the cycles I_n , put

$$R = \mathbf{k}[x, y, z^{\pm 1}]/(xy - z + 1).$$

Suppose that n is invertible in $\mathbf{k}$ and that $\mathbf{k}$ contains a primitive n th root of unity. Theorem 1.1 and proposition 3.11 give

$$\mathcal{W}(M_{I_n}) \simeq \text{Perf}(R \rtimes \mu_n) \simeq \text{Perf}([\text{Spec } R/\mu_n]),$$

where μ_n acts on x, y, z with weights $1, -1, 0$. The algebraic properties of the cycle algebras are established in [KS23, Theorem 1.2 and Section 6.2]. For $n = 1$, this mirror equivalence is due to Pascaleff [Pas14]; the general case follows from a covering argument (cf. [CU13]).

For the four affine stars, let (E, G) be a pair consisting of an elliptic curve E and a cyclic group $G \cong \mu_N$ of automorphisms fixing its origin. Let s be the number of cone points of E/G . The group order and cone-point orders are:

Q	N	$(d_1, \dots, d_s)$
$\tilde{D}_4$	2	(2, 2, 2, 2)
$\tilde{E}_6$	3	(3, 3, 3)
$\tilde{E}_7$	4	(2, 4, 4)
$\tilde{E}_8$	6	(2, 3, 6)

Label the central vertex by 0 and the vertices on the j th arm by (j, p) , $1 \leq p < d_j$, starting next to the centre, and orient the arrows towards the centre. If $\mathbf{k}$ contains primitive d_j th roots ζ_j , define the bulk class $\mathbf{b}_{\text{grp}}$ by

$$q_0 = 1, \quad q_{(j,p)} = \zeta_j,$$

and denote this parameter tuple by $\mathbf{q}_{\text{grp}}$. Let $\mathbb{T}$ be the underlying real torus of E , and put

$$\Gamma = \pi_1(\mathbb{T}) \cong \mathbb{Z}^2, \quad \mathbb{G}_{m,\mathbf{k}}^2 = \text{Spec } \mathbf{k}[\Gamma],$$

with the induced G -action.

Corollary 1.7. *Suppose that $\text{char } \mathbf{k} \nmid N$ and that the primitive roots ζ_j lie in $\mathbf{k}$. For any such choice,*

$$\mathcal{W}_{\mathbf{b}_{\text{grp}}}(M_Q) \simeq \mathcal{W}_G(T^*\mathbb{T}) \simeq \text{Perf}([\mathbb{G}_{m,\mathbf{k}}^2/G]).$$

Here $\mathcal{W}_G$ denotes the equivariant wrapped Fukaya category, with the grading and relative spin background specified in Section 6.4. All three categories have the algebraic model $\text{Perf}(\mathbf{k}[\Gamma] \rtimes G)$.

The equivalence between the equivariant cotangent category and the quotient-stack category requires only $\text{char } \mathbf{k} \nmid N$; the roots are needed for the comparison with the plumbing. Proofs of Corollaries 1.4 and 1.7 are given in Section 6.

1.1. Crepant resolutions and Hecke algebras. For the four affine stars, we choose the complex model

$$M_Q = \text{Res}(T^*E/G),$$

the minimal resolution of T^*E/G , identified with the completion above by an exact symplectomorphism (Proposition 6.2). The exceptional curves over a cone point of order d_j form the A_{d_j-1} arm, while the strict transform of the zero section is the central component. In the cotangent coordinate v , the function v^N defines a proper map $f : M_Q \rightarrow \mathbb{C}$.

Our choice of roots of unity for the bulk parameters on the exceptional spheres parallels the crepant resolution conjecture of Ruan [Rua06, Section 2] and Bryan–Graber [BG09, Conjecture 1.2]. That conjecture relates the genus-zero Gromov–Witten theories of a quotient orbifold and its crepant resolution: after analytic continuation and a change of cohomological variables, the parameters associated with exceptional curves are specialised to roots of unity, while the remaining parameters stay free. For the local A_{d_j-1} singularities, this comparison is proved in equivariant quantum cohomology, with exceptional parameters $e^{-2\pi i/d_j}$ [Coa+09, Theorem A.1 and Corollary A.2]; related ADE computations appear in [BG08]. Our wrapped Fukaya category calculation exhibits the same pattern of parameter specialisation, but does not establish the Gromov–Witten correspondence for M_Q .

Allowing the bulk parameter on the central sphere to vary deforms the quotient-torus mirror to a quantum torus. Write $q = q_0$. The parameters $q_{(j,p)}$ belong to the exceptional spheres, and q belongs to the central sphere. Set $q_{(j,p)} = \zeta_j$ and allow arbitrary $q \in \mathbf{k}^\times$; denote the resulting bulk class by $\mathbf{b}_{\text{grp},q}$. Under the field hypotheses of Corollary 1.7, we obtain

$$(1.8) \quad \mathcal{W}_{\mathbf{b}_{\text{grp},q}}(M_Q) \simeq \text{Perf}(\mathcal{A}_{q^N} \rtimes G), \quad \mathcal{A}_\eta = \mathbf{k}\langle X^{\pm 1}, Y^{\pm 1} \rangle / (YX - \eta XY).$$

The monomial action is the one described in [EOR07, Section 4.1]. At $q = 1$ this is Corollary 1.7.

Near the group-algebra point, there is also a comparison over a formal power series ring in the bulk parameters, with coefficients in $\mathbb{C}$. In this setting, [Hon+26, Theorem 1.6 and Example 7.4] identifies the cohomology of the bulk-deformed regular cotangent fibre for $[E/G]$ with Etingof’s orbifold Hecke algebra [Eti17], with constant terms -1 . It is concentrated in degree zero since E is a $K(\pi, 1)$. This computation concerns the fibre algebra; split-generation of that formal orbifold category is a further question.

The central corner $H = e_0 \Lambda^q(Q) e_0$ is the rank-one generalised double affine Hecke algebra of Etingof, Oblomkov and Rains [EOR07]: with $\alpha_{ja} = \prod_{p=1}^a q_{(j,p)}^{-1}$, it is generated by $U_1^{\pm 1}, \dots, U_s^{\pm 1}$ subject to

$$\prod_{a=0}^{d_j-1} (U_j - \alpha_{ja}) = 0 \quad (1 \leq j \leq s), \quad U_1 \cdots U_s = q,$$

see Section 6.1. Section 6.4 gives the parameter dictionary relating this presentation to Etingof's formally deformed orbifold Hecke algebra.

1.2. Hilbert schemes and higher rank. For $n \geq 1$, put

$$Y_n = \text{Hilb}^n(M_Q), \quad G_n = G^{\times n} \rtimes S_n, \quad \mathbb{T}_n = \mathbb{T}^{\times n}.$$

The Hilbert scheme is smooth of complex dimension $2n$, with a projective symplectic resolution

$$Y_n \longrightarrow \text{Sym}^n(T^*E/G) \cong (T^*E)^n/G_n.$$

The holomorphic symplectic form Ω_n induced from M_Q has weight one under cotangent scaling. If V_n generates positive real scaling, then $\lambda_n = \iota_{V_n} \text{Re } \Omega_n$ is a finite-type Liouville form. Its skeleton is the underlying set of the zero fibre of the proper map

$$h_n : Y_n \longrightarrow \text{Sym}^n(M_Q) \longrightarrow \text{Sym}^n(\mathbb{C}) \cong \mathbb{C}^n.$$

The first arrow is the Hilbert-Chow map, sending a length- n subscheme to its support counted with multiplicities; the second is induced by f . The skeleton includes nonreduced subschemes with transverse directions when $n > 1$. Groechenig's description [Gro14, Theorems 4.1 and 5.1] realises these complex spaces as moduli of stable parabolic Higgs bundles of rank nN .

The higher-rank generalised double affine Hecke algebra $H_n(\mathbf{u}, \tau)$ of Etingof–Gan–Oblomkov [EGO06] has an interaction parameter τ . Put

$$\Lambda_n^q = (\Lambda^q(Q))^{\otimes n} \rtimes S_n.$$

At $\tau = 1$, over every coefficient field,

$$(1.9) \quad H_n(\mathbf{u}, 1) \cong e_0^{\otimes n} \Lambda_n^q e_0^{\otimes n}.$$

Section 6.4 gives the presentations, parameter dictionary ($\mathbf{u} \leftrightarrow \mathbf{q}$) and corresponding dg formality statement.

When the roots ζ_j lie in $\mathbf{k}$, the corner at $\mathbf{q}_{\text{grp}}$ is full and isomorphic to $\mathbf{k}[\Gamma^n] \rtimes G_n$. Theorem 1.10 gives the corresponding equivariant mirror equivalence when $|G_n|$ is invertible in $\mathbf{k}$.

Theorem 1.10. *For every affine star and every $n \geq 1$, suppose that $N^n n! = |G_n|$ is invertible in $\mathbf{k}$. Then*

$$\mathcal{W}_{G_n}(T^*\mathbb{T}_n) \simeq \text{Perf}(\mathbf{k}[\Gamma^n] \rtimes G_n) \simeq \text{Perf}([\mathbb{G}_{m,\mathbf{k}}^{2n}/G_n]).$$

The brane data are the canonical cotangent grading and the equivariant relative spin background $\pi^ T^*\mathbb{T}_n$, where $\pi : T^*\mathbb{T}_n \rightarrow \mathbb{T}_n$ is the cotangent projection. If $\mathbf{k}$ contains the roots ζ_j , these categories are also equivalent to $\text{Perf}(\Lambda_n^{\mathbf{q}_{\text{grp}}})$.*

The theorem concerns the equivariant cotangent category. For the smooth resolution Y_n we formulate a different comparison. Let $\mathcal{V}_n$ be the tautological rank- n bundle, whose fibre at Z is $H^0(Z, \mathcal{O}_Z)$, and define

$$(1.11) \quad \mathbf{b}_{\text{coll}}(\beta) = (-1)^{\langle c_1(\det \mathcal{V}_n), \beta \rangle}, \quad \beta \in H_2(Y_n; \mathbb{Z}).$$

This character is 1 on classes coming from $H_2(M_Q; \mathbb{Z})$ and -1 on a Hilbert-Chow exceptional line. Here the collision sign is implemented by the relative-spin background $c_1(\det \mathcal{V}_n) \bmod 2$, as specified in Section 6.4.

The collision weight -1 is motivated by the Hilbert-Chow crepant resolution correspondence: in the local model, the undeformed orbifold point corresponds to exceptional quantum parameter -1 [BG09, Theorem 3.11]. The character $\mathbf{b}_{\text{coll}}$ gives the corresponding sign on collision classes.

Conjecture 1.12. *Assume the characteristic restrictions of Corollary 1.4 and that $n!$ is invertible in $\mathbf{k}$. Put $Y_n^\vee = \text{Hilb}^n(\tilde{S}_Q)$ over $\mathbf{k}$. Then*

$$\mathcal{W}_{\mathbf{b}_{\text{coll}}}(Y_n) \simeq \text{Perf}(Y_n^\vee).$$

The algebraic side of the conjecture is established in Section 6.4:

$$\mathrm{Perf}(Y_n^\vee) \simeq \mathrm{Perf}(\Lambda_n^1).$$

Here $\tau = 1$ and all vertex parameters equal 1. The idempotent $e_0^{\otimes n}$ is not full, so the Hecke corner does not recover the category of perfect Λ_n^1 -modules. The remaining step is the symplectic comparison across the Hilbert-Chow exceptional locus. For $n = 1$, the collision class is trivial and the conjecture reduces to Corollary 1.4.

Section 6.4 also explains the collision weight -1 through the local Hecke relation. For general interaction parameter τ , the tensor-product algebra must itself be deformed; constructing the corresponding dg model and computing the wrapped category remain open.

1.3. Organisation. Section 2 constructs the Weinstein neighbourhoods and identifies their surgery descriptions. Section 3 computes the associated bulk-deformed algebras. Section 4 proves that the cocores' wrapped Floer cohomology is concentrated in degree zero, and Section 5 completes the proof of Theorem 1.1 using surgery and generation. Section 6 proves the mirror corollaries, the higher-rank equivariant theorem and the algebraic side of the Hilbert-scheme conjecture.

Conventions. The geometric models are defined over $\mathbb{C}$; Floer complexes and algebras have coefficients in $\mathbf{k}$. The planes have their unique spin structures, trivial rank-one local systems, and compatible bulk trivialisations. Signed curve counts are interpreted in $\mathbf{k}$, and wrapping is unstopped. We fix an arbitrary $\mathbf{b}$ throughout; the ordinary category is its trivial specialisation. Products are read from right to left: vu means u followed by v , and $e_j A_F(\mathbf{q}) e_i$ is the space of morphisms from L_i to L_j .

For Liouville and Weinstein domains we follow [CE12, Section 11.1 and Definition 11.10]. Our sign convention is $\iota_Z \omega = \lambda$, where $d\lambda = \omega$. The completion of a Liouville domain W is

$$\widehat{W} = W \cup ([0, \infty) \times \partial W), \quad \lambda = e^\rho \lambda|_{\partial W}$$

on the cylindrical end. A Lagrangian disk $D \subset W$ with $\lambda|_D = 0$ near its boundary extends along the Liouville flow to an exact cylindrical Lagrangian plane $\widehat{D} \subset \widehat{W}$.

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2. NEIGHBOURHOODS OF SINGULAR FIBRES AS LEGENDRIAN SURGERIES

We construct a Weinstein neighbourhood of the singular fibre and identify its Legendrian surgery description, with transverse holomorphic disks completing to the cocores.

Let $\pi : X \rightarrow \Delta$ be a proper relatively minimal complex elliptic fibration with smooth total space and a single singular fibre

$$F = \pi^{-1}(0) = \sum_{i=1}^k m_i C_i.$$

Fix a nowhere-vanishing holomorphic two-form Ω on X and put $\omega = \mathrm{Re} \Omega$. The existence of Ω forces F to be nonmultiple, by adjunction and the fact that the normal bundle of the reduced fibre of a multiple fibre mD has exact order m [Pet95, Section 15, Lemmas 1 and 3 and Corollary 4]. Complex curves are Lagrangian for ω , whose symplectic orientation agrees with the complex orientation.

Write $F_{\text{red}} = \bigcup_i C_i$, let Σ be its singular locus, and let $\nu_i : \tilde{C}_i \rightarrow C_i$ be the normalisations. Choose pairwise disjoint holomorphic disks D_i meeting F only at points $x_i \in C_i \setminus \Sigma$, transversally to C_i . Put

$$W_\epsilon = \{|\pi| \leq \epsilon\}, \quad \mu = \log |\pi| : X \setminus F \longrightarrow \mathbb{R}.$$

Properness implies that the W_ϵ form a neighbourhood basis of F . For sufficiently small $\epsilon > 0$, each W_ϵ is a compact manifold with boundary $\{|\pi| = \epsilon\}$, and each $D_i \cap W_\epsilon$ is a disk.

Figure 1 shows the situation schematically for a fibre of type I_0^* .

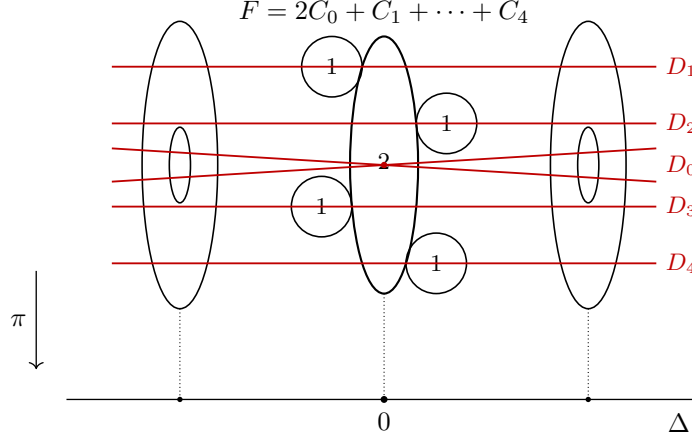


FIGURE 1. Schematic picture of the fibration near a fibre of type I_0^* . The central component C_0 has multiplicity 2 and the four other components multiplicity 1; nearby fibres are tori. The holomorphic disks $D_1, \dots, D_4$ (red) through the multiplicity-one components are local sections, whereas the disk D_0 through C_0 projects to Δ with degree 2 and meets each nearby fibre twice.

For the normal crossing fibres, the dual graph Γ_F has one vertex for each irreducible component and one edge for each node. Kodaira's classification [Kod63], see also [Bar+04, Section V.7], gives:

Fibre type	Dual graph Γ_F
$I_n, n \geq 1$	$\tilde{A}_{n-1}$
$I_n^*, n \geq 0$	$\tilde{D}_{n+4}$
IV^*	$\tilde{E}_6$
III^*	$\tilde{E}_7$
II^*	$\tilde{E}_8$

Here I_1 is a rational curve with one node, represented by a single vertex with a loop; I_2 has two vertices joined by two edges. In the remaining normal crossing cases, the components are smooth rational curves meeting transversally.

The remaining types II , III , IV have all multiplicities equal to one and a single singular point p . They consist, respectively, of a rational cuspidal curve, two smooth rational curves meeting at p with intersection multiplicity two, and three smooth rational curves meeting pairwise transversally at p .

Theorem 2.1. *For all sufficiently small $\epsilon > 0$, W_ϵ carries a Weinstein structure (λ, ϕ) with $d\lambda = \omega$ and skeleton F_{red} . The form λ restricts to zero on F_{red} and on each $D_i \cap W_\epsilon$. Up to Weinstein homotopy, this structure has the following surgery descriptions, with the completed disks $\hat{D}_i$ as the cocores corresponding to the components C_i .*

- (1) *For the normal crossing types, W_ϵ is the positive plumbing of disk cotangent bundles of two-spheres according to Γ_F , with $D_i \cap W_\epsilon$ corresponding to the cotangent fibre at x_i . Its completion is Legendrian surgery on*

$$L_{\Gamma_F} \subset \#^{b_1(\Gamma_F)}(S^1 \times S^2, \xi_{\text{std}}),$$

shown in Figures 2 to 4. Here $b_1(\Gamma_F) = 1$ for I_n and 0 otherwise.

- (2) For types II, III, IV, W_ϵ is obtained from a Darboux ball by attaching one Weinstein 2-handle for each component C_i . The attaching link $\Lambda_F = \bigsqcup_i \Lambda_i$ is Legendrian isotopic to the rainbow closure of

$$\sigma_1^3, \quad \sigma_1^4, \quad (\sigma_1 \sigma_2)^3,$$

respectively, shown in Figure 5. For III and IV, the isotopy can realise any prescribed correspondence between the labelled components.

The proof of this theorem occupies the rest of this section.

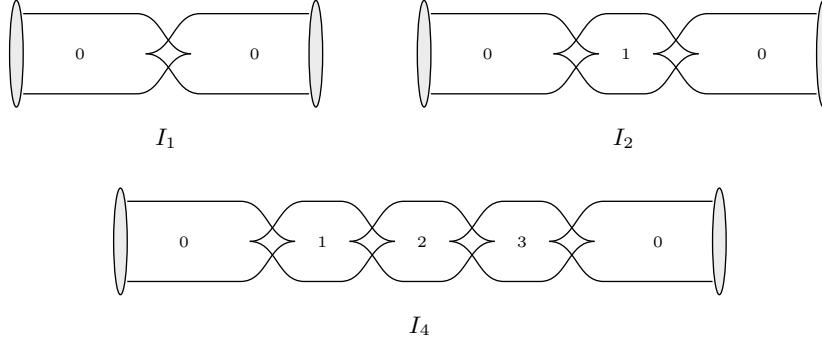


FIGURE 2. Legendrian surgery diagrams for the cycle plumblings of type I_n , drawn for $n = 1, 2, 4$.

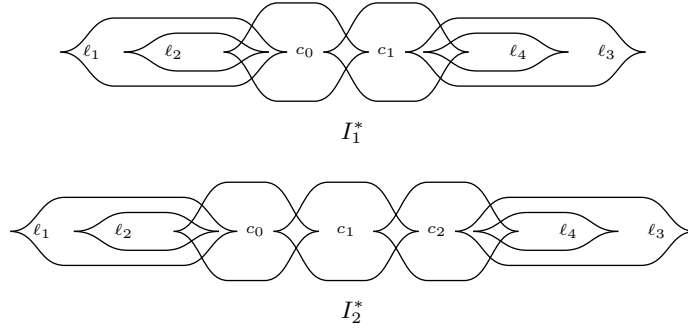


FIGURE 3. Legendrian surgery diagrams for the plumblings of type I_n^* with $n > 0$, drawn for $n = 1, 2$, with the labels of Figure 7.

The completion, together with the planes $\widehat{D}_i$, does not depend on the choices made in the proof; see Proposition 2.13.

We construct the Liouville field using weighted radial models near the singular points and cotangent models along the smooth parts of the components. We then join these models while preserving $Z \log |\pi| > 0$. This ensures that the resulting field points outward along every sufficiently small boundary $\{|\pi| = \epsilon\}$.

Lemma 2.2. *Let $p \in \Sigma$. There are holomorphic coordinates (x, y) centred at p in which $\Omega = dx \wedge dy$ and F_{red} is the zero set of*

$$g_p = \begin{cases} xy & \text{at a node,} \\ y^2 - x^3 & \text{for II,} \\ y(y - x^2) & \text{for III,} \\ xy(x - y) & \text{for IV.} \end{cases}$$

Similarly, for each i there are holomorphic coordinates (u, v) centred at x_i with $\Omega = du \wedge dv$, $C_i = \{v = 0\}$ and $D_i = \{u = 0\}$.

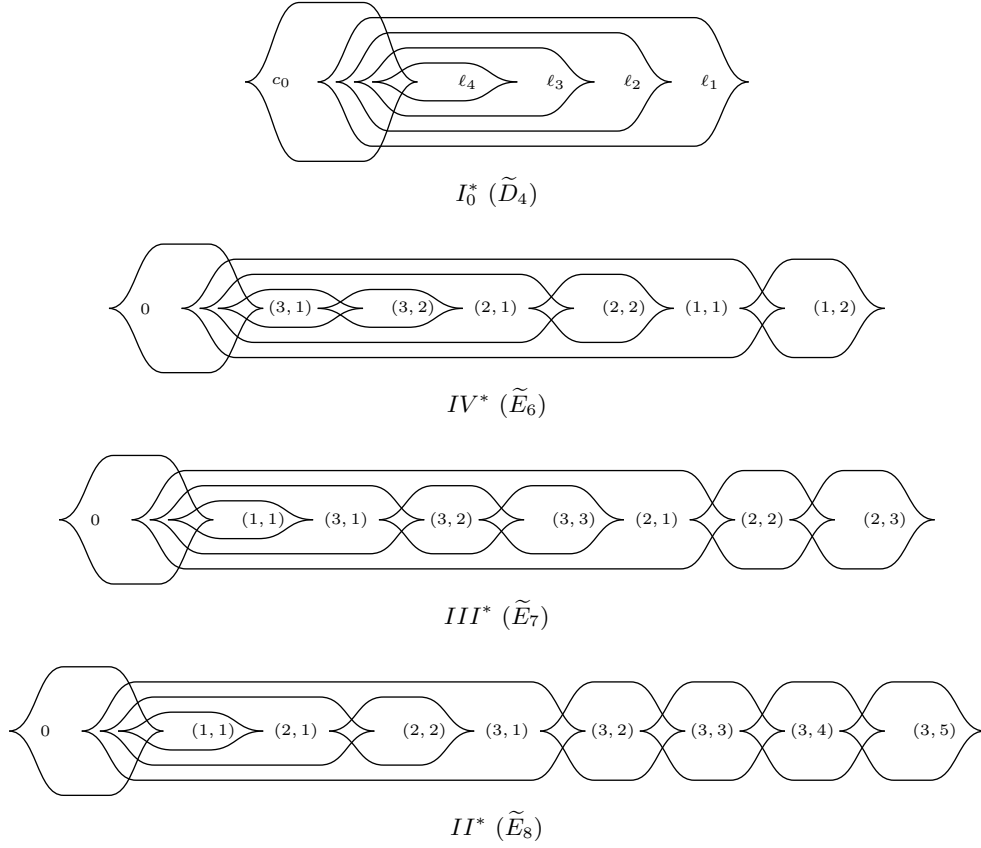


FIGURE 4. Legendrian surgery diagrams for the star-shaped plumbings of types I_0^* , IV^* , III^* and II^* .

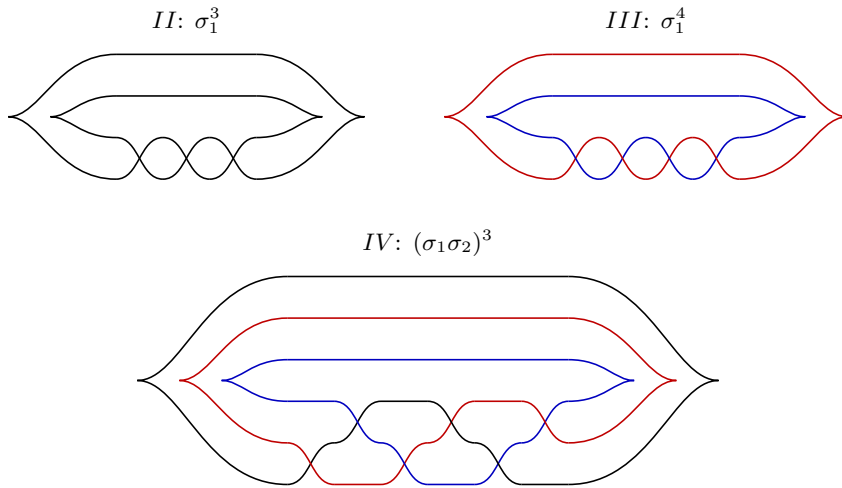


FIGURE 5. Front projections of the attaching links Λ_{II} , Λ_{III} and Λ_{IV} : the rainbow closures of σ_1^3 , σ_1^4 and $(\sigma_1 \sigma_2)^3$.

Proof. Choose holomorphic coordinates giving the stated curve equations and write $\Omega = f(X, Y) dX \wedge dY$. At a node and in types II , III and IV , respectively, put

$$(2.3) \quad (a_p, b_p) = \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{2}{5}, \frac{3}{5}\right), \left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{2}, \frac{1}{2}\right).$$

For $(a, b) = (a_p, b_p)$, set

$$h(X, Y) = \int_0^1 f(t^a X, t^b Y) dt, \quad x = Xh^a, \quad y = Yh^b.$$

Here $h(0) = f(0) \neq 0$, so the powers are defined using a local holomorphic logarithm. Since $(1 + aX\partial_X + bY\partial_Y)h = f$, we have $dx \wedge dy = f dX \wedge dY = \Omega$, while $g_p(x, y) = h^d g_p(X, Y)$, where d is the weighted degree of g_p . Thus (x, y) are the required coordinates. At x_i , apply the same construction with $a = b = \frac{1}{2}$ after straightening the transverse curves C_i and D_i to the coordinate axes; the rescaling preserves each axis. $\square$

Each branch B is defined by an irreducible factor g_B of g_p , which is quasi-homogeneous of positive degree d_B for the weights (2.3).

For real numbers a, b with $a + b = 1$ put

$$\xi_{a,b} = ax \partial_x + by \partial_y.$$

Consider the vector field $Z_{a,b} = \xi_{a,b} + \overline{\xi_{a,b}}$ on $\mathbb{C}^2$ generating the flow $(x, y) \mapsto (e^{as}x, e^{bs}y)$. Since $\iota_{\xi}\Omega = 0$,

$$(2.4) \quad \lambda_{a,b} := \iota_{Z_{a,b}}\omega = \operatorname{Re}(ax dy - by dx), \quad d\lambda_{a,b} = \omega.$$

In the real coordinates

$$(2.5) \quad q = (\operatorname{Re} x, \operatorname{Im} x), \quad p = (-\operatorname{Re} y, \operatorname{Im} y)$$

we have $\omega = dp \wedge dq$, and a direct computation gives

$$(2.6) \quad \lambda_{a,b} = p dq - d\langle a, p, q \rangle, \quad Z_{a,b} = a q \partial_q + b p \partial_p.$$

This is a special case of the following cotangent model. Let S be a surface and V a vector field on S . On T^*S , with local coordinates q on S , dual coordinates p and $\omega = dp \wedge dq$, put

$$\lambda_V = p dq - d\langle p, V(q) \rangle, \quad Z_V = E + L_V,$$

where $E = p \partial_p$ is the fibrewise Euler field. Writing $V = V^1(q) \partial_{q_1} + V^2(q) \partial_{q_2}$, its cotangent lift is

$$\begin{aligned} L_V &= V^1(q) \frac{\partial}{\partial q_1} + V^2(q) \frac{\partial}{\partial q_2} \\ &\quad - \left(p_1 \frac{\partial V^1}{\partial q_1} + p_2 \frac{\partial V^2}{\partial q_1} \right) \frac{\partial}{\partial p_1} - \left(p_1 \frac{\partial V^1}{\partial q_2} + p_2 \frac{\partial V^2}{\partial q_2} \right) \frac{\partial}{\partial p_2}. \end{aligned}$$

One checks $\iota_E \omega = p dq$ and $\iota_{L_V} \omega = -d\langle p, V \rangle$, so Z_V is the Liouville field of λ_V . Along the zero section $Z_V = V$, and λ_V vanishes on the zero section and on the cotangent fibres over zeros of V . By (2.6), $\lambda_{a,b} = \lambda_V$ for $V = a q \partial_q$. For a function β on S ,

$$(2.7) \quad Z_{\beta V} = \beta Z_V + (1 - \beta)E - \langle p, V \rangle \nabla \beta \cdot \partial_p,$$

where $\nabla \beta \cdot \partial_p = \sum_j (\partial_{q_j} \beta) \partial_{p_j}$.

We next straighten each branch at a singular point equivariantly. For $0 < \eta \leq \frac{1}{2}$ and $k > 0$ write

$$T_k(\eta) = \{(u, v) \in \mathbb{C}^2 : 0 < |u| < 1, |v| < \eta |u|^k\}.$$

Lemma 2.8. *Let B be a branch of $\{g_p = 0\}$ at p , in the coordinates of Lemma 2.2, and let Φ_B , c_B and k_B be given by the table below. Then Φ_B is injective on $T_{k_B}(\eta)$, $\Phi_B^*(dx \wedge dy) = du \wedge dv$, the curve $u \mapsto \Phi_B(u, 0)$ parametrises $B \setminus \{p\}$ near p and is the normalisation of B , and*

$$(\Phi_B)^* \xi_{a_p, b_p} = c_B u \partial_u + (1 - c_B) v \partial_v.$$

type	B	$\Phi_B(u, v)$	c_B	k_B
node	$\{y = 0\}$	(u, v)	$\frac{1}{2}$	1
node	$\{x = 0\}$	$(-v, u)$	$\frac{1}{2}$	1
II	$\{y^2 = x^3\}$	$(u^2, u^3 + v/2u)$	$\frac{1}{5}$	4
III	$\{y = 0\}$	(u, v)	$\frac{1}{3}$	2
III	$\{y = x^2\}$	$(u, u^2 + v)$	$\frac{1}{3}$	2
IV	$\{y = 0\}$	(u, v)	$\frac{1}{2}$	1
IV	$\{x = 0\}$	$(-v, u)$	$\frac{1}{2}$	1
IV	$\{y = x\}$	$(u, u + v)$	$\frac{1}{2}$	1

Moreover, on $T_{k_B}(\eta)$,

$$(2.9) \quad |v \partial_v \log(g_B \circ \Phi_B) - 1| \leq 2\eta, \quad |v \partial_v \log(g_{B'} \circ \Phi_B)| \leq 2\eta \quad (B' \neq B),$$

and each $g_{B'} \circ \Phi_B$ with $B' \neq B$ is nonvanishing. The expression $v \partial_v \log(g_B \circ \Phi_B)$ in the first estimate is interpreted by its removable extension at $v = 0$.

Proof. The Jacobian determinants are one, and the weight identities follow by substitution. Injectivity is immediate except for the cusp chart. There, equality of the images of two distinct points forces

$$u' = -u, \quad v + v' = -4u^4,$$

contradicting $|v|, |v'| < \eta|u|^4$.

For the estimates, $g_B \circ \Phi_B = \pm v$, except in the cusp case, where

$$(y^2 - x^3) \circ \Phi_B = u^2 v \left(1 + \frac{v}{4u^4}\right).$$

The remaining factors are, up to sign, u or $u^j \pm v$, with $j = 1, 2$. Their nonvanishing and the bounds follow from these expressions and

$$\frac{|v|}{|u^j \pm v|} \leq \frac{\eta}{1 - \eta} \leq 2\eta.$$

□

In the real coordinates (2.5) attached to (u, v) , (2.6) and Lemma 2.8 give

$$(2.10) \quad \Phi_B^* \omega = dp \wedge dq, \quad \Phi_B^* \lambda_{a_p, b_p} = \lambda_{V_B}, \quad V_B = c_B q \partial_q.$$

Thus near each branch the linear Liouville structure at p is the cotangent model of a radial vector field on the branch.

Proposition 2.11. *For all sufficiently small $\epsilon > 0$ there is a Weinstein structure (λ, ϕ) on W_ϵ with $d\lambda = \omega$ such that:*

- (1) λ vanishes on F_{red} and on each $D_i \cap W_\epsilon$, and $Z\mu > 0$ on $W_\epsilon \setminus F$;
- (2) the zeros of Z are the points of Σ , of index zero, the points x_i , of index two, and $e_i - 1$ points of index one on each $C_i \setminus \Sigma$, where e_i is the number of points of $\tilde{C}_i$ lying over Σ ;
- (3) near $p \in \Sigma$, in the coordinates of Lemma 2.2, $\lambda = \lambda_{a_p, b_p}$ and $\phi = |x|^2 + |y|^2$;
- (4) the skeleton is F_{red} , the unstable manifold of x_i is $D_i \cap W_\epsilon$, and all stable manifolds are contained in F_{red} .

The count in (2) comes from a vector field on each normalisation $\tilde{C}_i$ with e_i sources at the preimages of Σ , one sink at x_i , and $e_i - 1$ saddles. These counts describe the auxiliary handle decomposition used in the construction.

Proof. Near $p \in \Sigma$ use $\lambda_p = \lambda_{a_p, b_p}$, and near x_i use $\lambda_{x_i} = \lambda_{-1/2, 3/2}$ in the coordinates (u, v) with $C_i = \{v = 0\}$ and $D_i = \{u = 0\}$. Writing $\pi = w_p \prod_B g_B^{m_B}$ with w_p a unit gives

$$Z_p \mu = \sum_B m_B d_B + \operatorname{Re} \frac{\xi_{a_p, b_p} w_p}{w_p} \geq \kappa_p > 0$$

after shrinking the charts. Similarly, $\pi = v^{m_i} w_i$ near x_i gives $Z_{x_i} \mu \geq m_i$.

Each normalisation $\tilde{C}_i$ is a sphere. On it, choose a smooth function f_i and a vector field Y_i , with $Y_i f_i > 0$ away from the zeros of Y_i , sources at the e_i points above Σ , a sink at x_i , and $e_i - 1$ saddles. Use the local models

$$(f_i, Y_i) = \begin{cases} ((|x|^2 + |y|^2) \circ \nu_i, c_B q \partial_q), & \text{near a source on branch } B, \\ (c_i - |q|^2, -\frac{1}{2} q \partial_q), & \text{near } x_i, \\ (c_s - q_1^2 + q_2^2, -q_1 \partial_{q_1} + q_2 \partial_{q_2}), & \text{near a saddle } s, \end{cases}$$

with suitable constants c_i, c_s .

Choose $R > 0$ so that the source and sink models hold on disjoint disks $\{|q| < R\}$ around the marked points $\nu_i^{-1}(\Sigma) \cup \{x_i\}$, identifying x_i with its unique lift to $\tilde{C}_i$. Set

$$\mathcal{A} = \{R e^{-L} < |q| < R\}, \quad A_i = \tilde{C}_i \setminus \bigcup_{\text{marked points}} \{|q| < R\}.$$

Each collar $\mathcal{A}$ has logarithmic length L , while A_i and Y_i remain fixed as L increases.

To ensure $Z\mu > 0$ away from the prescribed local models, we rescale Y_i . For $0 < \delta < 1$, choose smooth functions $\beta_i : \tilde{C}_i \rightarrow [\delta, 1]$ and put

$$V_i = \beta_i Y_i, \quad \beta_i = \begin{cases} 1 & \text{near the marked points,} \\ \delta & \text{near } A_i, \end{cases} \quad |q| |\nabla \beta_i| \leq C/L \quad \text{on } \mathcal{A}.$$

This preserves $V_i f_i > 0$ away from the zeros. Place the overlaps with the marked-point neighbourhoods where $\beta_i = 1$ for every δ .

The relative Lagrangian neighbourhood theorem gives cotangent charts Θ_i along the normalisations with small disks about the marked points removed, agreeing with the branch charts of Lemma 2.8 and the charts at x_i on their collars. Use $\Theta_{i*} \lambda_{V_i}$ on the tubes and λ_p, λ_{x_i} near the marked points. These forms agree on every overlap by the choice of β_i and equations (2.10) and (2.6), defining a primitive λ of ω near F_{red} . Its restriction vanishes on F_{red} and on the prescribed transverse disks near x_i . Measure the tube width by η , using $|v| < \eta |u|^{k_B}$ on the branch collars and $|p| < \eta$ over A_i .

We verify $Z\mu > 0$ first over A_i , where $Z = E + \delta L_{Y_i}$. A transverse holomorphic coordinate τ satisfies $\tau \circ \Theta_i(q, p) = \ell_q(p) + O(|p|^2)$ with ℓ_q a real linear isomorphism. Since $\pi = \tau^{m_i} \cdot (\text{unit})$, we have $E\mu = m_i + O(|p|)$ and $L_{Y_i} \mu = O(1)$, uniformly over A_i . Thus

$$Z\mu = m_i + O(\delta + \eta) > 0$$

for sufficiently small δ, η once the charts are fixed.

On each collar, (2.7) gives

$$Z = \beta_i Z_{Y_i} + (1 - \beta_i) E - \langle p, Y_i \rangle \nabla \beta_i \cdot \partial_p.$$

The first two fields satisfy $Z_{Y_i} \mu \geq c > 0$ and $E\mu \geq m_B/2$ for small η , by (2.9); at x_i use $\pi = v^{m_i} w_i$. The same bounds give $|p| |\partial_p \mu| \leq C$, so the last term contributes at most C/L in absolute value. Thus, for small η , $Z\mu \geq c_0 - C/L$, with c_0, C independent of L, δ . Choose L large, fix the cotangent charts, and then choose δ, η successively small so that both lower bounds are positive.

Extend the f_i to a smooth function f near F_{red} , using $f_i + |p|^2$ near the sinks and saddles and $|x|^2 + |y|^2$ near Σ . At a saddle the normal eigenvalues of Z are $1 \pm \delta > 0$, giving index one; the other models give index zero at Σ and index two at x_i . The local models and the inequality $V_i f_i > 0$ on the compact remainder of F_{red} give $Zf > 0$ away from the zeros in a neighbourhood of F_{red} . Shrink ϵ so that W_ϵ lies there. Near the boundary, set $\phi = (1 - \chi(\mu))f + \chi(\mu)(K + \mu)$, where χ increases from 0 to 1 and $K + \mu > f$ on the transition region. Then $Z\phi > 0$ away from the zeros, ϕ is constant on the boundary, and its local Morse models are unchanged.

Finally, F_{red} is compact and invariant, whereas every forward trajectory off F exits W_ϵ : until it exits, it lies in a compact annulus where $Z\mu$ has a positive minimum. Thus the skeleton is F_{red} , which contains every stable manifold. For small ϵ , $D_i \cap W_\epsilon$ lies in the linear model at x_i and is its unstable disk. $\square$

Remark 2.12. Since $\mu = \log |\pi|$, the hypersurface $\pi^{-1}(\{|t| = r\})$ is the level set $\mu^{-1}(\log r)$. Thus $Z\mu = d\mu(Z) > 0$ means that Z is transverse to this hypersurface and points outward from $W_r = \{|\pi| \leq r\}$, for every sufficiently small $r > 0$.

Proof of Theorem 2.1. Let (λ, ϕ) be as in Proposition 2.11. The first assertions are parts (1), (2) and (4) of that proposition.

For the normal crossing types, Proposition 2.11 constructs the positive cotangent plumbing along Γ_F , including self-plumbing for I_1 (see [ST03, Proposition 7.3] for trees).

Choose disjoint balls $B_p = \{|x|^2 + |y|^2 \leq \rho_p^2\}$ in the node charts. Let $\Delta_{i,a} \subset \tilde{C}_i$, $1 \leq a \leq e_i$, be the disk components of $\nu_i^{-1}(\bigcup_p B_p)$, and put $K_{i,a} = \nu_i(\partial\Delta_{i,a})$. Each ∂B_p contains a standard Legendrian Hopf pair. Choose Morse data joining these disks by $e_i - 1$ bands into a disk Δ_i with complementary disk containing x_i . Attach the corresponding Weinstein 1-handles to the balls. The band sum $\Lambda_i = \nu_i(\partial\Delta_i)$ of the $K_{i,a}$ is the Legendrian attaching circle of a 2-handle whose core is the complementary disk, with its Liouville collar, and whose cocore is $D_i \cap W_\epsilon$. Let G have the balls B_p as vertices and the 1-handles as edges. Cancelling 0–1 pairs along a spanning tree of G [CE12, Proposition 12.22] leaves the handle counts $(h_0, h_1, h_2) = (1, b_1(\Gamma_F), k)$. Bands through cancelled handles give Legendrian connected sums, preserving the Hopf clasps. Ordering the bands as in [EL19, Section 3] gives L_{Γ_F} . The 2-handle charts remain fixed, so their cocores complete to $\hat{D}_i$.

For *II*, *III* and *IV*, Proposition 2.11 gives one minimum p and the index-two points x_i . Since $\phi = |x|^2 + |y|^2$ near p , a sufficiently small sublevel is the ball

$$B = \{\phi \leq r^2\} = \{|x|^2 + |y|^2 \leq r^2\}.$$

By Weinstein's handle theory [Wei91] (see [CE12, Section 11.4 and Proposition 12.12]), W_ϵ is obtained from (B, λ_p) by attaching Weinstein 2-handles along the Legendrian circles

$$\Lambda_i = W^s(x_i) \cap \partial B = C_i \cap \partial B,$$

with cocores $W^u(x_i) = D_i \cap W_\epsilon$.

The path $\lambda_t = \lambda_{a_t, b_t}$ with $(a_t, b_t) = (1-t)(a_p, b_p) + t(\frac{1}{2}, \frac{1}{2})$ consists of Liouville forms whose Liouville fields are transverse to the round sphere ∂B . At $t = 1$ it is the standard radial Liouville form of the linear symplectic form ω , so (B, λ_1) is a Darboux ball. By [CE12, Lemma 12.7(i)], this deformation extends across a collar of ∂B with no critical points, fixing the structure beyond the collar up to positive scaling. Comparing the initial and final collar holonomies [CE12, Lemma 11.4] gives a contactomorphism from $(\partial B, \ker \lambda_p)$ to $(\partial B, \ker \lambda_1) \cong (S^3, \xi_{\text{std}})$ isotopic to the identity. Thus W_ϵ is Weinstein homotopic to the Legendrian surgery on the image Λ_F of $\bigsqcup_i \Lambda_i$, with the same cocores since scaling leaves the Liouville field unchanged. As a smooth oriented link Λ_F is the link of the plane curve singularity $\{g_p = 0\}$, oriented by the complex orientation of its branches: the positive torus link $T(2, 3)$, $T(2, 4)$ or $T(3, 3)$.

The handle classes are $[C_i]$, so the Weinstein framing convention [Gom98] gives

$$\text{tb}(\Lambda_i) = C_i^2 + 1, \quad \text{lk}(\Lambda_i, \Lambda_j) = C_i \cdot C_j \quad (i \neq j).$$

Since $F \cdot C_i = 0$, the component self-intersections are 0 in type *II* and -2 in types *III*, *IV*. Thus the links $T(2, 3)$, $T(2, 4)$ and $T(3, 3)$ have component Thurston–Bennequin invariants 1, -1 and -1 , respectively. The rainbow closures in Figure 5 have these oriented link types and component Thurston–Bennequin invariants. The Legendrian classifications [EH01, Section 4.1] and [DET24, Theorem 1.1(1)] therefore identify them with Λ_F . In particular, all component rotation numbers vanish. For *III* and *IV*, the ordered classification [DET24, Theorem 1.2(1)] allows any prescribed correspondence between components. $\square$

The Liouville completion is independent of the choices made above.

Proposition 2.13. *Let λ_0, λ_1 be primitives of ω on W_ϵ whose Liouville fields point outward along ∂W_ϵ , which vanish on F_{red} , and which vanish on $D_i \cap \{\epsilon' \leq |\pi| \leq \epsilon\}$ for some $\epsilon' < \epsilon$ and all i . Then there is an exact symplectomorphism between the completions of (W_ϵ, λ_0) and (W_ϵ, λ_1) carrying each completed plane $\hat{D}_i$ for λ_0 to an exact Lagrangian cylindrically isotopic to $\hat{D}_i$ for λ_1 .*

Proof. The closed form $\lambda_1 - \lambda_0$ has zero periods: W_ϵ retracts onto F_{red} , and both primitives vanish on loops in F_{red} . Hence $\lambda_1 - \lambda_0 = dg$. The path

$$\lambda_t = \lambda_0 + t dg$$

has Liouville field $Z_t = (1-t)Z_0 + tZ_1$, which remains outward-pointing along ∂W_ϵ . Each $D_i \cap W_\epsilon$ remains an exact Lagrangian disk, cylindrical near the boundary, since λ_t vanishes there. By [CE12, Lemma 11.6 and Proposition 11.8], the completions admit smoothly varying exact symplectic identifications preserving the Liouville forms near infinity. Pulling back the completed disks under these identifications gives an exact cylindrical Lagrangian isotopy, proving the claim. $\square$

3. CHEKANOV–ELIASHBURG ALGEBRAS ASSOCIATED WITH KODAIRA FIBRES

We compute the Chekanov–Eliashberg dg algebras with bulk parameters and identify their degree-zero cohomology with $A_F(\mathbf{q})$. For the normal crossing types we use the plumbing calculation of [EL19]; for *II*, *III*, *IV*, the rainbow-closure model of [CS26]. The comparison with wrapped Floer cohomology and the vanishing in negative degrees are established in Sections 4 and 5.

3.1. Legendrian dg algebras and basepoint variables. Let $\Lambda = \bigsqcup_v \Lambda_v$ be a Legendrian link in $\#^k(S^1 \times S^2, \xi_{\text{std}})$, presented in Gompf’s standard form, with one basepoint on each component. We write $\text{CE}^*(\Lambda)$ for its fully noncommutative Chekanov–Eliashberg dg algebra with cohomological grading, in the form given in [EL19, Section 2] following Ekholm and Ng. It is an algebra over $S = \bigoplus_v \mathbf{k}e_v$, freely generated over S by the Reeb chords of Λ (the crossings of the resolved diagram), the internal generators of the 1-handles, and invertible basepoint variables $t_v^{\pm 1}$ with $e_v t_v e_v = t_v$ and $dt_v = 0$. The differential of a chord counts holomorphic disks, each weighted by the sign and by the time-ordered product of the basepoint variables encountered along its boundary. When $k = 0$ the internal generators are absent and $\text{CE}^*(\Lambda)$ is the Chekanov–Eliashberg algebra of a link in (S^3, ξ_{std}) with basepoints [CS26, Section 2.1]. The components of Λ carry their bounding spin structures, which are those of the cocores; relative to the formulas of Ekholm and Ng this amounts to the substitution $t_v \mapsto -t_v$ [EL19, footnote 2], already incorporated below. All components of the links in Theorem 2.1 have rotation number zero, so $\text{CE}^*(\Lambda)$ is $\mathbb{Z}$ -graded once a Maslov potential is chosen on each component. We use the gradings of [EL19; CS26]. The explicit models below have generators in degrees 0 and -1 . Their comparison with the holomorphic gradings of the planes is made in the proof of Proposition 5.3.

Since t_v is closed of degree zero, the two-sided ideal generated by the elements $t_v - q_v e_v$ is a dg ideal. For $\mathbf{q} \in (\mathbf{k}^\times)^r$ we put

$$(3.1) \quad \text{CE}_{\mathbf{q}}^*(\Lambda) = \text{CE}^*(\Lambda) / (t_v - q_v e_v : v).$$

Its differential is obtained by substituting $q_v e_v$ for t_v ; the trivial bulk class corresponds to $\mathbf{q} = \mathbf{1}$. The identification $q_v = \mathfrak{b}([C_v])$ is justified by the capping argument in the proof of Proposition 5.3. For plumbings, this bulk interpretation was proposed in [EL19, Remark 2].

A stable tame isomorphism fixing the basepoint variables descends under $t_v = q_v e_v$: both the changes of generators and the cancelling generator pairs remain valid after substitution. For cycle plumbings, we also need the reduction of the internal 1-handle algebra described in the proof of Theorem 3.5.

3.2. Normal crossing types: the plumbing dg algebra. Let F be of normal crossing type and let Q be the graph Γ_F , so that Q is $\hat{A}_{n-1}$, $\hat{D}_{n+4}$, $\hat{E}_6$, $\hat{E}_7$ or $\hat{E}_8$. The graph for I_1 has one vertex and one loop, and that for I_2 has two vertices and two distinct edges. Orient the edges and order them, and write $\bar{Q}$ for the double of Q : for each arrow a of Q it has the reverse arrow a^* . For the cycles we use the orientation $a_i : i \rightarrow i+1$ and the spanning tree T obtained by deleting a_{n-1} ; for the trees $T = Q$. The conventions for each type are drawn in Figures 6 to 8. Set

$$S = \bigoplus_{i \in Q_0} \mathbf{k}e_i, \quad \mathcal{L}_Q = \mathbf{k}\bar{Q}[(1 + aa^*)^{-1}, (1 + a^*a)^{-1} : a \in Q_1].$$

Here $1 = \sum_i e_i$, and the inverses are algebraic universal localisations. Put

$$(3.2) \quad \begin{aligned} A_i &= \prod_{t(a)=i}^{\rightarrow} (e_i + aa^*), & B_i &= \prod_{s(a)=i}^{\rightarrow} (e_i + a^*a), \\ \Lambda^q(Q) &= \mathcal{L}_Q / (A_i - q_i B_i : i \in Q_0). \end{aligned}$$

An empty product is e_i . Both products follow the chosen arrow order. This is the multiplicative preprojective algebra of Crawley-Boevey and Shaw with parameters q_i ; up to isomorphism it does not depend on the orientation or the order [CS06, Theorem 1.4]. For the affine fibres set $A_F(\mathbf{q}) = \Lambda^q(Q)$.

The Chekanov–Eliashberg algebra of the link L_Q of Theorem 2.1(1) was computed in [EL19, Theorem 17]. In the notation of Section 3.1, the result is a quasi-isomorphism between $\mathrm{CE}^*(L_Q)$ and the dg algebra $\mathcal{L}_Q$ over $S\langle t_i^{\pm 1} \rangle$ generated by

$$c_a, c_a^* \ (a \in Q_1), \quad z_a^{\pm 1}, \zeta_a \ (a \in Q_1 \setminus T), \quad \tau_i \ (i \in Q_0),$$

with $c_a = e_{t(a)}c_a e_{s(a)}$, $c_a^* = e_{s(a)}c_a^* e_{t(a)}$, $z_a = e_{s(a)}z_a e_{s(a)}$, $\zeta_a = e_{s(a)}\zeta_a e_{s(a)}$ and $\tau_i = e_i \tau_i e_i$, in degrees $|c_a| = |c_a^*| = |z_a| = 0$ and $|\zeta_a| = |\tau_i| = -1$, and with differential

$$(3.3) \quad \begin{aligned} d\zeta_a &= e_{s(a)} + c_a^* c_a - z_a, \\ d\tau_i &= \prod_{t(a)=i}^{\rightarrow} (e_i + c_a c_a^*) - t_i \prod_{s(a)=i, a \notin T}^{\rightarrow} z_a \prod_{s(a)=i, a \in T}^{\rightarrow} (e_i + c_a^* c_a). \end{aligned}$$

The generators c_a, c_a^* are the two crossings between the components of L_Q attached to the ends of a ; τ_i is the self-crossing of the i th component; and z_a, ζ_a come from the 1-handle of the edge $a \notin T$, which is present only for the cycles.

Definition 3.4. For $\mathbf{q} \in (\mathbf{k}^\times)^{Q_0}$, let $\mathcal{B}_Q^{\mathbf{q}}$ be the dg algebra obtained from $\mathcal{L}_Q$ by setting $t_i = q_i e_i$ in Equation (3.3).

Theorem 3.5. For every normal crossing type F , every field $\mathbf{k}$, and every nonzero parameter tuple $\mathbf{q}$, the dg algebra $\mathrm{CE}_q^*(L_Q)$ is quasi-isomorphic to $\mathcal{B}_Q^{\mathbf{q}}$.

Proof. For cycle plumbings, [EL19, Theorem 12 and Proposition 14] replace the internal two-strand algebra by its quasi-isomorphic Laurent model $\mathbf{k}[z^{\pm 1}]$. This reduction does not involve the basepoint variables. After specialising $t_i = q_i e_i$, the remaining chord generators form a semifree dg extension of the internal algebra, so the reduction remains a quasi-isomorphism. The subsequent changes of generators and destabilisations in [EL19, Lemma 16 and Theorem 17] fix the basepoint variables and give the displayed differential. For trees, no internal-algebra reduction is needed. $\square$

The dg algebra $\mathcal{B}_Q^{\mathbf{q}}$ is concentrated in nonpositive degrees. A word of degree -1 contains exactly one of the generators ζ_a, τ_i , so its differential is a two-sided multiple of $d\zeta_a$ or $d\tau_i$. Hence

$$H^0(\mathcal{B}_Q^{\mathbf{q}}) = S\langle c_a, c_a^*, z_a^{\pm 1} \rangle / (d\zeta_a, d\tau_i), \quad H^k(\mathcal{B}_Q^{\mathbf{q}}) = 0 \quad (k > 0).$$

Lemma 3.6. In $H^0(\mathcal{B}_Q^{\mathbf{q}})$ the elements $e_{t(a)} + c_a c_a^*$ and $e_{s(a)} + c_a^* c_a$, $a \in Q_1$, are invertible in their respective corners. Consequently

$$H^0(\mathcal{B}_Q^{\mathbf{q}}) \cong \Lambda^q(Q),$$

by the map $c_a \mapsto a$, $c_a^* \mapsto a^*$, $z_a \mapsto e_{s(a)} + a^*a$.

Proof. For $a \notin T$, the relation $d\zeta_a = 0$ identifies $e_{s(a)} + c_a^* c_a$ with the invertible element z_a . The identity

$$(e_{t(a)} + c_a c_a^*)^{-1} = e_{t(a)} - c_a (e_{s(a)} + c_a^* c_a)^{-1} c_a^*$$

gives invertibility at the other endpoint. Now remove the edges of T successively at leaves. At each leaf, the vertex relation contains just one factor whose invertibility is not yet known; all other factors are invertible, so this factor is invertible as well. The displayed identity, with the arrows exchanged if necessary, gives invertibility at its other endpoint. This proves the assertion for every edge. Eliminating the z_a then gives the defining presentation of $\Lambda^q(Q)$. $\square$

To compare with the dg multiplicative preprojective algebra, adjoin $z_a^{\pm 1}, \zeta_a$ also for $a \in T$, with

$$d\zeta_a = e_{s(a)} + c_a^* c_a - z_a.$$

This models derived localisation at the remaining factors. Since their classes are already invertible in H^0 , the localisation map is a quasi-isomorphism [BCL18, Section 3]. The argument of [KS23, Remark 4.7] now applies to this model with localisation variables for every arrow and gives

$$\mathcal{B}_Q^q \simeq \Lambda^{\text{dg}, q}(Q).$$

In the presentations below, the idempotent e_i corresponds to the component C_i , hence to the plane $\widehat{D}_i$; the parameter q_i is the value of $\mathfrak{b}$ on the class γ_i of C_i . The doubled quivers of each type are drawn with their presentations below.

3.3. Affine A diagrams. For I_n , orient the cycle by $a_i : i \rightarrow i+1$, with indices in $\mathbb{Z}/n$. When $n = 2$, the arrow a_1 and the formal reverse a_0^* remain distinct. The conventions are drawn in Figure 6. The relation at vertex i is

$$(3.7) \quad e_i + a_{i-1} a_{i-1}^* = q_i (e_i + a_i^* a_i).$$

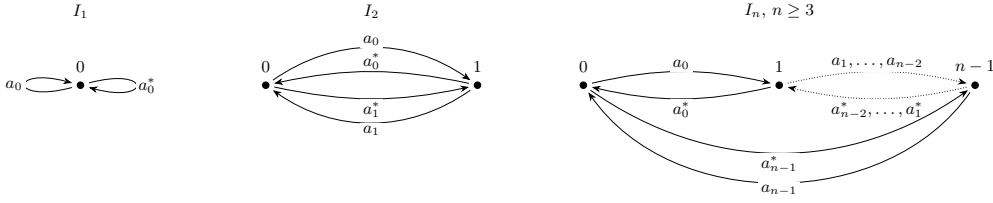


FIGURE 6. Doubled quivers for the cyclic fibres.

Write $x = \sum_i a_i$, $y = \sum_i a_i^*$, and $z = 1 + yx$. Put

$$D = \bigoplus_{i \in \mathbb{Z}/n} \mathbf{k}[z^{\pm 1}]e_i, \quad \mathbf{q} = \sum_i q_i e_i, \quad \sigma(e_i) = e_{i+1}, \quad \sigma(z) = \mathbf{q}z.$$

These formulas define an automorphism of D . Its iterates account for the individual vertex parameters; for example, $\sigma^n(z) = (\prod_i q_i)z$. Set $t = z - 1$.

Proposition 3.8. *The algebra $\Lambda^q(\tilde{A}_{n-1})$ has generators D, x, y and relations*

$$(3.9) \quad xd = \sigma(d)x, \quad yd = \sigma^{-1}(d)y \quad (d \in D), \quad yx = t, \quad xy = \sigma(t).$$

Set $U_p = x^p$ for $p \geq 0$ and $U_p = y^{-p}$ for $p < 0$. Each corner $e_j \Lambda^q(\tilde{A}_{n-1}) e_i$ has basis

$$(3.10) \quad \{b_{p,r}^i = U_p z^r e_i : p, r \in \mathbb{Z}, p \equiv j - i \pmod{n}\}.$$

Proof. Summing Equation (3.7) gives $yx = z - 1$, $xy = \mathbf{q}z - 1$, and $xz = \mathbf{q}zx$. Together with the endpoints of the arrows, these are Equation (3.9). Conversely, in that presentation $a_i = xe_i$ and $a_i^* = ye_{i+1}$ satisfy the vertex relations and all localisation factors are invertible: their nontrivial corner entries are ze_i or $q_i ze_i$.

Moving D to the right and eliminating adjacent xy and yx pairs spans the algebra by $U_p D$. To prove independence, use the skew Laurent ring $D[s^{\pm 1}; \sigma]$, where $sd = \sigma(d)s$, and send x to s , y to ts^{-1} . The image of U_p is s^p for $p \geq 0$ and

$$\left(\prod_{j=0}^{-p-1} \sigma^{-j}(t) \right) s^p \quad \text{for } p < 0.$$

Every factor is a non-zero-divisor in D : in each vertex corner it is a nonzero scalar times z , minus 1. Distinct powers of s are independent, and multiplication by these factors is injective. This proves the basis assertion over every field and for every nonzero parameter tuple. $\square$

For a single vertex,

$$\Lambda^q(\tilde{A}_0) = \mathbf{k}\langle x, y \rangle[(1 + yx)^{-1}]/(xy - qyx - q + 1).$$

At $q = 1$, this is the commutative algebra $\mathbf{k}[x, y, (1 + xy)^{-1}]$.

Proposition 3.11. *At $q = 1$, give*

$$R = \mathbf{k}[x, y, z^{\pm 1}]/(xy - z + 1)$$

the $\mathbb{Z}/n$ -grading $\deg x = 1$, $\deg y = -1$, and $\deg z = 0$. Then, over every field,

$$\Lambda^1(\tilde{A}_{n-1}) \cong \{(r_{ji}) \in M_n(R) : r_{ji} \in R_{j-i}\}.$$

The isomorphism sends e_i to E_{ii} , a_i to $xE_{i+1,i}$, and a_i^ to $yE_{i,i+1}$, where E_{ji} are the matrix units.*

Proof. The assignments satisfy the vertex relations and send every localisation factor to an invertible diagonal matrix with entries 1 or z . They therefore define a homomorphism from $\Lambda^1(\tilde{A}_{n-1})$ to the displayed matrix algebra.

Set $V_p = x^p$ for $p \geq 0$ and $V_p = y^{-p}$ for $p < 0$. By the single-vertex case of Proposition 3.8, the elements $V_p z^r$, $p, r \in \mathbb{Z}$, form a basis of R , with degree p modulo n . The homomorphism sends the basis from Proposition 3.8 bijectively to

$$V_p z^r E_{ji}, \quad p \equiv j - i \pmod{n}.$$

Hence it is an isomorphism. $\square$

If n is invertible in $\mathbf{k}$ and $\mathbf{k}$ contains a primitive n -th root of unity ζ , the algebra at $q = 1$ also has the skew group description

$$\Lambda^1(\tilde{A}_{n-1}) \cong R \rtimes \mu_n(\mathbf{k}),$$

where $\mu_n(\mathbf{k}) = \langle \zeta \rangle$ acts on x, y, z with weights $1, -1, 0$. Indeed, $g = \sum_i \zeta^i e_i$ satisfies $gx = \zeta xg$, $gy = \zeta^{-1}yg$, and $g^n = 1$. Conversely, in the skew group algebra the formulas

$$e_i = \frac{1}{n} \sum_{r=0}^{n-1} \zeta^{-ir} g^r, \quad a_i = x e_i, \quad a_i^* = y e_{i+1}$$

recover the quiver generators. These constructions are inverse.

Kaplan and Schedler prove that $\Lambda^q(\tilde{A}_{n-1})$ is 2-Calabi–Yau and prime over every field and at every parameter tuple. At the trivial bulk class it is a noncommutative crepant resolution of its centre, whose spectrum is the affine surface $z^n + xy + xyz = 0$ [KS23, Theorem 1.2 and Section 6.2]. For $n > 1$, this surface has an A_{n-1} singularity; for $n = 1$ it is smooth.

3.4. Affine D diagrams. For I_n^* with $n > 0$, label the chain by $c_0, \dots, c_n$ and orient its arrows as $b_k : c_{k-1} \rightarrow c_k$. Orient the four leaf arrows a_1, a_2 towards c_0 and a_3, a_4 towards c_n , and order the arrows as $a_1, a_2, b_1, \dots, b_n, a_3, a_4$. Write $u_i = a_i a_i^*$, using 1 for the identity in the relevant vertex corner. The arrow labels in Figure 7 follow this convention.

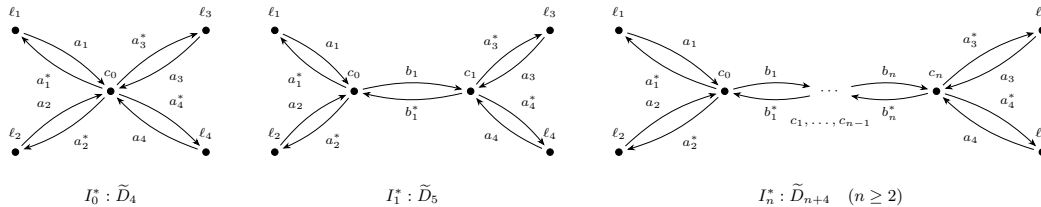


FIGURE 7. The doubled affine D quivers.

Write λ_i for the parameter at leaf i and ν_k for that at c_k .

Proposition 3.12. *For every nonzero parameter tuple, the algebra $\Lambda^q(\tilde{D}_{n+4})$ has the following presentation. For $n > 0$ its relations are*

$$(3.13) \quad \begin{aligned} a_i^* a_i &= (\lambda_i^{-1} - 1) e_i \quad (1 \leq i \leq 4), \\ (1 + u_1)(1 + u_2) &= \nu_0(1 + b_1^* b_1), \\ 1 + b_k b_k^* &= \nu_k(1 + b_{k+1}^* b_{k+1}) \quad (1 \leq k < n), \\ (1 + b_n b_n^*)(1 + u_3)(1 + u_4) &= \nu_n. \end{aligned}$$

For $n = 0$ the last three relations are replaced by

$$(3.14) \quad (1 + u_1)(1 + u_2)(1 + u_3)(1 + u_4) = \nu_0.$$

Proof. The displayed relations are the vertex equations. In the polynomial quotient they define, the leaf equations give

$$u_i^2 = (\lambda_i^{-1} - 1) u_i, \quad (1 + u_i)^{-1} = 1 - \lambda_i u_i.$$

For $n > 0$, the relation at c_0 therefore makes $1 + b_1^* b_1$ invertible. Applying

$$(1 + AB)^{-1} = 1 - A(1 + BA)^{-1}B$$

and then the successive chain relations shows that every localisation factor has a polynomial inverse. For $n = 0$, the leaf inverses suffice. The universal property of localisation gives the claimed presentation. $\square$

3.5. The affine stars. For I_0^* , IV^* , III^* and II^* the graph is a star with arms of lengths $d_j - 1$, where $(d_1, \dots, d_s)$ is $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$ or $(2, 3, 6)$. Label the central vertex by 0 and the vertices of the j th arm by (j, p) , $1 \leq p < d_j$, starting next to the centre, orient the arrows towards the centre, $c_{jp} : (j, p) \rightarrow (j, p-1)$ with $(j, 0) = 0$, and order them by j and then by p . Write $q = q_0$, as in Section 6. The doubled quivers of IV^* , III^* and II^* are drawn in Figure 8; that of I_0^* is in Figure 7.

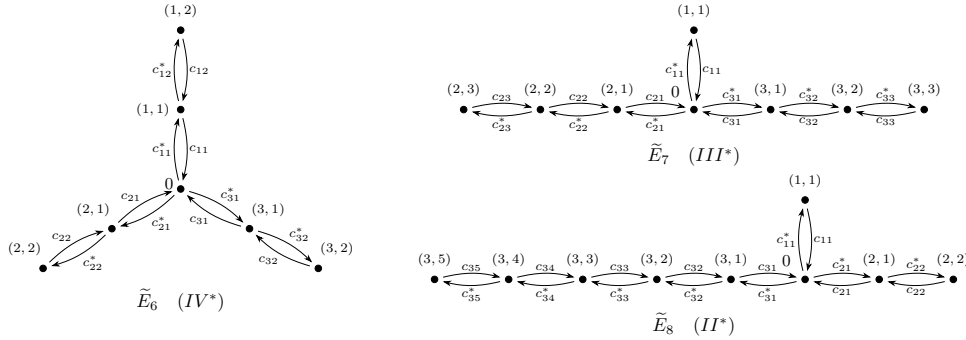


FIGURE 8. The doubled affine quivers for IV^* , III^* and II^* .

Proposition 3.15. *For every nonzero parameter tuple, $\Lambda^q(Q)$ is the quotient of the path algebra $\mathbf{k}\overline{Q}$ by the relations*

$$\begin{aligned} c_{j,d_j-1}^* c_{j,d_j-1} &= (q_{(j,d_j-1)}^{-1} - 1) e_{(j,d_j-1)}, \\ e_{(j,p)} + c_{j,p+1} c_{j,p+1}^* &= q_{(j,p)} (e_{(j,p)} + c_{jp}^* c_{jp}), \quad (1 \leq p \leq d_j - 2), \\ \prod_{j=1}^s (e_0 + c_{j1} c_{j1}^*) &= q e_0. \end{aligned}$$

No localisation is needed. For I_0^* this is Equation (3.14) with $u_j = c_{j1} c_{j1}^*$.

Proof. The relations are the vertex equations Equation (3.2). The tip relation makes $e + c_{j,d_j-1}^* c_{j,d_j-1}$ a nonzero scalar; the identity $(e + cc^*)^{-1} = e - c(e + c^*c)^{-1}c^*$ and the interior relations then express the inverse of every localisation factor as a polynomial in the arrows, moving from the tips to the centre. The universal property of localisation gives the presentation. $\square$

The central corner $e_0 \Lambda^q(Q) e_0$ is the rank-one generalised double affine Hecke algebra with presentation (6.4); this and the polynomial arm relations (6.3) are used in Section 6.

3.6. The three nontransverse fibres. By Theorem 2.1(2), the attaching links for II , III , IV are the rainbow closures of σ_1^3 , σ_1^4 , $(\sigma_1 \sigma_2)^3$, respectively. We use the model $G(\beta)$ of [CS26, Definition 2.2 and Lemma 2.3], retaining one idempotent and one bulk parameter per component.

For $\beta = \sigma_{i_1} \cdots \sigma_{i_m}$, let

$$P_\beta = P_{i_1}(z_1) \cdots P_{i_m}(z_m),$$

where $P_i(z)$ is the identity except for the block $\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$ in positions $i, i+1$. The model has differential

$$(3.16) \quad dB = P_\beta^{-1} + DLU,$$

where L, U are lower and upper unitriangular matrices, B has degree -1 , and the remaining generators have degree zero.

The comparison in [CS26, Lemma 2.3] fixes the basepoint variables. Our conventions give $D = -\text{diag}(q_i e_i)$ for III , IV . For II , we use $D = \text{diag}(-q, 1)$, retaining one basepoint as in [CS26, Section 4].

By [CS26, Lemma 2.4], eliminating the entries of L, U from $P_\beta^{-1} + DLU = 0$ gives the following presentations of $H^0(G(\beta)_q)$. The crossing generators are labelled alphabetically in braid order, as in Figure 9.

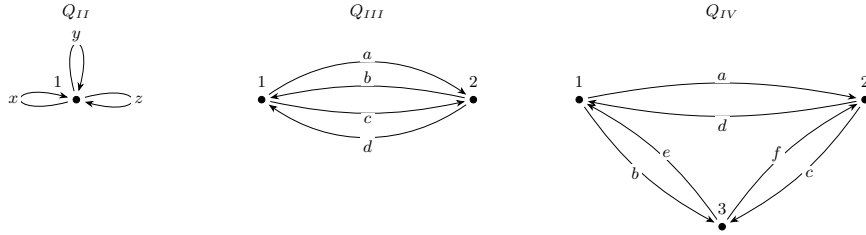


FIGURE 9. Quivers for the nontransverse fibres. The arrows are the degree-zero generators of $A_{II}(q)$, $A_{III}(q_1, q_2)$, and $A_{IV}(q_1, q_2, q_3)$, with relations (3.17), (3.18), and (3.19). The cusp loops commute at $q = 1$.

For type II , the two diagonal relations give

$$(3.17) \quad A_{II}(q) = \mathbf{k}\langle x, y, z \rangle / (q + x + z + zyx, 1 + x + z + xyz).$$

At $q = 1$, this algebra is commutative and equals $\mathbf{k}[x, y, z] / (xyz + x + z + 1)$ [EL23, Section 6.1.5]. For type III , with $a, c : 1 \rightarrow 2$ and $b, d : 2 \rightarrow 1$,

$$(3.18) \quad A_{III}(q_1, q_2) = \mathbf{k}Q_{III} / ((1 - q_1)e_1 + ba + da + dc + dcba, (1 - q_2^{-1})e_2 + ab + ad + cd + abcd).$$

For type IV , with $a : 1 \rightarrow 2$, $b : 1 \rightarrow 3$, $c : 2 \rightarrow 3$, $d : 2 \rightarrow 1$, $e : 3 \rightarrow 1$, $f : 3 \rightarrow 2$, put $A = ec - d$, $B = fb - a - fca$ and $C = ca - b$. Then

$$(3.19) \quad \begin{aligned} A_{IV}(q_1, q_2, q_3) &= \mathbf{k}Q_{IV} / (R_1^q, R_2^q, R_3^q), \\ R_1^q &= (1 - q_1)e_1 + da + eb - eca, \\ R_2^q &= (1 - q_2)e_2 + fc - Bq_1^{-1}A, \\ R_3^q &= (1 - q_3)e_3 + Cq_1^{-1}e - (-c - Cq_1^{-1}A)q_2^{-1}(-f + Bq_1^{-1}e). \end{aligned}$$

Each R_i^q lies in $e_i \mathbf{k}Q_{IV} e_i$, and the only inverses used in the elimination are the scalars q_i^{-1} . At $q = 1$ the relations reduce to $da + eb - eca$, $fc - BA$ and $Ce - (-c - CA)(-f + Be)$. In all three algebras multiplication is concatenation of paths, read from right to left as in the conventions of the introduction, and products of noncomposable paths are zero.

For each Kodaira type F , let $\mathcal{B}_F^q$ denote the dg algebra obtained above: the plumbing algebra $\mathcal{B}_Q^q$ of Definition 3.4 for the normal crossing types, and the rainbow model $G(\beta)$ of Equation (3.16) specialised at $D = \text{diag}(-q, 1)$ for II and $D = -\text{diag}(q_i e_i)$ for III, IV . In all cases

$$(3.20) \quad \text{CE}_q^*(\Lambda_F) \simeq \mathcal{B}_F^q, \quad H^0(\mathcal{B}_F^q) = A_F(q), \quad H^k(\mathcal{B}_F^q) = 0 \quad (k > 0).$$

Here $A_F(q)$ is the algebra listed in Table 1. The dg algebra $\mathcal{B}_F^q$ is concentrated in nonpositive degrees.

Remark 3.21. For the rainbow closures, Casals–Simons establish vanishing with the basepoint variables retained [CS26, Main Theorem]. Here we establish vanishing in negative degrees after every bulk specialisation by the wrapped Floer computation of Section 4 and the surgery comparison of Section 5.

4. WRAPPED FLOER COHOMOLOGY

We prove that the wrapped Floer cohomology between the completed transverse disks is concentrated in degree zero. Relative periods express chord degrees as Morse indices; positivity of their angular variation makes these indices zero. We then construct cofinal wrapping Hamiltonians to obtain the wrapped Floer statement. Throughout, the completed disks carry the bulk trivialisations of Section 5.1; these change the coefficients of Floer operations but not the chords or their degrees.

4.1. Holomorphic gradings and relative periods. We use $\iota_{X_H}\omega = -dH$. In coordinates with $\omega = dp \wedge dq$, the flow of $H(p)$ is $(q, p) \mapsto (q + \nabla H(p), p)$. Gradings follow [Sei00, Section 2d]: the degree of a chord from L_i to L_j is the absolute index of $d\phi_H^1(TL_i)$ and TL_j , with the first grading transported along the Hamiltonian flow. A small Hamiltonian with a nondegenerate minimum on a Lagrangian gives a degree-zero constant chord.

Lemma 4.1. *A complex symplectic surface (X, I, Ω) with symplectic form $\text{Re } \Omega$ has a grading structure in which every holomorphic curve has phase zero. In holomorphic coordinates $\Omega = dw \wedge dt$, let L_i, L_j be holomorphic graphs $w = w_i(t), w_j(t)$, both graded by zero. For a real Hamiltonian $H(t)$ and a nondegenerate chord from L_i to L_j lying over $t = t_0$, the point t_0 is a nondegenerate critical point of*

$$H(t) + \text{Re } A(t), \quad A'(t) = w_j(t) - w_i(t),$$

and the degree of the chord is the Morse index of this function at t_0 . Here A is any local holomorphic primitive of $w_j - w_i$.

Proof. Choose an I -Hermitian metric whose fundamental form α_I satisfies

$$\alpha_I^2 = (\text{Re } \Omega)^2.$$

These forms determine an almost complex structure compatible with $\text{Re } \Omega$, with complex volume form

$$\Psi = \alpha_I - i \text{Im } \Omega.$$

On every holomorphic curve, Ψ restricts to the positive area form α_I , so the curve has phase zero. The space of normalised metrics is contractible, making the grading structure independent of this choice up to homotopy.

Put $q = (\text{Re } w, \text{Im } w)$ and $p = (-\text{Re } t, \text{Im } t)$, so that $\omega = dp \wedge dq$. Let F_i be a local holomorphic primitive of $w_i(t)$, and put $G_i(p) = \text{Re } F_i(t(p))$. Then G_i is harmonic, and the graph $w = w_i(t)$ is $q = -\nabla G_i(p)$. The grading transported from the vertical plane $\{\delta q = 0\}$ through

$$\{\delta q = -u D^2 G_i \delta p\}, \quad 0 \leq u \leq 1,$$

has phase lift $-\sum_a \arctan \lambda_a(D^2 G_i) = 0$: since G_i is harmonic, the two eigenvalues of $D^2 G_i$ are opposite. It therefore agrees with the holomorphic grading. At a chord, the tangent planes to $\phi_H^1(L_i)$ and L_j are the graphs of $D^2 H - D^2 G_i$ and $-D^2 G_j$. The linear symplectic shear

$$(\delta q, \delta p) \mapsto (\delta q + D^2 G_j \delta p, \delta p)$$

subtracts the target graph, sending these planes to the graph of $D^2(H + G_j - G_i)$ and the vertical plane. Transporting both gradings along this shear, the graph-index rule [Sei00, Section 2d, (iii)–(v)] identifies the chord degree with the Morse index of $H + G_j - G_i$. $\square$

Extend the grading structure of Lemma 4.1 to $\widehat{W}_\epsilon$ by Liouville transport along the cylindrical end. Transport it to the abstract surgery through a fixed identification in Theorem 2.1. For I_n , where $H^1(M; \mathbb{Z}) = \mathbb{Z}$, this identification is part of our grading convention. We give each completed disk $\widehat{D}_i$ the grading which is zero on its holomorphic disk $D_i \cap W_\epsilon$. This extends uniquely over the contractible plane and is called its **holomorphic grading**.

Let t denote the coordinate on the base disk Δ and its pullback to X under π . On a simply connected sector of Δ^* , define the fibrewise differential η_t by $\Omega = \eta_t \wedge dt$. Integrating η_t on the universal covers of the elliptic fibres gives w with $\Omega = dw \wedge dt$. Each disk D_i gives m_i holomorphic branches over the sector, since $t|_{D_i}$ has degree m_i near the fibre. Choose lifts of two branches, written $w = w_i(t)$ and $w = w_j(t)$, as in Lemma 4.1.

A relative period is the integral of η_t along a path in the fibre from the first branch to the second, with the path class transported over the sector. The chosen lifts give the integral $w_j(t) - w_i(t)$. Changing the path by going around either of the two basic cycles of the torus adds the corresponding period. Thus all relative periods between these branches have the form

$$(4.2) \quad P(t) = w_j(t) - w_i(t) + a\omega_1(t) + b\omega_2(t), \quad a, b \in \mathbb{Z},$$

where ω_1, ω_2 are the integrals of η_t over a chosen homology basis of the fibre. The integers a, b record the additional traversals of these cycles. Changing lifts only relabels these integers. The identically zero expression corresponds to the constant path on one branch and will be excluded.

A chord of a Hamiltonian $H(t)$ in this path class lifts to a chord from $w = w_i(t)$ to $w = w_j(t) + a\omega_1(t) + b\omega_2(t)$. Their difference is $P(t)$, so Lemma 4.1 computes the degree of a nondegenerate chord as the Morse index of

$$H(t) + \operatorname{Re} A(t), \quad A'(t) = P(t).$$

The relative periods allow us to apply the same local index calculation to chords in every winding class.

The next lemma shows that the argument of $tP(t)$ increases at a uniformly positive rate as t travels counterclockwise around a small circle; this gives the positivity needed in the chord-index calculation. The main ingredient is semistable reduction: after a finite base change, the central fibre is smooth or multiplicative, and the periods have explicit local forms.

Lemma 4.3. *There are $\epsilon_0, \kappa > 0$ such that every relative period P which is not identically zero is nowhere zero on $0 < |t| < \epsilon_0$ and satisfies*

$$(4.4) \quad \operatorname{Re} K_P(t) \geq \kappa, \quad K_P(t) = 1 + t \frac{P'(t)}{P(t)}.$$

The constants are uniform over all branches and winding classes.

Moreover, after a finite base change $t = s^N$, there are an integer $d \geq 1$ and a nowhere-zero holomorphic function u on the covering disk with the following property. Fix two lifted branches and the integer b in (4.2). For every $a \in \mathbb{Z}$, the corresponding relative period satisfies

$$Ns^{N-1}P_a(s^N) = s^{d-1}u(s)(a + \delta_b(s)),$$

where δ_b is independent of a . On each simply connected sector,

$$\delta_b(s) = \ell_b \log s + v_b(s),$$

with $\ell_b \in \mathbb{C}$ constant and v_b holomorphic at 0. (For types II, III, IV, I_0^ , IV^* , III^* , II^* , semistable reduction has a smooth central fibre, so $\ell_b = 0$.)*

Proof. Take a semistable base change $t = s^N$ [SS10, Section 5.2], with every $m_i \mid N$, and pass to its relatively minimal regular model. The disks split into sections $\sigma_\ell(s)$; choose one as the zero section $e(s)$. The central fibre is smooth or of type I_n . On a simply connected sector of the punctured covering disk, normalise the periods to 1, $\tau(s)$, with the period 1 on a monodromy-invariant cycle (the vanishing cycle in the I_n case) and $\operatorname{Im} \tau(s) > 0$. Then

$$E_s \cong \mathbb{C}/(\mathbb{Z} + \tau(s)\mathbb{Z}) \cong \mathbb{C}^*/q(s)\mathbb{Z}, \quad q(s) = e^{2\pi i \tau(s)}, \quad z = e^{2\pi i \zeta}, \quad \eta_s = d\zeta = \frac{dz}{2\pi i z}.$$

The pulled-back form Ω descends holomorphically to this model by extension across the contraction points. Its coefficient relative to $\eta_s \wedge ds$ is constant on smooth compact fibres and holomorphic in s , since this frame is nonvanishing at a smooth point of the central fibre. Thus, after shrinking,

$$\Omega = s^{d-1}u(s)\eta_s \wedge ds, \quad d \geq 1, \quad u(s) \neq 0.$$

Let $\Pi(s)$ be the corresponding relative period of η_s . Then

$$(4.5) \quad \begin{aligned} Ns^{N-1}P(s^N) &= s^{d-1}u(s)\Pi(s), \\ NK_P(s^N) &= d + s \frac{u'(s)}{u(s)} + s \frac{\Pi'(s)}{\Pi(s)}. \end{aligned}$$

Since $su'/u = O(|s|)$, we study the zeros and logarithmic derivatives of Π .

Choose paths $\gamma_\ell(s)$ from $e(s)$ to $\sigma_\ell(s)$ continuously over the sector. The Abel–Jacobi lifts give

$$\zeta_\ell(s) = \int_{\gamma_\ell(s)} \eta_s, \quad z_\ell(s) = e^{2\pi i \zeta_\ell(s)}, \quad \sigma_\ell(s) = [z_\ell(s)] \in E_s.$$

Following γ_i backwards and then γ_j gives the integral $\zeta_j - \zeta_i$. Adding the periods $a + b\tau$ yields

$$\Pi_{a,b}(s) = a + b\tau(s) + \zeta_j(s) - \zeta_i(s).$$

If the central fibre is smooth, τ extends holomorphically to 0 with $\text{Im } \tau(0) > 0$, and the paths can be chosen over the whole disk. If the central fibre is I_n , Kodaira’s local description [Kod63, (7.7) and Section 8(v)] gives

$$\tau(s) = \frac{n}{2\pi i} \log s + \tau_0(s),$$

with τ_0 holomorphic at 0. Sections avoid the nodes: a section of $xy = s$ cannot pass through $x = y = 0$. For σ_ℓ meeting the k_ℓ th component, Kodaira’s charts [Kod63, (8.42)–(8.45)] give a lift $z_\ell = s^{k_\ell}v_\ell(s)$, with $v_\ell(0) \neq 0$. Taking logarithms compatible with the Abel–Jacobi lifts gives

$$\zeta_j(s) - \zeta_i(s) = \frac{k_j - k_i}{2\pi i} \log s + \frac{1}{2\pi i} \log \frac{v_j(s)}{v_i(s)}.$$

Here $\log(v_j/v_i)$ is holomorphic at 0. Substituting $\log s/(2\pi i) = (\tau(s) - \tau_0(s))/n$ gives

$$\Pi_{a,b}(s) = a + c\tau(s) + \beta(s), \quad c = b + \frac{k_j - k_i}{n},$$

where β ranges over a finite list of functions holomorphic at 0, one for each pair of sections, independent of a, b . In either case, fixing b gives the asserted form of δ_b .

We now estimate the logarithmic derivatives uniformly. In the smooth case, for $\Pi_{a,b}(0) \neq 0$, discreteness of the finitely many lattice translates gives $|\Pi_{a,b}(0)| \geq \delta > 0$. Taking imaginary parts gives $|b| \text{Im } \tau(0) \leq |\Pi_{a,b}(0)| + |\text{Im}(\zeta_j(0) - \zeta_i(0))|$, hence $1 + |b| \leq C|\Pi_{a,b}(0)|$ uniformly. Both $\Pi_{a,b}(s) - \Pi_{a,b}(0)$ and $s\Pi'_{a,b}(s)$ are bounded by $C|s|(1 + |b|)$. For small s , these bounds give $\Pi_{a,b}(s) \neq 0$ and

$$\frac{s\Pi'_{a,b}(s)}{\Pi_{a,b}(s)} = O(|s|) \quad \text{uniformly in } a, b.$$

In the multiplicative case, work on finitely many sectors with bounded argument. If $c \neq 0$, then $|c| \geq 1/n$, and for small s ,

$$|\Pi(s)| \geq \frac{1}{2}|c| \text{Im } \tau(s), \quad \left| \frac{s\Pi'(s)}{\Pi(s)} \right| \leq \frac{C}{|\log |s||},$$

uniformly in a, b , since $s\tau'(s) = n/(2\pi i) + O(|s|)$. If $c = 0$, then b is fixed and $\Pi = a + \beta(s)$; when $\Pi(0) \neq 0$, the same translated-lattice argument gives nonvanishing and $s\Pi'/\Pi = O(|s|)$.

Only finitely many holomorphic periods vanish at 0: at most one winding class for each pair of sections. Each such period that is not identically zero has the form $\Pi = s^k v$, with $k \geq 1$ and $v(0) \neq 0$. These periods are therefore nonvanishing on a common punctured disk and satisfy $s\Pi'/\Pi = k + O(|s|)$. By (4.5), $NK_P(s^N)$ therefore tends uniformly to d for the remaining periods, and to $d + k$ for these finitely many exceptions. Since $d \geq 1$, shrinking the disk gives $\text{Re } K_P \geq \kappa > 0$ uniformly over all branches and winding classes. $\square$

4.2. Boundary chords. Choose $\epsilon < \epsilon_0$ small enough for Proposition 2.11, and put

$$Y = \partial W_\epsilon, \quad \alpha = \lambda|_Y, \quad \Gamma_i = D_i \cap Y.$$

Since $\lambda|_{D_i} = 0$, the circles Γ_i are Legendrian. Let $r = e^\rho$ be the Liouville radial coordinate, where ρ is Liouville flow time from Y , so $r = 1$ on Y . For $\mu = \log |t|$, the Reeb field is

$$(4.6) \quad R_\alpha = \frac{X_\mu}{Z_\mu}, \quad X_\mu : \quad \dot{w} = -\frac{1}{t}, \quad \dot{t} = 0.$$

Indeed, X_μ spans the characteristic line of Y and $\lambda(X_\mu) = Z_\mu > 0$. A positive chord in the relative class P therefore satisfies

$$(4.7) \quad tP(t) = -c, \quad c > 0,$$

where c is its flow time for X_μ .

Proposition 4.8. *Every positive Reeb chord between the Γ_i is nondegenerate. Its realisation by a convex radial Hamiltonian $H = h(r)$ on the completion, with $h'' > 0$ at the chord, has degree zero with the holomorphic gradings of the completed disks.*

Proof. Fix a Reeb chord γ from $p_- \in \Gamma_i$ to $p_+ \in \Gamma_j$. By (4.6) it is a flow line of X_μ of some time $c > 0$, and its relative period P satisfies (4.7). Lift Γ_i and Γ_j to the graphs $w = w_i(t)$ and $w = w_i(t) + P(t)$ over $|t| = \epsilon$, as after (4.2). Put $t_0 = \pi(p_-)$, so that $\gamma \subset \pi^{-1}(t_0)$, and $K = K_P(t_0)$. The X_μ flow lines from Γ_i form the surface $S(\theta, u) = (w_i(t) - u/t, t)$, $t = \epsilon e^{i\theta}$, in Y , and Γ_j is cut out by $w - w_i(t) - P(t)$, which restricts to S as $-(tP(t) + u)/t$. As $\partial_\theta(tP) = itPK_P = -icK$ at t_0 , the map $(\theta, u) \mapsto tP(t) + u$ has Jacobian determinant $c \operatorname{Re} K \neq 0$ at the parameter $(\arg t_0, c)$ of p_+ , by Lemma 4.3, so S is transverse to Γ_j in Y at p_+ . If γ is a flow line of time τ of a positive multiple X of R_α , then $T_{p_+}S = D\phi_X^\tau(T_{p_-}\Gamma_i) + \mathbb{R}R_\alpha(p_+)$, since X sweeps out the same surface; hence

$$(4.9) \quad T_{p_+}Y = D\phi_X^\tau(T_{p_-}\Gamma_i) \oplus \mathbb{R}R_\alpha(p_+) \oplus T_{p_+}\Gamma_j.$$

For $X = R_\alpha$, whose flow preserves the contact planes, this is the nondegeneracy of γ .

Since Lemma 4.1 needs a Hamiltonian in t , we interpolate. Put $\mu_0 = \log \epsilon$ and, for $0 \leq v \leq 1$,

$$f_v = (1-v)(\mu - \mu_0) + v(r-1), \quad g_v = (1-v)Z\mu + v > 0,$$

with μ extended smoothly across Y ; only its 2-jet along γ enters the linearised flows below. On Y , where μ and r are constant and $dr(Z) = 1$, we have $f_v = 0$ and $df_v = g_v dr$; as $X_r = R_\alpha$ on the collar, where $\lambda = r\alpha$, this gives $X_{f_v} = g_v R_\alpha$ along Y . So γ is a flow line of X_{f_v} of a time c_v continuous in v , with $c_0 = c$ and c_1 its flow time for R_α . For $\sigma > 0$, fixed below, put $H_v = c_v f_v + \frac{\sigma}{2} f_v^2$. As f_v is constant along its flow, $\phi_{H_v}^1(x) = \phi_{f_v}^{c_v + \sigma f_v(x)}(x)$, which is $\phi_{f_v}^{c_v}$ on Y . Thus γ , suitably parametrised, is a time-one chord of every H_v ; here $H_0 = c(\mu - \mu_0) + \frac{\sigma}{2}(\mu - \mu_0)^2$ is a function of t , and $H_1 = c_1(r-1) + \frac{\sigma}{2}(r-1)^2$ is radial.

Let $B_v = D\phi_{f_v}^{c_v}$ at p_- and $0 \neq e \in T_{p_-}\Gamma_i$. We have $T_{p_-}\widehat{D}_i = \mathbb{R}e \oplus \mathbb{R}Z$ and $T_{p_+}\widehat{D}_j = T_{p_+}\Gamma_j \oplus \mathbb{R}Z$. Differentiating $\phi_{H_v}^1$ at p_- , where $df_v(e) = 0$ and $df_v(Z) = g_v(p_-)$,

$$D\phi_{H_v}^1(e) = B_v e, \quad D\phi_{H_v}^1(Z(p_-)) = B_v Z(p_-) + \sigma g_v(p_-) X_{f_v}(p_+).$$

As $\sigma \rightarrow \infty$, the image tangent plane converges to

$$\operatorname{span}\{B_v e, X_{f_v}(p_+)\} = T_{p_+}S.$$

By (4.9), the limiting plane $T_{p_+}S$ is transverse to $T_{p_+}\widehat{D}_j$. The convergence is uniform for $v \in [0, 1]$, by continuity and compactness, so the image tangent planes are transverse for all sufficiently large σ . Thus γ is nondegenerate for every H_v .

For $v = 1$, the Reeb flow commutes with the Liouville flow, so $B_1 Z(p_-) = Z(p_+)$. The image tangent plane is therefore

$$\operatorname{span}\{B_1 e, Z(p_+) + \sigma R_\alpha(p_+)\},$$

which is transverse to $T_{p_+}\widehat{D}_j$ for every $\sigma > 0$, by (4.9).

Choose such a σ with $\sigma > c|K|^2/\operatorname{Re} K$. The degree of γ is then independent of v . For H_0 , Lemma 4.1 applied to the lifted chord gives it as the Morse index at t_0 of $\Phi = H_0 + \operatorname{Re} A$, $A' = P$. In the coordinates (μ, θ) , $\partial_\mu A = tP$, $\partial_\theta A = itP$ and $t\partial_t(tP) = tPK_P = -cK$ at t_0 , so

$$D^2\Phi = \begin{pmatrix} \sigma - c\operatorname{Re} K & c\operatorname{Im} K \\ c\operatorname{Im} K & c\operatorname{Re} K \end{pmatrix}, \quad \det D^2\Phi = c\sigma \operatorname{Re} K - c^2|K|^2 > 0,$$

which, as $c\operatorname{Re} K > 0$, is positive definite. So γ has degree zero for H_0 , hence for H_1 .

A radial $H = h(r)$ with $h'(1) = c_1$ and $h''(1) > 0$ has the linearised flow of H_1 along γ with σ replaced by $h''(1)$. So γ is a nondegenerate chord of H by the preceding transversality calculation. Its degree is constant on the connected family $h''(1) \in (0, \infty)$, hence zero. Inverse Liouville flow carries any chord of $h(r)$ at $r = r_0 > 1$ to a chord at $r = 1$ of $g(r) = h(r_0 r)/r_0$, preserving its degree. If $h''(r_0) > 0$, then $g''(1) = r_0 h''(r_0) > 0$, so the chord has degree zero. $\square$

4.3. Cofinal wrapping.

Theorem 4.10. *Let M be the completed surgery of Theorem 2.1, with the grading conventions fixed above, and let $\widehat{D}_i$ be its holomorphically graded cocores. For every field $\mathbf{k}$, every $\mathbf{b} \in H^2(M; \mathbf{k}^\times)$, and all i, j ,*

$$\operatorname{HW}_{\mathbf{b}}^*(\widehat{D}_i, \widehat{D}_j; \mathbf{k}) = 0 \quad (* \neq 0), \quad \operatorname{HW}_{\mathbf{b}}^0(\widehat{D}_i, \widehat{D}_j; \mathbf{k}) \neq 0.$$

The wrapped complexes can be computed by cofinal Hamiltonians all of whose chords have degree zero.

Proof. Use Theorem 2.1 to work on $\widehat{W}_\epsilon$. The set of positive flow times of Reeb chords between the circles $\Gamma_i \subset Y$ is locally finite. Indeed, the circles Γ_i are disjoint and transverse to the Reeb field, so these times are bounded away from zero. The solutions with bounded time form a compact set, and Proposition 4.8 makes them isolated.

Choose increasing slopes T_k outside this spectrum, tending to infinity. Take Hamiltonians H_k which are fixed and sufficiently small on W_ϵ and equal to $h_k(r)$ on the end. Choose increasing convex radial profiles h_k , strictly convex at every radius supporting a Hamiltonian chord, and eventually linear with slope T_k . On the compact part, use the standard cotangent-neighbourhood perturbation associated to a small Morse function with one nondegenerate minimum y_i on each disk. Its only self-chord is the fixed minimum y_i , of degree zero. Choose the perturbation to agree near Y with a radial Hamiltonian whose common slope lies strictly between zero and the first positive Reeb-chord time. For sufficiently small compact data, there are no chords between distinct disks, and the radial collar prevents chords from crossing Y . All end chords have degree zero by Proposition 4.8. The final linear region has no chords by the choice of T_k .

Choose regular contact-type Floer data and the bulk weights of Section 5.1. Every complex is concentrated in degree zero, so its differential vanishes. Taking the direct limit, as in Section 5.1, proves vanishing outside degree zero. To prove nonvanishing, we first show that the continuation maps are injective, and then produce a chord for every pair of disks.

Choose the profiles with $H_{k+1} \geq H_k$ everywhere and $H_{k+1} = H_k$ on a growing compact set containing every chord of H_k , with the Floer data fixed there. Project the continuation images onto the span of these old chords and order them by action. The continuation action inequality makes the resulting square matrix triangular. Equality of actions forces a constant strip, whose coefficient is a unit; its bulk weight is $\mathbf{b}(0) = 1$. Thus the triangular matrix has invertible diagonal entries, and the continuation maps are injective.

To find a positive boundary chord between any two branches, use the cover of Lemma 4.3. On the fixed circle $|s| = \epsilon^{1/N}$, the function $s^d u(s)$ winds $d \geq 1$ times around zero. Choose a compact arc on which its real part is negative and its imaginary part has opposite signs at the endpoints. With b fixed on a sector containing this arc,

$$\frac{Ns^N P_a(s^N)}{a} = s^d u(s) \left(1 + \frac{\delta_b(s)}{a} \right) \longrightarrow s^d u(s)$$

uniformly on the arc. For a sufficiently large positive integer a , the negative real part and the endpoint signs persist. The intermediate value theorem therefore gives a point where $tP_a(t)$ is negative real, with $t = s^N$. This is a positive boundary chord by (4.7). Its degree-zero generator survives continuation, proving nonvanishing for every pair. $\square$

5. SURGERY AND GENERATION

We now identify the algebras of Section 3 with the bulk-deformed wrapped endomorphisms of the cocores, using the surgery formula of Bourgeois, Ekholm and Eliashberg [BEE12] in the form of [EL23], and we prove that the cocores generate the bulk-deformed category, following [Cha+22]. Both arguments are proved in the literature for the ordinary wrapped Fukaya category. These arguments extend to the bulk-deformed categories of Section 5.1: bulk weights respect gluing, so the algebraic identities remain valid, and we check below that the comparison maps retain invertible leading coefficients.

5.1. Bulk-deformed branes and weights. Let M be the completion of a neighbourhood of F , identified with the Legendrian surgery of Theorem 2.1, and let r be the number of components. Since M retracts onto F_{red} ,

$$H_2(M; \mathbb{Z}) = \bigoplus_{i=1}^r \mathbb{Z}\gamma_i, \quad \gamma_i = [C_i], \quad H_1(M; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & F = I_n, \\ 0, & \text{otherwise,} \end{cases}$$

where C_i carries its complex orientation. By the proof of Theorem 2.1, γ_i is the class of the i th handle: the core sphere for the plumbings, and the core capped in the 0-handle for *II*, *III* and *IV*. Reversing the orientation of C_i inverts q_i .

View the bulk class as a character and write its coordinates

$$(5.1) \quad \mathfrak{b} : H_2(M; \mathbb{Z}) \longrightarrow \mathbf{k}^\times, \quad q_i = \mathfrak{b}(\gamma_i).$$

The parameter space is the full torus $(\mathbf{k}^\times)^r$, including roots of unity and the trivial character. Since $H_1(M; \mathbb{Z})$ is free, the universal coefficient theorem identifies these characters with $H^2(M; \mathbf{k}^\times)$.

Without giving full details, we briefly indicate what $\mathcal{W}_{\mathfrak{b}}(M)$ is. Choose a multiplicative singular two-cocycle η representing $\mathfrak{b}$. A bulk-deformed brane is a properly embedded exact graded spin Lagrangian, conical outside a compact set, equipped with a trivialisation of η along it and a finite-rank local system. Each chord carries the Hom space between the endpoint fibres of the local systems, tensored with a one-dimensional bulk coefficient line. After choosing a basis vector in each bulk coefficient line, a polygon contributes a scalar bulk weight multiplying the usual composition and parallel-transport maps. This weight is the product of the cocycle values on a triangulation, corrected by the boundary trivialisations; it is independent of subdivision and multiplicative under gluing. Changes of trivialisation are absorbed by tensoring the local systems with rank-one local systems, and changing the representative η gives an equivalent category. We write $\mathcal{W}_{\mathfrak{b}}(M)$ for the split-closed triangulated envelope of the resulting wrapped A_∞ category. The contractible cocores admit the required trivialisations, unique up to isomorphism.

We use the Hamiltonian model of wrapped Floer cohomology:

$$\text{HW}_{\mathfrak{b}}^*(L_0, L_1; \mathbf{k}) = \varinjlim_k \text{HF}_{\mathfrak{b}}^*(L_0, L_1; H_k; \mathbf{k}).$$

Here H_k is an increasing cofinal sequence of Hamiltonians that are eventually linear in the Liouville radial coordinate, with slopes tending to infinity outside the Reeb chord spectrum; the maps are monotone continuation maps. The comparison with the geometric wrapped model is [EL23, Appendix B.1.1, Lemmas 79–80]. Its bulk-deformed version follows from the gluing rule for weights, as in Lemma 5.2.

The trivialisation of η is part of the brane data. For a closed two-form, the analogous datum is a primitive of its restriction [Sie21, Section 3.1, Remark 3.2]. Here bulk deformation means a multiplicative degree-two twist, rather than a series of operations with arbitrary interior insertions.

For the cocores, a capped polygon of class β carries the universal weight $[\beta] \in \mathbb{Z}[H_2(M; \mathbb{Z})]$. Evaluating $[\beta] \mapsto \mathfrak{b}(\beta)$ gives its bulk weight. For cyclic plumbings, choose reference paths recording chord winding, compatibly with gluing. We cap the polygon boundary as a whole; individual noncontractible chords need not admit capping disks.

Weights do not change the Floer equations, degrees, energy bounds, or maximum principle. Exactness excludes nonconstant sphere bubbles and bubbles on an embedded exact boundary component. Fixed asymptotic data give finite zero-dimensional counts, so the coefficients are finite sums in this group algebra. This is the bulk-deformed version of [Sie21, Sections 3.1 and 4.3.2]; its construction uses only multiplication of weights, not that paper's particular choice of an exponential coefficient field.

5.2. Bulk weights in Floer-theoretic arguments. Write $w(u)$ for the scalar bulk weight of a holomorphic curve u , defined in Section 5.1. Bulk-deformed operations count rigid curves with their signed weights $\pm w(u)$.

Lemma 5.2. *Let $\mathcal{M}$ be a compactified one-dimensional moduli space of holomorphic curves with boundary on exact branes carrying bulk trivialisations, in a Liouville manifold or cobordism. Suppose its boundary consists of broken configurations of rigid curves. The bulk weight is constant on each connected component and equals the product of the weights of the pieces at every breaking. Consequently, the signed boundary-count identities hold with bulk weights. The signed scalar weight of each rigid curve is a unit.*

Proof. The cocycle identity and the boundary trivialisations make the weight invariant under deformation and multiplicative under gluing. The signed weighted boundary count therefore vanishes on each compact one-dimensional component. Each signed scalar weight is a unit in the coefficient ring. $\square$

This applies to the A_∞ and continuation identities and to the curve-count identities used in Propositions 5.3 and 5.6, with the coefficients of Section 5.1.

5.3. The surgery comparison. Let F be one of the Kodaira types, M the completion of the surgery of Theorem 2.1, Λ_F its attaching link and $\widehat{D}_i$ its cocores with their holomorphic gradings. The grading of the cocores induces a Maslov potential on each component of Λ_F , and hence a $\mathbb{Z}$ -grading on $\text{CE}_q^*(\Lambda_F)$; for the cycles it also fixes the trivialisation along the 1-handle. We call this the induced grading. The geometric gradings compared below differ by an integer shift on each component, which changes the degree of a chord from Λ_i to Λ_j by the difference of the shifts, and, for the cycles, by an element $m \in H^1(M; \mathbb{Z}) = \mathbb{Z}$, which changes the degree of a word by m times its winding number around the cycle.

Proposition 5.3. *For every bulk class $\mathfrak{b}$ as in Equation (5.1), there is an A_∞ quasi-isomorphism respecting the idempotents and gradings,*

$$(5.4) \quad \text{CW}_{\mathfrak{b}}^*\left(\bigoplus_i \widehat{D}_i, \bigoplus_i \widehat{D}_i\right) \simeq \mathcal{B}_F^q,$$

where the induced grading on $\mathcal{B}_F^q$ agrees with that of Section 3.

The projection $\mathcal{B}_F^q \rightarrow A_F(q)$ is a quasi-isomorphism. Thus $A_F(q)$ is a minimal model for the cocore endomorphism algebra, and

$$(5.5) \quad \text{HW}_{\mathfrak{b}}^a(\widehat{D}_i, \widehat{D}_j; \mathbf{k}) = \begin{cases} e_j A_F(q) e_i, & a = 0, \\ 0, & a \neq 0. \end{cases}$$

Proof. By [EL23, Theorem 2 and Theorem 83], following [BEE12, Section 5.4 and Remark 5.9], there is an A_∞ quasi-isomorphism $\Phi : \text{CW}^*(\bigoplus_i \widehat{D}_i) \rightarrow \text{CE}^*(\Lambda_F)$ for the ordinary wrapped Floer complex, whose components count holomorphic disks in the surgery cobordism with boundary on the cocores and on the cores. It is defined for links in the boundary of any subcritical Weinstein manifold. For the cyclic plumbings, we use the fixed integer grading of Section 4.1 and compatible basepoint paths as in [EL23, Appendix A, Remark 72], together with the coefficient choices of Section 5.1. The map preserves gradings when $\text{CE}^*(\Lambda_F)$ carries the induced grading. Equip every cocore with its bulk trivialisation and count every disk with its weight. The target differential counts disks in the symplectisation of the boundary before surgery.

Choose reference arcs on each attaching circle cut at its basepoint, with compatible reference paths and capping chains as in Section 5.1. Closing a disk u through the handle cores, a positive passage through the i th basepoint contributes the oriented capped core class $\gamma_i = [C_i]$, and a negative passage contributes $-\gamma_i$. Thus, if n_i is the signed number of passages, then

$$[u_{\text{cap}}] = \sum_i n_i [C_i], \quad w(u) = \mathfrak{b}([u_{\text{cap}}]) = \prod_i q_i^{n_i}.$$

Here the coefficient-line bases are chosen compatibly with the cappings; changing chord cappings rescales these bases. The weight is therefore exactly that obtained by substituting $t_i = q_i e_i$ in the basepoint word.

Hence the weighted target is $\mathrm{CE}_q^*(\Lambda_F)$. By Lemma 5.2, the weighted surgery map Φ_b satisfies the A_∞ relations.

We show that Φ_b is a quasi-isomorphism following [EL23, Theorem 83], using the chord correspondence and disk counts of [Ekh19]. As noted in [Ekh19, Introduction], these results also apply without the simple-connectivity assumption; see [EO17, Section 7.3]. Fix an action cutoff $A > 0$. For sufficiently thin surgery handles, the cocore chords below this cutoff correspond to composable words of chords of Λ_F [Ekh19, Theorem 1.2]. Their actions agree up to an error tending to zero with the handle width. The unweighted linear term sends each chord to its corresponding word with coefficient ± 1 , plus terms of smaller action [Ekh19, Theorem 7.5(a)]. Let d_w denote this diagonal coefficient with weights. For a one-letter word, the local disk is unique [Ekh19, Lemma 7.4]; the deformation in [Ekh19, Lemma 7.6] has a compact one-dimensional moduli space with no broken ends, so it preserves the weighted count and d_w is a unit. For a concatenation $w = uv$, let $p_{u,v}$ be the weighted two-input disk count of [Ekh19, Theorem 7.5(c)]. The boundary identity in the proof of that theorem, weighted by Lemma 5.2, gives

$$p_{u,v}d_{uv} = \pm d_u d_v.$$

Induction on word length makes the right-hand side a unit, hence d_{uv} is a unit. The constant generators have the local unit-disk coefficients of [EL23, Remark 82], which are units as well. Thus, for each cutoff and sufficiently thin handles, $\Phi_{b,1}$ is triangular with invertible diagonal on the finitely many generators below the cutoff, hence is an isomorphism there. As the cutoff increases, shrink the handles and use the compatible cobordism maps of [EL23, Theorem 83]. These are continuation quasi-isomorphisms, and their compatibility with the surgery maps persists with bulk weights by Lemma 5.2. Passing to the direct limit therefore proves that Φ_b is a quasi-isomorphism. The identification of the wrapped complexes with their Hamiltonian versions is [EL23, Appendix B.1.1, Lemmas 79–80], which is again a comparison of counts of rigid disks and continuation maps and holds with weights.

Composing Φ_b with the quasi-isomorphism of Theorem 3.5, respectively the stable tame isomorphism of Section 3.6, gives Equation (5.4). By Theorem 4.10, the cohomology of the left-hand side is concentrated in degree zero, so $H^*(\mathcal{B}_F^q)$ is concentrated in induced degree zero.

We now compare the induced grading with that of Section 3. Let $\mathcal{B}$ be $\mathcal{B}_F^q$ with the grading of Section 3. It is concentrated in nonpositive degrees, so $e_j H^*(\mathcal{B}) e_i$ lies in degrees ≤ 0 for all i, j . The induced grading differs from it by shifts s_i and, for the cycles, by $m \in \mathbb{Z}$ as above: an element of $e_j H^*(\mathcal{B}) e_i$ of degree d and winding number ν has induced degree $d + s_j - s_i + m\nu$. By the surgery comparison just established and Theorem 4.10, $e_j H^*(\mathcal{B}) e_i$ is nonzero and supported only in induced degree zero for all i, j . For the trees and the rainbow closures, $H^1(M; \mathbb{Z}) = 0$, so m is absent. Take $i \neq j$ and a nonzero element of $e_j H^*(\mathcal{B}) e_i$, of degree $d \leq 0$; then $d + s_j - s_i = 0$, so $s_j - s_i \geq 0$. Exchanging i and j gives $s_i - s_j \geq 0$. Hence all shifts are equal, and the induced grading coincides with that of Section 3, whose degrees do not change under a common shift. For the cycle I_n , the element $x^n e_0 = a_{n-1} \cdots a_0$ of $e_0 \Lambda^q(\tilde{A}_{n-1}) e_0$ is nonzero by Proposition 3.8, has degree 0 and winding number 1, so its induced degree is m . This degree must be zero, so $m = 0$. With $m = 0$, the same argument using nonzero elements in opposite corners shows that all shifts s_i are equal.

Thus $H^k(\mathcal{B}_F^q) = 0$ for $k \neq 0$, while Equation (3.20) identifies $H^0(\mathcal{B}_F^q)$ with $A_F(q)$. Since $\mathcal{B}_F^q$ is nonpositively graded, the canonical projection

$$\mathcal{B}_F^q \longrightarrow H^0(\mathcal{B}_F^q) = A_F(q)$$

is a dg algebra quasi-isomorphism. Hence $A_F(q)$ is a minimal model for the cocore endomorphism algebra. Restricting to the corners gives Equation (5.5). $\square$

5.4. Generation in the bulk-deformed category. Proposition 5.3 identifies the split-closed subcategory generated by the bulk-deformed cocores with $\mathrm{Perf}(A_F(q))$. The corresponding statement for all bulk-deformed branes requires a generation argument. We give the coefficient extension of the geometric proof of [Cha+22, Theorem 1.1 and Remark 1.3], which uses only the ingredients of Lemma 5.2, together with a single invertible triangle map at each step of the construction of the augmentation.

Proposition 5.6. *For every Kodaira type F and every bulk class $\mathbf{b}$, the cocores $\widehat{D}_i$ generate $\mathcal{W}_{\mathbf{b}}(M)$, with the bulk-deformed brane convention of Section 5.1.*

Proof. We use linear wrapping throughout this argument, so the cocycle is transported by continuation rather than kept fixed under a Liouville rescaling. All branes, isotopies, and cobordisms below carry the corresponding transported boundary trivialisations, and all identities between curve counts are used in their weighted form (Lemma 5.2). Let (L, E) be a brane with a finite-rank local system. Make L transverse to the skeleton and take a Hamiltonian translate D_i of a cocore at each intersection a_i . Shift the grading of each D_i so that its short Legendrian chord a_i has degree zero. Equip D_i with the constant local system $V_i = E_{a_i}$, and write $\mathfrak{D}_i = (D_i, V_i)$. Transport the cocycle trivialisations along these isotopies. The preparations in [Cha+22, Section 9.1 and Lemma 9.4] use only the geometry, actions, and regularity of the curves and thus are unaffected by weights. They order the cocores so that the quotient of the Legendrian algebra by order-reversing chords has generators a_i, b_{ij}^m, c_{ij}^m and an action filtration. The chords a_i and b_{ij}^m run from D_i to L , while c_{ij}^m runs from D_i to D_j , for $i < j$. The distinguished triangle contributing to db_{ij}^m has negative boundary word $a_j c_{ij}^m$.

We seek an augmentation ε assigning to each chord a map between its endpoint fibres. Disk words are evaluated by composing these maps with parallel transport along the boundary arcs. We use the triangular differential of [Cha+22, Lemma 9.5], proceeding by decreasing i and then increasing action. Set $\varepsilon(a_i) = \text{id}_{V_i}$ and $\varepsilon(b_{ij}^m) = 0$. Since $da_i = 0$, their augmentation equations reduce to

$$0 = \varepsilon(db_{ij}^m) = u_{ij}^m P_{ij}^m \varepsilon(c_{ij}^m) + \varepsilon(R_{ij}^m),$$

where R_{ij}^m and dc_{ij}^m involve only generators treated earlier in this order. The coefficient $u_{ij}^m \in \mathbf{k}^\times$ is the signed weight of the unique triangle of [Cha+22, Lemma 9.4]. Here P_{ij}^m is parallel transport in E along its boundary arc from a_j to the endpoint of b_{ij}^m on L . The triangular structure follows from endpoints and action inequalities, which bulk weights do not change.

Since P_{ij}^m is invertible, this determines

$$\varepsilon(c_{ij}^m) = -(u_{ij}^m)^{-1} (P_{ij}^m)^{-1} \varepsilon(R_{ij}^m) \in \text{Hom}(V_i, V_j).$$

Applying ε to $d^2 b_{ij}^m = 0$, the previously verified augmentation equations give $u_{ij}^m P_{ij}^m \varepsilon(dc_{ij}^m) = 0$, hence $\varepsilon(dc_{ij}^m) = 0$. The construction respects degrees: the a_i have degree zero, the differential is homogeneous, and bulk weights and parallel transport have degree zero. Setting all order-reversing chords to zero completes the augmentation.

Perform surgery at the a_i , using the identity maps to glue the local systems and choosing bulk trivialisations so that the local scalar coefficients are 1. These data extend across the surgery handles, and the weighted cobordism maps transfer the augmentation. The comparison of [Cha+22, Proposition 8.16] applies with bulk weights: Lemma 5.2 supplies the required identities, and the unique local triangles of [Cha+22, Lemma 4.14 and Corollary 4.18] contribute invertible coefficients. Thus, naturally for every test brane T ,

$$\text{CW}_{\mathbf{b}}(T, \overline{L}) \simeq \text{hom}_{\mathcal{W}_{\mathbf{b}}}(T, \mathfrak{L}),$$

where $\overline{L}$ is the surgered immersion with its glued local system and transferred augmentation and $\mathfrak{L}$ is a finite twisted complex on $\mathfrak{D}_1, \dots, \mathfrak{D}_k, (L, E)$, with (L, E) its final term.

Since $\overline{L}$ misses the skeleton, its wrapped morphisms vanish by the displacement and action argument of [Cha+22, Proposition 7.6], which is unaffected by bulk weights. Hence $\mathfrak{L}$ is zero, expressing (L, E) as a twisted complex of the $\mathfrak{D}_i$. Each $\mathfrak{D}_i$ is a direct sum of $\dim V_i$ copies of D_i , and continuation identifies D_i with the corresponding grading shift of its original cocore. All resulting twisted complexes are finite. $\square$

Proof of Theorem 1.1. By Theorem 2.1, the transverse holomorphic disks complete to the cocores $\widehat{D}_i$. Proposition 5.3 computes their bulk-deformed wrapped cohomology and gives the minimal model $A_F(\mathbf{q})$ for their endomorphism algebra. Since the cocores generate by Proposition 5.6, the Yoneda functor gives

$$\mathcal{W}_{\mathbf{b}}(M) \simeq \text{Perf}(A_F(\mathbf{q})).$$

$\square$

6. MIRROR DESCRIPTIONS AT SPECIAL BULK CLASSES

We prove Corollaries 1.4 and 1.7 using Theorem 1.1 and Corollary 1.2. At the trivial bulk class, the mirror descriptions concern all affine D and E plumbings and types II, III, IV . At the root-of-unity bulk class, they concern the four stars $\tilde{D}_4, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, whose quotient models we first describe.

Let $N \in \{2, 3, 4, 6\}$ and let $E = \mathbb{C}/\Lambda$ admit multiplication by $\zeta_N = e^{2\pi i/N}$. Set

$$X_N = \text{Res}((E \times \mathbb{C}_z)/\mu_N), \quad (w, z) \mapsto (\zeta_N w, \zeta_N^{-1} z).$$

The invariant function z^N induces a proper map $t : X_N \rightarrow \mathbb{C}$, and $dw \wedge dz$ extends to a nonvanishing holomorphic two-form Ω on X_N . The elliptic curve is arbitrary for $N = 2$, has $j = 1728$ for $N = 4$, and has $j = 0$ for $N = 3, 6$. The stabilisers and central fibres are listed below.

N	F	stabiliser orders	singularities before resolution
2	I_0^*	(2, 2, 2, 2)	$4A_1$
3	IV^*	(3, 3, 3)	$3A_2$
4	III^*	(2, 4, 4)	$A_1 + 2A_3$
6	II^*	(2, 3, 6)	$A_1 + A_2 + A_5$

The stabilisers are computed from the kernels of $1 - \zeta_N^k$. The central component E/μ_N has multiplicity N , and an order- d stabiliser contributes an arm with multiplicities

$$\frac{N}{d}, \frac{2N}{d}, \dots, \frac{(d-1)N}{d}$$

from the tip towards the centre, as shown in Figure 10.

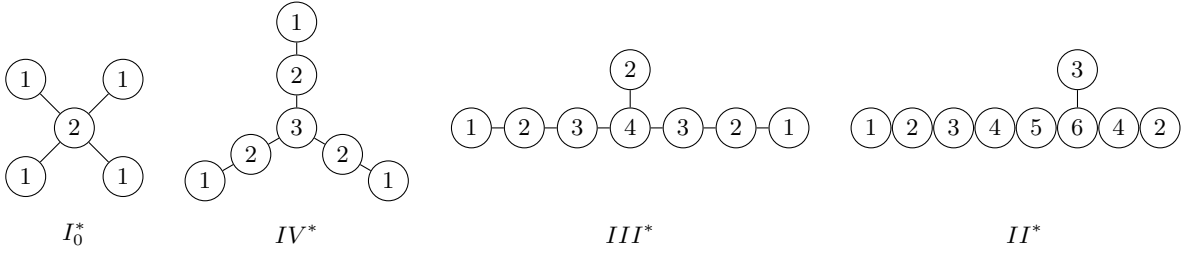


FIGURE 10. The numbers are component multiplicities.

Proposition 6.2. *The surface X_N , with $\omega = \text{Re } \Omega$, admits a complete Liouville structure exact symplectomorphic to the completed affine plumbing of type $\tilde{D}_4, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ for $N = 2, 3, 4, 6$, respectively.*

Proof. The real dilation of z lifts to a complete field Z with $\mathcal{L}_Z \Omega = \Omega$. Its primitive $\lambda = \iota_Z \text{Re } \Omega$ vanishes on F_{red} , and $Z \log |t| = N > 0$ away from F . Properness of t identifies X_N with the completion of (W_ϵ, λ) . The difference between λ and the primitive of Proposition 2.11 is closed and vanishes on the fibre; since W_ϵ retracts onto F_{red} , it is exact. The convex path between the primitives remains outward along ∂W_ϵ , so their completions are exact symplectomorphic by [CE12, Lemma 11.6 and Proposition 11.8]. Now apply Theorem 2.1(1). $\square$

6.1. The central corner. Let Q be one of the four affine stars, with the inward orientation and labels of Section 3.5. Put

$$A = \Lambda^q(Q), \quad H = e_0 A e_0, \quad q = q_0.$$

We identify H with a generalised double affine Hecke algebra and give a criterion for e_0 to be full, so that A and H are Morita equivalent.

Write $c_{jp} : (j, p) \rightarrow (j, p-1)$, with $(j, 0) = 0$, and set

$$B_{jp} = e_{(j,p)} + c_{jp}^* c_{jp}, \quad U_j = e_0 + c_{j1} c_{j1}^*, \quad \alpha_{ja} = \prod_{p=1}^a q_{(j,p)}^{-1}, \quad \alpha_{j0} = 1.$$

The tip and interior relations are

$$B_{j,d_j-1} = q_{(j,d_j-1)}^{-1} e_{(j,d_j-1)}, \quad e_{(j,p)} + c_{j,p+1} c_{j,p+1}^* = q_{(j,p)} B_{jp},$$

and the central relation is $U_1 \cdots U_s = q e_0$, with this order fixed. If $C = 1 + c c^*$ and $B = 1 + c^* c$, then $(C - 1)f(C) = c f(B) c^*$ for every polynomial f . Induction from the tip therefore gives

$$(6.3) \quad \prod_{a=p}^{d_j-1} (B_{jp} - (q_{(j,p)} \cdots q_{(j,a)})^{-1} e_{(j,p)}) = 0, \quad \prod_{a=0}^{d_j-1} (U_j - \alpha_{ja} e_0) = 0.$$

The central-corner calculation of Crawley-Boevey and Shaw [EOR07, Appendix 1, Lemma 10.1] gives

$$(6.4) \quad H \cong \frac{\mathbf{k}\langle U_1^{\pm 1}, \dots, U_s^{\pm 1} \rangle}{\left(\prod_{a=0}^{d_j-1} (U_j - \alpha_{ja}) \ (1 \leq j \leq s), \ U_1 \cdots U_s - q \right)}.$$

Thus H is a rank-one generalised double affine Hecke algebra. The calculation is over $\mathbf{k}$ and requires no splitting field.

Lemma 6.5. *If no nonempty consecutive product $q_{(j,p)} \cdots q_{(j,a)}$ on an arm equals 1, then e_0 is full in A .*

Proof. This is [EOR07, Appendix 2, Proposition 11.2] over $\mathbb{C}$; the following argument works over every field. In A/Ae_0A , the first arrow pair on each arm vanishes, so $B_{j1} = e_{(j,1)}$. The first equation in Equation (6.3) then gives a nonzero scalar multiple of $e_{(j,1)}$ equal to zero. Repeat along the arm. Every vertex idempotent vanishes in the quotient, proving $Ae_0A = A$. $\square$

At $\mathfrak{b}_{\text{grp}}$, the numbers α_{ja} run through the d_j th roots of unity, and all consecutive products in Lemma 6.5 differ from 1. Consequently,

$$(6.6) \quad H \cong \mathbf{k}\langle U_1, \dots, U_s \rangle / (U_j^{d_j} - 1, \ U_1 \cdots U_s - 1) \cong \mathbf{k}[\Gamma \rtimes G].$$

The last presentation is that of the orbifold fundamental group of E/G . The orbifold fundamental group is an extension of G by $\Gamma = \pi_1(E)$, split by the fixed origin of E . Hence A is Morita equivalent to $\mathbf{k}[\Gamma] \rtimes G$ at these parameters.

6.2. The homological consequence of formality. Kaplan–Schedler show that exactness of the Crawley-Boevey–Shaw bimodule complex implies the 2-Calabi–Yau property, and establish this exactness for quivers containing a cycle [KS23, Corollaries 3.19 and 3.20]. For affine D and E trees, Corollary 1.2 supplies the required exactness, giving the following consequence.

Lemma 6.7. *Let Q be an affine tree of type D or E and $A = \Lambda^q(Q)$. For every field and every nonzero parameter tuple, A has global dimension at most two and is bimodule 2-Calabi–Yau. In particular, for every finite-dimensional left or right A -module V ,*

$$(6.8) \quad \text{RHom}_A(V, A) \simeq \text{Hom}_{\mathbf{k}}(V, \mathbf{k})[-2],$$

with the corresponding opposite module structure.

Proof. Let $\mathcal{B} = \mathcal{B}_Q^q$ be the dg algebra of Definition 3.4. By Lemma 3.6, $H^0(\mathcal{B}) = A$, and Corollary 1.2 makes the projection $\mathcal{B} \rightarrow A$ a quasi-isomorphism. Hence $A \otimes_{\mathcal{B}}^{\mathbb{L}} A \simeq A$. Derived base change of the standard diagonal resolution of the semifree dg algebra $\mathcal{B}$, constructed using universal derivations, therefore gives a projective bimodule resolution

$$0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0,$$

where

$$P_2 = P_0 = \bigoplus_{v \in Q_0} A e_v \otimes_{\mathbf{k}} e_v A, \quad P_1 = \bigoplus_{a \in \overline{Q}_1} A e_{t(a)} \otimes_{\mathbf{k}} e_{s(a)} A.$$

The degree-zero arrows contribute P_1 , and the degree- -1 generators τ_v contribute P_2 . The degree-two differential differentiates the relations $A_v - q_v B_v$. As a resolution of right A -modules, this sequence splits; tensoring with any left A -module therefore gives a projective resolution of length at most two. The

right-module argument is identical. Multiplying the relation generator at v by B_v^{-1} identifies this complex with the multiplicative preprojective complex of Crawley-Boevey and Shaw [CS06, Lemma 3.1]: after evaluation in A ,

$$\partial((A_v - q_v B_v)B_v^{-1}) = \partial(A_v - q_v B_v)B_v^{-1}.$$

Its self-duality, proved by Kaplan and Schedler [KS23, Theorem 3.17 and Corollary 3.19], therefore gives

$$\mathrm{RHom}_{A \otimes A^{\mathrm{op}}}(A, A \otimes A^{\mathrm{op}}) \simeq A[-2].$$

Tensor–Hom duality applied to the finite projective resolution of V yields Equation (6.8). $\square$

6.3. Noncommutative crepant resolutions. For a normal Gorenstein domain R , a noncommutative crepant resolution (NCCR) is an algebra $\mathrm{End}_R(M)$, with M a finitely generated reflexive R -module, which is maximal Cohen–Macaulay over R and has finite global dimension [Van04a, Definition 4.1 and Lemma 4.2]. We construct such resolutions for the surface algebras arising above.

Let Q be $\tilde{D}_n$ ($n \geq 4$), $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$, put $A = \Lambda^1(Q)$, and choose the idempotent e at the tip of a longest arm; this is an extending vertex. Shaw’s corner calculation [Sha05, Theorem 4.1.1], also recorded in [KS23, Theorem 6.4], identifies $R = eAe$ with $\mathbf{k}[x, y, z]/(f_Q)$, with the following choice of coordinate signs:

Q	f_Q	weights of (x, y, z)
$\tilde{D}_n$	$xyz + xy^2 - p_{n-5}x^2y + p_{n-4}xz - z^2$	$(2, n-2, n-1)$
$\tilde{E}_6$	$xyz + z^2 + x^2z + y^3$	$(3, 4, 6)$
$\tilde{E}_7$	$xyz + z^2 + y^3 + x^3y$	$(4, 6, 9)$
$\tilde{E}_8$	$xyz + z^2 + y^3 + x^5$	$(6, 10, 15)$.

Here $p_i = p_i(x)$ is defined by $p_{-1} = -1$, $p_0 = 0$ and $p_{i+1} = x(p_{i-1} + p_i)$. Simultaneous induction gives, for $k \geq 1$,

$$(6.9) \quad p_{2k-1} = -x^k + O(x^{k+1}), \quad p_{2k} = -kx^{k+1} + O(x^{k+2}).$$

In particular, for $n \geq 4$,

$$(6.10) \quad x(p_{n-4} - 2p_{n-5})^2 - (x+4)p_{n-4}^2 = -4x^{n-3}\delta_n(x)$$

defines a polynomial δ_n with $\delta_n(0) = \pm 1$: the left-hand side has order exactly $n-3$, with leading coefficient ∓ 4 . The weights in the table are those of the initial forms below.

Lemma 6.11. *Let $\mathrm{char} \mathbf{k}$ be different from 2 in type D , from 2, 3 in types E_6, E_7 , and from 2, 3, 5 in type E_8 . Then R is a geometrically integral, normal, Gorenstein surface with finitely many singular points, and the origin is a du Val singularity of type D_n , E_6 , E_7 or E_8 , respectively. For $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ the origin is the only singular point. For $\tilde{D}_n$ the singular points other than the origin lie over multiple roots of δ_n in $\bar{\mathbf{k}}$; if δ_n is separable, the origin is the only singular point.*

Proof. In types E and $\tilde{D}_4$, each polynomial is quadratic in z with leading coefficient ± 1 and nonsquare discriminant: in types E_6, E_8 the discriminant has odd degree in y , and in types D_4, E_7 it has a factor x or y , respectively, with odd valuation. This remains true over an algebraic closure, so R is geometrically integral. Write g_Q for the weighted initial form, obtained by omitting the term xyz . The weight of xyz exceeds that of g_Q by 1, so the weighted Euler identity forces $xyz = 0$ at a singular point of $f_Q = 0$; substituting in the three partial derivatives leaves only the origin.

In type $\tilde{D}_n$, $n \geq 4$, write $f_Q = -z^2 + Pz - Q_0$ with $P = (p_{n-4} + y)x$ and $Q_0 = p_{n-5}x^2y - xy^2$. The discriminant is

$$P^2 - 4Q_0 = xE, \quad E = (x+4)y^2 + 2x(p_{n-4} - 2p_{n-5})y + xp_{n-4}^2.$$

Since $E(0, y) = 4y^2$, the factor x divides $P^2 - 4Q_0$ exactly once, so the discriminant is not a square over $\bar{\mathbf{k}}$ and f_Q is geometrically irreducible. As $\mathrm{char} \mathbf{k} \neq 2$, the singular points of the surface are the points $z = P/2$ over the singular points of the plane curve $xE = 0$. That curve is reduced: $x \nmid E$, and E is not a square because its discriminant with respect to y is $\Delta = -16x^{n-2}\delta_n$ by Equation (6.10), which is nonzero. Hence the surface has finitely many singular points, and being a hypersurface with isolated

singularities it is normal and Gorenstein. We locate the singular points of $xE = 0$. On $x = 0$ we have $E = 4y^2$, so only the origin occurs. At a point with $x_0 \neq 0, -4$, the quadratic $E(x_0, y)$ has a repeated root exactly when $\Delta(x_0) = 0$, and the curve $E = 0$ is singular there exactly when moreover $\Delta'(x_0) = 0$; since $\delta_n(0) \neq 0$, this means that x_0 is a multiple root of δ_n . Finally $x_0 = -4$ does not occur: the recursion gives $p_i(-4) = (-1)^{i+1}i2^{i+1}$, so $p_{n-4}(-4)$ and $(p_{n-4} - 2p_{n-5})(-4)$ are $\pm(n-4)2^{n-3}$ and $\pm(2n-9)2^{n-3}$, which do not vanish simultaneously in any characteristic other than 2. Thus $E(-4, y)$ is either a nonzero constant or a linear polynomial with nonzero derivative. Neither case gives a singular point.

At the origin, substitute $z = z' + P/2$, so that $f_Q = -z'^2 + xE/4$, where

$$\frac{xE}{4} = xy^2 + \frac{1}{4}x^2y^2 - \left(p_{n-5} - \frac{1}{2}p_{n-4}\right)x^2y + \frac{1}{4}p_{n-4}^2x^2.$$

With weights $2, n-2, n-1$ on x, y, z' , Equation (6.9) gives the initial form

$$g_Q = \begin{cases} -z'^2 + xy^2 + \frac{1}{4}x^{n-1}, & n \text{ odd}, \\ -z'^2 + x\left(y + \frac{1}{2}x^{n/2-1}\right)^2 - \frac{1}{4}x^{n-1}, & n \text{ even}. \end{cases}$$

These are du Val forms of type D_n .

In types E , the initial form is obtained by omitting xyz . These are the corresponding du Val forms; for E_6 , completing the square gives $z'^2 - x^4/4 + y^3$.

In every case, the weighted Tjurina algebra

$$\mathbf{k}[x, y, z]/(g_Q, \partial_x g_Q, \partial_y g_Q, \partial_z g_Q)$$

has top nonzero weight strictly below the weight of g_Q : these weights are $2(n-2) < 2(n-1)$ in type D_n , and $10 < 12, 16 < 18, 28 < 30$ in types E_6, E_7, E_8 . At each successive weight, the error lies in the ideal generated by g_Q and its partial derivatives. Multiplication by a unit and a coordinate change remove that weighted piece; the corrections have strictly higher weight than the terms they modify, so iteration converges in the completed local ring. Hence the origin has the asserted du Val type. Including g_Q in the Tjurina ideal also covers type D_4 in characteristic 3. $\square$

The following proposition establishes the NCCR and Satake assertions of [KS23, Conjecture 1.4] for affine D and E . The new input is Corollary 1.2, which supplies the exact bimodule resolution used in Lemma 6.7.

Proposition 6.12. *Let Q be an affine Dynkin quiver of type D or E , and assume the characteristic hypotheses of Lemma 6.11. Set $A = \Lambda^1(Q)$, let e be an extending vertex idempotent, and put $R = eAe$. Then*

$$A \cong \text{End}_R(Ae)$$

is a noncommutative crepant resolution (NCCR) of R , and the Satake map

$$Z(A) \longrightarrow R, \quad z \longmapsto ez,$$

is an isomorphism.

Proof. Deleting e leaves a finite Dynkin diagram, whose multiplicative preprojective algebra is isomorphic to the ordinary one in the stated characteristics [Kap23, Theorem 1.2]. Thus $B = A/AeA$ is finite-dimensional, and Lemma 6.7 gives, on both sides,

$$\text{Ext}_A^i(B, A) = 0 \quad (i = 0, 1).$$

Since B is finite-dimensional and A is finitely generated over $\mathbf{k}$, B is finitely presented. Hence $I = AeA$ is finitely generated as a left ideal. Expressing its generators as finite sums aeb shows that $eA = eI$ is finite over R . The right-sided argument gives the same conclusion for Ae .

Applying $\text{Hom}_A(-, A)$ to $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ gives

$$A^{\text{op}} \cong \text{Hom}_A(I, A) \cong \text{End}_R(eA).$$

For the second identification, an R -linear map $h : eA \rightarrow eA$ extends to $Ae \otimes_R eA$ by $ae \otimes x \mapsto aeh(x)$. The kernel of multiplication onto I is annihilated by e , hence is a B -module and has no nonzero maps to A . The extension therefore descends uniquely to I . Applying the same argument on the other side gives $A \cong \text{End}_R(Ae)$.

Taking the summands with target $R = eAe$ identifies Ae and eA as each other's R -duals through multiplication. Both are therefore reflexive. By [KS23, Lemma 6.2], $A = \text{End}_R(Ae)$ is maximal Cohen–Macaulay over the normal surface R . Lemma 6.7 gives finite global dimension, so A is a noncommutative crepant resolution of R , as in [KS23, Definition 6.1].

By [KS23, Lemma 6.13], the map $Z(A) \rightarrow eAe$, $z \mapsto ez$, is an isomorphism. Hence $Z(A) \cong R$ and $S_Q = \text{Spec } R$. $\square$

Remark 6.13. For the four stars, the generalised DAHA $H = e_0 A e_0$ at the unipotent point also has centre R : it is the endomorphism ring of the nonzero reflexive R -module $e_0 A e_0$, so normality gives $Z(H) = R$. However, e_0 is not full, since it annihilates every arm vertex-simple module.

We now determine when S_Q has an additional singular point in type $\tilde{D}_n$. Such points correspond to repeated roots of δ_n , by Lemma 6.11. Put $m = n - 4 \geq 1$ and

$$(6.14) \quad g_m(s) = 2s^m - (1-s)^{m-1} \in \mathbb{Z}[s], \quad N_m = 2m^m - (1-m)^{m-1} = g_m(m).$$

Proposition 6.15. *Let $n \geq 5$ and let $\mathbf{k}$ be a field of characteristic $p \neq 2$. Then δ_n is separable over $\mathbf{k}$ if and only if $p = 0$, or $p \nmid N_m$, or $p \mid 2m - 1$. In particular δ_n is separable in characteristic zero. When δ_n is not separable, it has exactly one multiple root, at $x_0 = 1/(m(m-1))$, and this root is double. The surface S_Q then has exactly one singular point other than the origin; it lies over x_0 and is a $\mathbf{k}$ -rational split A_1 singularity.*

Proof. For $m = 1$, we have $\delta_5 = N_1 = 1$, so assume $m \geq 2$. Put $x = 1/(s(s-1))$. Solving the recurrence for p_i gives

$$p_i = \frac{(-1)^i s^{-i} - (s-1)^{-i}}{2s-1} \quad (i \geq -1).$$

Since $g_m(\frac{1}{2}) = 0$, write $g_m(s) = (2s-1)q_m(s)$, where $q_m \in \mathbb{Z}[s]$ is monic of degree $m-1$. Substitution in Equation (6.10) gives

$$(6.16) \quad \delta_n(x) = x^{m-1} q_m(s) q_m(1-s), \quad x = \frac{1}{s(s-1)}.$$

The product $q_m(s)q_m(1-s)$ is invariant under $s \mapsto 1-s$, so it equals $R_m(s(s-1))$ for a polynomial $R_m \in \mathbb{Z}[w]$. Moreover, $R_m(0) = q_m(0)q_m(1) = 2$. Thus

$$\delta_n(x) = x^{m-1} R_m(1/x),$$

and inversion identifies the roots of δ_n and R_m with the same multiplicities.

We first locate the repeated roots of g_m . Neither 0 nor 1 is a root, and

$$s(1-s)g'_m(s) = m(1-s)g_m(s) + (m-s)(1-s)^{m-1}.$$

Hence a repeated root must be $s = m$. Since

$$g_m(m) = g'_m(m) = N_m,$$

such a root exists precisely when $N_m = 0$ in $\mathbf{k}$. In this case $m(1-m) \neq 0$, and differentiating the identity for g'_m gives

$$g''_m(m) = -\frac{(1-m)^{m-2}}{m} \neq 0.$$

Thus the root is exactly double.

We now pass to R_m . Away from $s = \frac{1}{2}$, the map $s \mapsto s(s-1)$ is unramified. Distinct roots of q_m cannot be identified by this map: if $g_m(s) = g_m(1-s) = 0$, multiplying the two equations gives $4s(1-s) = 1$, hence $s = \frac{1}{2}$.

At the remaining point,

$$g'_m(\tfrac{1}{2}) = 2^{2-m}(2m-1), \quad g''_m(\tfrac{1}{2}) = (m-1)2^{4-m}.$$

Thus $q_m(\tfrac{1}{2}) \neq 0$ unless $p \mid 2m-1$; in that exceptional case, $\tfrac{1}{2}$ is a simple root of q_m . It gives a simple root of R_m , since

$$s(s-1) + \tfrac{1}{4} = (s - \tfrac{1}{2})^2.$$

Consequently, R_m , and hence δ_n , has a repeated root exactly when $p \mid N_m$ and $p \nmid 2m-1$. It is the unique multiple root, is double, and lies at $x_0 = 1/(m(m-1))$. Finally, $N_m \neq 0$ in characteristic zero because $2m^m > (m-1)^{m-1}$.

In the exceptional case, $x_0 \neq 0$ and $x_0 + 4 = (2m-1)^2/(m(m-1)) \neq 0$. Near $x = x_0$, put

$$y_1 = y + \frac{x(p_m - 2p_{m-1})}{x+4}, \quad z_1 = z - \frac{x(p_m + y)}{2}.$$

Completing squares in the surface equation gives

$$f_Q = -z_1^2 + \frac{x(x+4)}{4}y_1^2 + \frac{x^{n-1}}{x+4}\delta_n(x).$$

The additional singular point is $x = x_0$, $y_1 = z_1 = 0$, so it is $\mathbf{k}$ -rational. Write $w = x - x_0$. The last term is w^2 times a unit, and the coefficient of y_1^2 has nonzero square constant term

$$\frac{x_0(x_0+4)}{4} = \left(\frac{2m-1}{2m(m-1)} \right)^2.$$

Taking its square root in $\mathbf{k}[[w]]$, factoring the two quadratic terms, and absorbing the remaining unit gives the completed local equation $uv = w^2$. This is a split A_1 singularity. $\square$

Remark 6.17. For $\tilde{D}_n$, all singularities of S_Q are rational double points in every characteristic other than 2. Since $N_1 = 1$, $N_2 = 9$ and $N_3 = 50$, the origin is the only singular point for $\tilde{D}_5$, $\tilde{D}_6$ and $\tilde{D}_7$; for $\tilde{D}_8$ one has $N_4 = 539 = 7^2 \cdot 11$, so S_Q acquires a second, split A_1 point in characteristic 11 and only there. Over $\mathbb{C}$, uniqueness of the singular point was also stated by Shaw [Sha05, Section 4.1].

The following standard type A model supplies the local charts for types III and IV . It is the quotient construction for the group scheme μ_n acting on $\mathbf{k}[r, s]$ with weights $(1, -1)$; see [Van04a, Example 1.1] in characteristic zero and [LY24, Theorems 1.5 and 1.7] in positive characteristic. The polynomial version over an arbitrary field follows by localisation, completion and faithfully flat descent.

Lemma 6.18. *Let $B_n = \Pi(\tilde{A}_{n-1})$, for $n = 2, 3$, be the ordinary preprojective algebra of the doubled cycle. Over every field it is a noncommutative crepant resolution of*

$$R_n = \mathbf{k}[Y, Z, t]/(YZ - t^n),$$

with $Z(B_n) = R_n$ and $eB_ne = R_n$ for each vertex idempotent e .

Proposition 6.19. *For $F = II, III, IV$, let $A = A_F(1)$ and let R_F be the coordinate ring of the corresponding cubic in Corollary 1.4. Over every field, R_F is normal and Gorenstein, and $Z(A) = R_F$. For type II the surface is smooth and $A = R_{II}$. For types III and IV the surface has a unique singular point, which is split of type A_1 and A_2 , respectively, and*

$$e_1 A e_1 = R_F, \quad A = \text{End}_{R_F}(A e_1)$$

is a noncommutative crepant resolution.

Proof. For II , the presentation in Section 3.6 gives $A = R_{II}$. A singular point would satisfy $xy = yz = -1$ and $xz = 0$, which is impossible over any field.

For types III and IV , we use two principal open sets. On the first, the algebra is a localisation of B_2 or B_3 from Lemma 6.18; on the second, it is a matrix algebra over a smooth surface. We identify their centres and check that the two descriptions agree on the overlap.

For *III*, write r_1, r_2 for the two relations in Equation (3.18) at the trivial bulk class. The identities

$$\begin{aligned} abc - cba &= r_2(a + c + cba) - (a + c + abc)r_1, \\ dab - bad &= dr_2 - r_1d - d(abc - cba)d \end{aligned}$$

give $abc = cba$ and $dab = bad$. Thus $u = ba + ab$ is central. Put $x = -1 - u$ and $c' = -a + cx$. The relations become

$$ba = ue_1, \quad ab = ue_2, \quad dc' = ue_1, \quad c'd = ue_2.$$

After inverting x , the substitution $c = (c' + a)/x$ identifies $A[x^{-1}]$ with $B_2[(1 + u)^{-1}]$ from Lemma 6.18. In the e_1 corner put $v = dc$ and $w = bc$. The two oriented cycles are

$$Y = da = -u + xv, \quad Z = bc' = -u + xw, \quad YZ = u^2.$$

Consequently its centre is the x -localisation of

$$(6.20) \quad R = \mathbf{k}[u, v, w]/(uv + uw + vw + uvw).$$

After inverting u , the arrows a, b, d, c' are invertible between the two vertices. Identifying the vertices by a , the remaining arrows are

$$a = 1, \quad b = u, \quad d = Y, \quad c = v/Y, \quad Y = -u - (1 + u)v.$$

These substitutions and their inverse give $A[u^{-1}] \cong \text{Mat}_2 \mathbf{k}[u^{\pm 1}, v, Y^{-1}]$. This is $\text{Mat}_2 R[u^{-1}]$: the relation $YZ = u^2$ makes Y invertible, and $w = uv/Y$. The change of variables $(x, y, z) = (-1 - u, -1 - v, 1 + w)$ turns Equation (6.20) into $xyz + x + y - z + 2 = 0$.

For *IV*, the substitution $h = c + bd$ and Gaussian elimination give the equivalent relations

$$(6.21) \quad \begin{aligned} r_1 &= ebda - eha + eb + da = 0, \\ r_2 &= adfh + aeh + fh - ad = 0, \\ r_3 &= hf + be = 0. \end{aligned}$$

Put $J = ebd + dfh$, $K = aeb + fha$, $\ell = d - eh + ebd$ and $m = d - eh - dfh$. The identities

$$J = \ell r_2 + r_1 m + e r_3 h, \quad K = a r_1 + r_2 a - a J a$$

show that $J = K = 0$. Together with $r_3 = 0$, they prove that

$$x = -(e_1 + eb) + (e_2 - fh) + (e_3 + be)$$

is central. Set $t = 1 + x$ and $d' = -xd - eh$. The paired products are

$$d'a = te_1, \quad ad' = te_2, \quad hf = te_3, \quad fh = te_2, \quad e(-b) = te_1, \quad (-b)e = te_3.$$

Hence $A[x^{-1}] = B_3[(1 - t)^{-1}]$, with oriented cycle a, h, e ; the inverse substitution is $d = -(d' + eh)/x$. In the e_1 corner put $p = da$ and $v = te_1 - dfb$. Then

$$Y = eha = -xp - t, \quad Z = -d'fb = -xv - t, \quad YZ = t^3.$$

Its centre is therefore the x -localisation of

$$(6.22) \quad R = \mathbf{k}[t, p, v]/(((1 - t)p - t)v - t(p - t)).$$

On $D(t)$ the arrows a, h, e are invertible between vertices. Identifying the vertices by a and e gives

$$a = e = 1, \quad b = -t, \quad d = p, \quad h = Y, \quad f = t/Y, \quad Y = (1 - t)p - t.$$

These formulas give $A[t^{-1}] \cong \text{Mat}_3 \mathbf{k}[t^{\pm 1}, p, Y^{-1}]$. This is $\text{Mat}_3 R[t^{-1}]$, since $YZ = t^3$ and $v = t(p - t)/Y$. Substituting $(x, y, z) = (t - 1, -1 - p, 1 + v)$ in Equation (6.22) gives $xyz + x^2 + 3x + y - z + 3 = 0$.

The two localisations cover: $(u, 1 + u) = (1)$ for *III*, and $(t, 1 - t) = (1)$ for *IV*. The corner identifications and central actions agree on the overlap, giving

$$e_1 A e_1 = Z(A) = R.$$

Finiteness, reflexivity, the identification $A = \text{End}_R(Ae_1)$, and the noncommutative crepant resolution conditions are local on this cover. They follow from Lemma 6.18 on the first chart and the matrix presentation on the second. The second chart is smooth, while the first has a unique singularity with local equation $YZ = u^2$ or $YZ = t^3$. Thus the singularities are split A_1 and A_2 in every characteristic. All

three coordinate rings are hypersurfaces with at most isolated singularities, hence are normal by Serre's criterion and Gorenstein. $\square$

Proof of Corollary 1.4. For the affine types, Lemma 6.11 and Proposition 6.15 give the stated singularities of S_Q , all of which are rational double points, including the additional A_1 point when it occurs. For types *II*, *III* and *IV*, the centres and singularities are computed in Proposition 6.19. In type *II* the surface is smooth and the equivalence follows directly from Theorem 1.1.

In the remaining cases, Propositions 6.12 and 6.19 show that A is a noncommutative crepant resolution of the normal Gorenstein surface $S = \text{Spec } R$. The equations define a surface S_0 over $k_0 = \mathbb{Q}$ or $\mathbb{F}_p$, according to the characteristic of $\mathbf{k}$. Its rational double points admit a projective crepant minimal resolution $\pi_0 : \tilde{S}_0 \rightarrow S_0$, with $R\pi_{0*}\mathcal{O}_{\tilde{S}_0} = \mathcal{O}_{S_0}$ [Lip69, Section 4 and Sections 23–24]. Since k_0 is perfect, $\tilde{S}_0$ is smooth. Base change to $\mathbf{k}$ therefore gives a smooth crepant resolution $\pi : \tilde{S} \rightarrow S$, and flat base change gives

$$R\pi_*\mathcal{O}_{\tilde{S}} = \mathcal{O}_S.$$

This resolution is minimal: crepancy and adjunction exclude exceptional (-1) -curves.

Here R is a finitely generated normal Gorenstein $\mathbf{k}$ -algebra of pure dimension two, and π is projective with fibres of dimension at most one. Thus Van den Bergh's tilting construction [Van04b, Section 3.2, Corollary 3.2.11] gives a noncommutative crepant resolution $A_{\tilde{S}}$ and an equivalence

$$\text{Perf}(\tilde{S}) \simeq \text{Perf}(A_{\tilde{S}}).$$

Since R is an equidimensional normal Gorenstein surface, $A_{\tilde{S}}$ and A are Morita equivalent by [IW13, Theorem 1.5(1)]. Together with Theorem 1.1, this gives

$$\mathcal{W}(M_F) \simeq \text{Perf}(A) \simeq \text{Perf}(\tilde{S}).$$

$\square$

6.4. Equivariant mirror symmetry and Hilbert schemes. For our inward-oriented arms, the vertex relations give $\alpha_{ja} = \prod_{p=1}^a q_{(j,p)}^{-1}$, so we label the roots at the group-algebra point by ζ_j^{-a} .

We consider the specialisation of Etingof's formal orbifold Hecke algebra for E/G [Eti17, Definition 3.3 and Example 3.12] in which each defining polynomial has constant term -1 . We compare it with the completed central corner of our multiplicative preprojective algebra. Put

$$R_{\tau} = \mathbb{C}[[\tau_{ja} : 1 \leq j \leq s, 1 \leq a < d_j]], \quad \tau_{j0} = - \sum_{a=1}^{d_j-1} \tau_{ja}.$$

The Hecke algebra is the (τ) -adic completion of

$$\frac{R_{\tau}\langle \gamma_1^{\pm 1}, \dots, \gamma_s^{\pm 1} \rangle}{\left(\prod_{a=0}^{d_j-1} (\gamma_j - \zeta_j^{-a} e^{\tau_{ja}}) \ (1 \leq j \leq s), \ \gamma_1 \cdots \gamma_s - 1 \right)}.$$

The condition $\sum_a \tau_{ja} = 0$ fixes the constant term of each Hecke polynomial to -1 . These root coordinates are related to the coefficient parameters $\hbar$ of [Hon+26, Definition 6.2] by the formal change of variables in [Hon+26, Lemma 6.3].

Rescale the generators and set

$$(6.23) \quad U_j = e^{-\tau_{j0}} \gamma_j, \quad q_{(j,p)}(\tau) = \zeta_j e^{\tau_{j,p-1} - \tau_{jp}}, \quad q(\tau) = e^{-\sum_j \tau_{j0}}.$$

The roots for U_j are then $\alpha_{ja} = \zeta_j^{-a} e^{\tau_{ja} - \tau_{j0}}$. By (6.4), the completed central corner is

$$e_0 \Lambda_{R_{\tau}}^{\widehat{q(\tau)}(Q)} e_0 \cong \left(\frac{R_{\tau}\langle U_1^{\pm 1}, \dots, U_s^{\pm 1} \rangle}{\left(\prod_{a=0}^{d_j-1} (U_j - \zeta_j^{-a} e^{\tau_{ja} - \tau_{j0}}) \ (1 \leq j \leq s), \ U_1 \cdots U_s - e^{-\sum_j \tau_{j0}} \right)} \right)^{\wedge}.$$

Both completions are with respect to (τ) . The substitution $\gamma_j = e^{\tau_{j0}} U_j$ identifies this algebra with the completed Hecke algebra above.

For arbitrary nonzero multiplicative preprojective parameters, the roots on each arm satisfy

$$\prod_{a=0}^{d_j-1} \alpha_{ja} = \prod_{p=1}^{d_j-1} q_{(j,p)}^{-(d_j-p)}.$$

After extending the coefficient field if necessary, choose

$$\kappa_j^{d_j} = (-1)^{d_j-1} \prod_a \alpha_{ja}, \quad \tilde{q} = \frac{q}{\prod_j \kappa_j}.$$

Since $\sum_j (1 - 1/d_j) = 2$, raising these equations to the powers N/d_j cancels the signs and gives the choice-independent identity

$$(6.24) \quad \tilde{q}^N = q^N \prod_{j=1}^s \prod_{p=1}^{d_j-1} q_{(j,p)}^{N(d_j-p)/d_j}.$$

For the formal family above, we may take $\kappa_j = e^{-\tau_{j0}}$, giving $\tilde{q} = 1$. Thus the family lies on the normalised $\tilde{q} = 1$ locus, while its central parameter q varies with the roots.

More generally, keeping $q_{(j,p)} = \zeta_j$ and allowing arbitrary $q \in \mathbf{k}^\times$, [EOR07, Sections 2 and 4.1] extend Equation (6.6) to

$$(6.25) \quad e_0 \Lambda^q(Q) e_0 \cong \mathcal{A}_{q^N} \rtimes \mu_N,$$

with the cyclic action described there. The presentation isomorphisms are defined over $\mathbb{Z}[q^{\pm 1}]$, so they apply over $\mathbf{k}$. Fullness of the central idempotent follows from Lemma 6.5; Theorem 1.1 then gives Equation (1.8).

The corresponding higher-rank generalised double affine Hecke algebra $H_n(\mathbf{u}, \tau)$ has parameters $u_{ja}, \tau \in \mathbf{k}^\times$, $1 \leq j \leq s$, $1 \leq a \leq d_j$. Its invertible generators $V_1, \dots, V_s, S_1, \dots, S_{n-1}$ satisfy the integral presentation [EGO06, Definition 3.2.1]

$$(6.26) \quad \begin{aligned} (V_1 \cdots V_s) J_n &= 1, & J_n &= S_1 \cdots S_{n-2} S_{n-1}^2 S_{n-2} \cdots S_1, \\ S_a S_{a+1} S_a &= S_{a+1} S_a S_{a+1} \quad (1 \leq a < n-1), \\ [S_a, S_b] &= 0 \quad (|a-b| > 1), & [V_j, S_a] &= 0 \quad (2 \leq a < n), \\ [V_j, S_1 V_j S_1] &= 0, & [V_i, S_1^{-1} V_j S_1] &= 0 \quad (i < j), \\ \prod_{a=1}^{d_j} (V_j - u_{ja}) &= 0 \quad (1 \leq j \leq s), & (S_a - \tau)(S_a + \tau^{-1}) &= 0 \quad (1 \leq a < n). \end{aligned}$$

Here brackets are commutators. For $n = 2$, $J_2 = S_1^2$; for $n = 1$, omit all S_a and their relations and set $J_1 = 1$. The parameter dictionary is

$$u_{ja} = \alpha_{j,a-1} \quad (j < s), \quad u_{sa} = q^{-1} \alpha_{s,a-1}.$$

Thus $V_j = U_j$ for $j < s$ and $V_s = q^{-1} U_s$ identify $H_1(\mathbf{u})$ with $e_0 \Lambda^q(Q) e_0$.

The quantum parameter in [EGO06, Section 5.1] is $\prod_{j,a} u_{ja}^{-N/d_j} = \tilde{q}^N$.

With Λ_n^q as in the introduction, put

$$\mathcal{B}_{Q,n}^q = (\mathcal{B}_Q^q)^{\otimes n} \rtimes S_n, \quad p_n = e_0^{\otimes n}.$$

The dg permutation action uses Koszul signs.

Proposition 6.27. *Over every field and at every multiplicative parameter,*

$$\mathcal{B}_{Q,n}^q \simeq \Lambda_n^q, \quad p_n \Lambda_n^q p_n \cong H_n(\mathbf{u}, 1).$$

If no nonempty consecutive product of parameters on a leg is one, p_n is full. At the unipotent point it is not full.

Proof. Tensor the quasi-isomorphism of Corollary 1.2 n times over the field. It commutes with permutations with Koszul signs. Taking a skew group algebra preserves this quasi-isomorphism: its underlying complex is a finite direct sum, so no exactness of invariants is needed. Since p_n is permutation-invariant, its corner is $(e_0 \Lambda^q(Q) e_0)^{\otimes n} \rtimes S_n$.

At $\tau = 1$, the standard identification [EGO06, Remark 3.2.2] gives

$$H_n(\mathbf{u}, 1) \cong H_1(\mathbf{u})^{\otimes n} \rtimes S_n.$$

It sends V_j to the first tensor factor and S_a to the adjacent transposition; its inverse places the rank-one generators in each tensor position by conjugation. These maps are defined over every field and require no division by $n!$. Together with the rank-one identification above, this proves the corner assertion.

Fullness follows by tensoring an expression $1 = \sum a_\ell e_0 b_\ell$ supplied by Lemma 6.5. At the unipotent point, a vertex-simple module supported on an arm has a nonzero n th tensor power with permutation action. The resulting Λ_n^1 -module is annihilated by p_n , which therefore cannot be full. $\square$

For general interaction parameter τ , an ordinary multiplicative analogue with corner $H_n(\mathbf{u}, \tau)$ was proposed in [EGO06, Introduction]. Constructing an interacting dg model and computing the corresponding wrapped category require further work.

At the group-algebra point let H_n^{grp} be the algebra defined by Equation (6.26) with its polynomial relations replaced by $V_j^{d_j} = 1$ and $S_a^2 = 1$. This definition needs no roots of unity. The same identification gives

$$H_n^{\text{grp}} \cong \mathbf{k}[(\Gamma \rtimes G)^n \rtimes S_n] \cong \mathbf{k}[\Gamma^n] \rtimes G_n.$$

Let $\mathcal{C}$ be a split-closed pretriangulated category with a coherent action of a finite group G and a split-generator F . If $|G| \in \mathbf{k}^\times$, then $\text{Ind } F = \bigoplus_{g \in G} gF$ split-generates $\mathcal{C}^G$. Indeed, induction expresses $\text{Ind}(\text{Forget } E)$ in terms of $\text{Ind } F$, and the coinduction unit divided by $|G|$ splits the counit onto E . In particular,

$$(6.28) \quad \text{Perf}(A)^G \simeq \text{Perf}(A \rtimes G).$$

Here the induced free right module has endomorphism algebra $A \rtimes G$. For commutative A , equivariant descent also identifies these categories with $\text{Perf}([\text{Spec } A/G])$.

Before the torus quotients of Corollary 1.7 and Theorem 1.10 we work out the simplest equivariant example, the reflection of the circle. Let $\text{char } \mathbf{k} \neq 2$ and $B = \mathbf{k}\langle u, v \rangle / (u^2 - 1, v^2 - 1)$. For $x = uv$, we have $uxu = x^{-1}$; conversely $v = ux$ recovers the two-involution presentation. Hence

$$(6.29) \quad B \cong \mathbf{k}[x^{\pm 1}] \rtimes \mu_2, \quad \text{Perf}(B) \simeq \text{Perf}([\mathbb{G}_{m, \mathbf{k}} / \mu_2]).$$

Its centre is $\mathbf{k}[x + x^{-1}]$: commuting with x kills the coefficient of u , and commuting with u imposes inversion invariance. The coarse quotient is an affine line; the stack also records the stabilisers at $x = \pm 1$.

The symplectic model is the reflection $(\theta, p) \mapsto (-\theta, -p)$ on $(T^*S^1, p d\theta)$. Use the vertical cotangent grading, zero equivariant background, and trivial Pin^+ structures in the flat frames. On both a fibre and the zero section choose the reflection lift ϵ with $\epsilon^2 = 1$. A fibre F over a nonfixed point x gives the equivariant object $F \oplus \sigma F$. For the flat metric and Hamiltonian $p^2/2$, every relative path class has a unique geodesic, of index zero. Thus its path-space Floer model has cohomology in degree zero [Abo11; Abo12].

The equivariant algebra has basis $(g, [\gamma])$, where $\gamma : gx \rightarrow x$, and multiplication

$$(g, [\gamma])(h, [\delta]) = (gh, [\gamma \circ (g\delta)]).$$

For a reflection-fixed point x_0 and a path $\eta : x_0 \rightarrow x$, the paths $\eta \circ (g\eta)^{-1}$ split the extension by $\pi_1(S^1)$. This identifies the path group with $\mathbb{Z} \rtimes \mu_2$ and its algebra with B . Degree concentration gives formality. Fibre generation and the preceding induction argument give

$$\mathcal{W}_{\mu_2}(T^*S^1) \simeq \text{Perf}(B) \simeq \text{Perf}([\mathbb{G}_{m, \mathbf{k}} / \mu_2]).$$

For the torus cotangent categories, $\mathcal{W}_{G_n}$ denotes the category of coherent G_n -equivariant objects in $\mathcal{W}(T^*\mathbb{T}_n)$. Here coherent equivariant objects means homotopy equivariant objects, in the sense discussed in [AS21, Section 6.1]. We use the canonical cotangent grading and the equivariant relative spin background $\pi^*T^*\mathbb{T}_n$, where $\pi : T^*\mathbb{T}_n \rightarrow \mathbb{T}_n$ is the projection. For $n = 1$, this gives the conventions for $\mathcal{W}_G(T^*\mathbb{T})$.

Proof of Corollary 1.7 and Theorem 1.10. Give $\mathbb{T}_n$ a product of G -invariant flat metrics. Each factor of $G_n = G^n \rtimes S_n$ preserves orientation, and swapping two real two-dimensional blocks has sign $(-1)^4 = 1$. The cotangent-fibre equivalence [Abo11; Abo12] and its coherent naturality for orientation-preserving base diffeomorphisms [OT24, Theorems 1.0.12 and 6.0.1] therefore apply to this action. These constructions allow the coefficient field $\mathbf{k}$ with the stated brane data [OT24, Choice 1.2.1].

The chosen relative spin background allows the G_n -action on the cotangent fibre to lift to its relative spin structure compatibly with the group law [Hon+26, Example 3.3]. At the fixed origin, put $V = T_0\mathbb{T}_n$. The relative spin representation is $V^* \oplus V \cong V \oplus V$, and the diagonal $SO(V)$ -action lifts to $\text{Spin}(V \oplus V)$: its map on fundamental groups is twice the usual map and hence zero modulo two. This gives coherent lifts of the G_n -action, including the permutations.

Projection to components gives an equivariant quasi-isomorphism

$$C_{-*}(\Omega_0\mathbb{T}_n; \mathbf{k}) \longrightarrow \mathbf{k}[\Gamma^n], \quad g(X^\gamma) = X^{g\gamma},$$

because each based-loop component is contractible. Since $N^n n!$ is invertible, Equation (6.28) applies: the induced fibre generates and has endomorphism algebra $\mathbf{k}[\Gamma^n] \rtimes G_n$. Equivariant descent proves the torus-stack equivalence of Theorem 1.10.

If the ζ_j belong to $\mathbf{k}$, the group corner of Equation (6.6) is full by Lemma 6.5. Proposition 6.27 makes p_n full with corner H_n^{grp} , proving the remaining equivalence. For $n = 1$, Theorem 1.1 at $\mathfrak{b}_{\text{grp}}$ also identifies the category of the smooth surface, which proves Corollary 1.7. $\square$

Recall that $Y_n = \text{Hilb}^n(M_Q)$, and that

$$h_n : Y_n \longrightarrow \text{Sym}^n(M_Q) \longrightarrow \text{Sym}^n(\mathbb{C})$$

is induced by $f : M_Q \rightarrow \mathbb{C}$. Let Ω_n be the induced holomorphic symplectic form and V_n the generator of positive real cotangent scaling.

Proposition 6.30. *The form $\lambda_n = \iota_{V_n} \text{Re } \Omega_n$ makes Y_n a finite-type Liouville manifold of real dimension $4n$, with skeleton the underlying set of $h_n^{-1}(0)$.*

Proof. The Hilbert scheme of a smooth holomorphic symplectic surface is smooth and holomorphic symplectic by the local construction in [Bea83, Section 6, Proposition 5]. Cotangent scaling lifts to it and has weight one on Ω_n . Cartan's formula gives $d\lambda_n = \text{Re } \Omega_n$, with complete Liouville vector field V_n induced by positive real scaling. Hilbert-Chow is projective and f is proper, so h_n is proper. If $b_1, \dots, b_n$ are elementary symmetric coordinates on its base, their weights are $N, 2N, \dots, nN$. Hence

$$\rho_n = \sum_{j=1}^n |b_j \circ h_n|^2, \quad V_n \rho_n = 2N \sum_{j=1}^n j |b_j \circ h_n|^2 > 0 \quad \text{off } h_n^{-1}(0).$$

Every positive sublevel is a compact Liouville domain. Outside the zero fibre, each flow line crosses every positive level exactly once; its forward trajectory escapes every compact set. The zero fibre is compact and invariant. Thus the completion is Y_n and the skeleton is precisely that fibre. $\square$

The Hilbert-Chow map introduces exceptional curves where points collide. In Conjecture 1.12, we assign weight -1 to a collision line and retain weight 1 on classes inherited from M_Q . We explain this choice, construct the class giving these weights, and specify the brane conventions.

Near a configuration with one double point and all other points distinct, Hilbert-Chow is locally an A_1 resolution times a smooth factor. The normalised Hecke generator $\tau^{-1}S_a$ has roots $1, -\tau^{-2}$. Comparing these with the local A_1 leg roots $1, t_{\text{coll}}^{-1}$ gives

$$t_{\text{coll}} = -\tau^2.$$

Thus the tensor-product case $\tau = 1$ suggests collision weight -1 , whereas collision weight 1 would give $\tau^2 = -1$. The same sign occurs at zero orbifold parameter in the Hilbert-Chow crepant resolution correspondence [BG09, Theorem 3.11].

Recall that the tautological rank- n bundle $\mathcal{V}_n$ on Y_n has fibre $H^0(Z, \mathcal{O}_Z)$ at a length- n subscheme Z . The collision class of Equation (1.11) is

$$\mathbf{b}_{\text{coll}}(\gamma) = (-1)^{\langle c_1(\det \mathcal{V}_n), \gamma \rangle}, \quad \gamma \in H_2(Y_n; \mathbb{Z}).$$

Its values can be computed directly. For $n \geq 2$, fix $n - 2$ distinct points and a further point x . The length-two subschemes supported at x form a collision line $\ell = \mathbb{P}(T_x M_Q)$. Along this line, $\mathcal{V}_n$ is the sum of $n - 2$ trivial summands and the rank-two bundle fitting into the evaluation sequence

$$0 \longrightarrow \mathcal{O}_\ell(1) \longrightarrow \mathcal{V}_2|_\ell \longrightarrow \mathcal{O}_\ell \longrightarrow 0.$$

Hence $\deg(\det \mathcal{V}_n|_\ell) = 1$ and $\mathbf{b}_{\text{coll}}([\ell]) = -1$.

To represent a class coming from $H_2(M_Q; \mathbb{Z})$, choose $n - 1$ distinct fixed points disjoint from a surface representing the given two-cycle in M_Q , and let the remaining point move along that surface. The tautological bundle is trivial along this cycle: a basis is given by the functions that are 1 at one labelled point and 0 at the others. Thus $c_1(\det \mathcal{V}_n)$ evaluates to zero, and the collision weight is 1.

In the category $\mathcal{W}_{\text{b}_{\text{coll}}}(Y_n)$, we implement these signs through the relative spin background

$$\beta_n = c_1(\det \mathcal{V}_n) \bmod 2 \in H^2(Y_n; \mathbb{Z}/2).$$

An oriented brane L carries a grading, a local system, and a spin structure on $TL \oplus (\det \mathcal{V}_n)_\mathbb{R}|_L$; the obstruction condition is $w_2(TL) = \beta_n|_L$. For the grading, choose a compatible almost hyperkähler structure extending Ω_n and use the complex volume form of the rotated almost complex structure compatible with $\text{Re } \Omega_n$. Where comparison with background zero is defined, the change of orientations contributes $(-1)^{\langle \beta_n, \gamma \rangle} = \mathbf{b}_{\text{coll}}(\gamma)$ for a compatibly capped class γ [Abo12, Section 2.2 and Lemma A.5]. We therefore insert no additional scalar copy of this sign.

This background admits the compact Lagrangians expected from the surface mirror. For a core sphere $C \cong \mathbb{P}^1$, put $L = \text{Hilb}^n(C) \cong \mathbb{P}^n$, which lies in the compact core. The form Ω_n vanishes on the dense locus of distinct points of C , hence on all of L , so L is Lagrangian. It is exact because $H^1(L; \mathbb{R}) = 0$. The universal divisor on $\mathbb{P}^1 \times \mathbb{P}^n$ has bidegree $(n, 1)$; pushing forward its defining sequence gives $\det \mathcal{V}_n|_L \cong \mathcal{O}(1 - n)$. Thus, for the mod-two hyperplane class h ,

$$\beta_n|_L = (n - 1)h = (n + 1)h = w_2(T\mathbb{P}^n).$$

Consequently L is relatively spin for β_n , including even n , when it is not spin for background zero.

There is also a check on endomorphisms. When $n! \in \mathbf{k}^\times$, the S_n -equivariant external tensor power of the mirror of C has endomorphism cohomology

$$(H^*(S^2; \mathbf{k})^{\otimes n})^{S_n} \cong \mathbf{k}[z]/(z^{n+1}), \quad |z| = 2.$$

Indeed, if x_i is the degree-two class in the i th factor, then $z = x_1 + \cdots + x_n$ satisfies $z^j = j!e_j(x_1, \dots, x_n)$, where e_j is the j th elementary symmetric polynomial. The classes $e_0, \dots, e_n$ form a basis of the invariant subspace. This agrees with the self-Floer cohomology $H^*(\mathbb{P}^n; \mathbf{k})$ of L .

We now identify the algebraic category on the right-hand side of Conjecture 1.12.

Proposition 6.31. *Under the field hypotheses of Conjecture 1.12,*

$$\text{Perf}(\text{Hilb}^n \tilde{S}_Q) \simeq \text{Perf}(\Lambda_n^1).$$

Proof. Put $Y = \tilde{S}_Q$ and $A = \Lambda^1(Q)$. The tilting construction and Morita equivalence used in the proof of Corollary 1.4 give a tilting bundle $\mathcal{T}$ on Y with endomorphism algebra A . Its external power $\mathcal{T}^{\boxtimes n}$ is a tilting generator on Y^n , with endomorphism algebra $A^{\otimes n}$ and the natural permutation linearisation. Since $n! \in \mathbf{k}^\times$, the equivariant-generation argument above gives

$$\text{Perf}([Y^n/S_n]) \simeq \text{Perf}(A^{\otimes n} \rtimes S_n) = \text{Perf}(\Lambda_n^1).$$

The surface Y is smooth and quasiprojective, being projective over the affine surface S_Q , and $\text{char } \mathbf{k}$ is zero or greater than n . The Hilbert-scheme derived McKay correspondence [Gro14, Theorem 3.11 and Corollary 3.12] therefore gives

$$\text{Perf}(\text{Hilb}^n Y) \simeq \text{Perf}([Y^n/S_n]).$$

The universal-kernel construction is defined over $\mathbf{k}$, and the equivalence can be checked after faithfully flat extension to $\bar{\mathbf{k}}$. Here smoothness and tameness allow passage to perfect categories. Combining the equivalences proves the proposition. $\square$

This identifies the algebraic mirror. A Floer comparison across the Hilbert-Chow exceptional locus with the specified collision background remains the assertion of Conjecture 1.12.

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