HOMOLOGICAL MIRROR SYMMETRY FOR MILNOR FIBERS VIA MODULI OF A_∞-STRUCTURES

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Abstract. We show that the base spaces of the semiuniversal unfoldings of some weighted homogeneous singularities can be identified with moduli spaces of A_∞-structures on the trivial extension algebras of the endomorphism algebras of the tilting objects. The same algebras also appear in the Fukaya categories of their mirrors. Based on these identifications, we discuss applications to homological mirror symmetry for Milnor fibers, and give a proof of homological mirror symmetry for an n-dimensional affine hypersurface of degree $n+2$ and the double cover of the n-dimensional affine space branched along a degree $2n+2$ hypersurface. Along the way, we also give a proof of a conjecture of Seidel from [69] which may be of independent interest.

1. Introduction

1.1. **Moduli of elliptic curves.** Our basic starting point is an algebraic variety with an isolated singularity admitting a $\mathbb{G}_\mathrm{m}$-action. The primordial example is the cusp singularity defined by

$$\left\{(x,y) \in \mathbb{A}^2 \;\middle|\; \mathbf{w}(x,y) := x^3 + y^2 = 0\right\}. \tag{1.1}$$

The main construction that we study in this paper originates from [46], where the case of the cusp singularity was studied in detail. We recall this construction in order to ease the reader to our topic before discussing higher-dimensional singularities with a $\mathbb{G}_\mathrm{m}$-action.

The cuspidal curve (1.1) has a $\mathbb{G}_\mathrm{m}$-action given by $t \cdot (x,y) = (t^2 x, t^3 y)$. Thus the coordinate ring gets a grading with $\deg(x) = 2$ and $\deg(y) = 3$. It can be compactified to the projective cone

$$\{[x:y:z] \in \mathbb{P}(2,3,1) \mid \mathbf{w}(x,y) = 0\} \tag{1.2}$$

by adding one point.

The semiuniversal unfolding of $\mathbf{w}$ is given by

$$\widetilde{\mathbf{w}}(x,y;u_4,u_6) := x^3 + y^2 + u_4 x + u_6, \tag{1.3}$$

whose homogenization

$$\mathbf{W}(x,y,z;u_4,u_6) := x^3 + y^2 + u_4 x z^4 + u_6 z^6 \tag{1.4}$$

defines the Weierstrass family $\pi_{\mathcal{Y}} \colon \mathcal{Y} \to U \coloneqq \operatorname{Spec} \mathbf{k}[u_4, u_6]$ of curves in $\mathbb{P}(2,3,1)$. Each curve $Y_u \coloneqq \pi_{\mathcal{Y}}^{-1}(u)$ is of arithmetic genus 1 and comes with a point $p \coloneqq \{z = 0\}$ at infinity and a section

$$\Omega_u \coloneqq \operatorname{Res} \frac{z dx \wedge dy}{\mathbf{W}(x, y, z, u_4, u_6)} \tag{1.5}$$

of the dualizing sheaf, which is given by $dx/\mathbf{W}_y = -dy/\mathbf{W}_x$ on the affine part. The $\mathbb{G}_\mathrm{m}$-action extends to the compactified family by

$$\begin{aligned} t \cdot ([x : y : z]; u_4, u_6) &= ([t^2 x : t^3 y : z]; t^4 u_4, t^6 u_6) && (1.6)\\ &= ([x : y : t^{-1} z]; t^4 u_4, t^6 u_6), && (1.7) \end{aligned}$$

which preserves the section $z = 0$ and satisfies

$$t^*(\Omega_{t \cdot u}) = t^{-1} \Omega_u. \tag{1.8}$$

The curves Y_u are elliptic curves outside the discriminant

$$\Delta \coloneqq \left\{ (u_4, u_6) \in U \;\middle|\; 4u_4^3 - 27u_6^2 = 0 \right\}. \tag{1.9}$$

If $u \in \Delta \setminus \mathbf{0}$, then Y_u is a rational curve with a single ordinary double point. Note that all curves above a $\mathbb{G}_\mathrm{m}$-orbit are isomorphic.

The base space U can be identified with the moduli space of triples (Y, p, Ω) consisting of a reduced connected curve Y of arithmetic genus 1, a smooth marked point p on Y such that $h^0(\mathcal{O}_Y(p)) = 1$ and $\mathcal{O}_Y(p)$ is ample, and a non-zero section Ω of the dualizing sheaf of Y (see [48, Theorem 1.4.2]). Furthermore, we have an isomorphism

$$\overline{\mathcal{M}}_{1,1} \cong [(U \setminus \mathbf{0})/\mathbb{G}_\mathrm{m}] \quad (\cong \mathbb{P}(4,6)) \tag{1.10}$$

with the moduli stack of stable curves of genus one with one marked point.

1.2. **Moduli of A_∞-structures.** The condition that $\mathcal{O}_{Y_u}(p)$ is ample is equivalent to

$$\mathcal{S}_u \coloneqq \mathcal{O}_{Y_u} \oplus \mathcal{O}_p \tag{1.11}$$

being a generator of the perfect derived category $\operatorname{perf} Y_u$. On the other hand, the fact that $h^0(\mathcal{O}_{Y_u}(p)) = 1$ implies that the isomorphism class of the Yoneda algebra

$$A \coloneqq \operatorname{End}(\mathcal{S}_u) \tag{1.12}$$

as a graded algebra is independent of $u \in U$. Indeed, it is easy to show that for any u, there is a canonical isomorphism (where we use the fixed basis Ω_u of $H^0(\omega_{Y_u})$) between A and the degree one trivial extension algebra of the path algebra of the A_2-quiver. More concretely, this is given by the quiver with relations given in Figure 1.1.

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, bend left, "u"] & \overset{2}{\bullet} \arrow[l, bend left, "v"] \end{tikzcd} \qquad |u| = 0,\ |v| = 1,\ uvu = vuv = 0$$

FIGURE 1.1. Quiver algebra description of A

Thus, considering the algebra A results in a dramatic loss of information hidden in perf Y_u, even though $\mathcal{S}_u$ is a generator. This is, of course, no surprise as we have forgotten to derive.

Recall that an A_∞-algebra $\mathcal{A}$ over $\mathbf{k}$ is a graded $\mathbf{k}$-module with a collection $\left(\mu^d\right)_{d=1}^\infty$ of $\mathbf{k}$-linear maps $\mu^d\colon \mathcal{A}^{\otimes d} \to \mathcal{A}[2-d]$ satisfying the A_∞-associativity equations

$$\sum_{m,n}(-1)^{|a_1|+\ldots+|a_n|-n}\mu^{d-m+1}(a_d,\ldots,a_{n+m+1},\mu^m(a_{n+m},\ldots,a_{n+1}),a_n,\ldots a_1)=0. \tag{1.13}$$

In particular, $\mu^1\colon \mathcal{A}\to\mathcal{A}[1]$ is a differential, i.e. $\mu^1\circ\mu^1=0$, and the product

$$a_2\cdot a_1=(-1)^{|a_1|}\mu^2(a_2,a_1) \tag{1.14}$$

on $\mathcal{A}$ is associative up to homotopy.

A *minimal* A_∞*-structure* on a graded associative $\mathbf{k}$-algebra A is an A_∞-structure $(\mu^k)_{k=1}^\infty$ on the graded vector space underlying A such that $\mu^1=0$ and μ^2 coincides with the given product on A. It is said to be *formal* if $\mu^k=0$ for $k>2$.

Recall that the Hochschild cochain complex of a graded algebra A has a bigrading, where $\mathrm{CC}^{r+s}(A)_s$ consists of maps $A^{\otimes r}\to A[s]$. The space of first-order deformations of A as a graded algebra is given by $\mathrm{HH}^2(A)_0$, and deformations to minimal A_∞-structures on A without changing μ^2 is controlled by $\mathrm{HH}^2(A)_{<0}\coloneqq\bigoplus_{i=1}^\infty\mathrm{HH}^2(A)_{-i}$. Moreover, if $\mathrm{HH}^1(A)_{<0}$ vanishes, then [63, Corollary 3.2.5] shows that the functor sending a $\mathbf{k}$-algebra R to the set of gauge equivalence classes of minimal A_∞-structures on $A\otimes R$ is represented by an affine scheme $\mathcal{U}_\infty(A)$, which is of finite type if $\dim\mathrm{HH}^2(A)_{<0}<\infty$. There is a natural $\mathbb{G}_\mathrm{m}$-action on $\mathcal{U}_\infty(A)$ given by

$$\mathbb{G}_\mathrm{m}\ni t\colon\ \left(\mu^d\right)_{d=2}^\infty\mapsto\left(t^{d-2}\mu^d\right)_{d=2}^\infty, \tag{1.15}$$

and the formal A_∞ structure on A is the fixed point of this action.

Returning back to the Weierstrass family, as explained in [47], the natural dg enhancement $\mathrm{end}(\mathcal{S})$ of $\mathrm{End}(\mathcal{S})$ gives a family $\mathcal{A}$ of minimal A_∞-structures on A over U, and hence a morphism

$$U\to\mathcal{U}_\infty(A). \tag{1.16}$$

We recall the following theorem from [47]. For simplicity, we state it over a field $\mathbf{k}$ with $\mathrm{char}\,\mathbf{k}\neq 2,3$, see [47] for a more general statement.

Theorem 1.1. *If* $\mathrm{char}\,\mathbf{k}\neq 2,3$*, then* (1.16) *is a* $\mathbb{G}_\mathrm{m}$*-equivariant isomorphism, sending the cuspidal curve* Y_0 *to the formal* A_∞*-structure on* A*.*

There are two main ingredients that enter in the proof of this result:

(i) The formality of the A_∞-algebra $\mathcal{A}_0$ for the cuspidal curve Y_0.

(ii) One has $\mathrm{HH}^1(A)_{<0}=0$, so that $\mathcal{U}_\infty(A)$ is an affine scheme, and

$$\mathrm{HH}^2(A)_{<0}=\mathbf{k}(4)\oplus\mathbf{k}(6), \tag{1.17}$$

so that (1.16) induces an isomorphism on tangent spaces at the fixed points of the $\mathbb{G}_\mathrm{m}$-action.

The Hochschild cohomology computation is done in two different ways in [46] and [47]. We will give yet another way in Section 3.4 below.

To elaborate on (i), first one shows the existence of a chain level $\mathbb{G}_{\mathrm{m}}$-action by taking the Čech complex with respect to a $\mathbb{G}_{\mathrm{m}}$-invariant affine cover. This gives a dg model for $\mathcal{A}_0$. Then, one arranges a $\mathbb{G}_{\mathrm{m}}$-equivariant homotopy to a minimal A_∞-structure, which follows from the fact that one can choose chain level representatives of a basis of $\operatorname{End}(\mathcal{S}_0)$ in such a way that each of them is in a one-dimensional representation of $\mathbb{G}_{\mathrm{m}}$. Finally, to deduce formality, one shows that the weight of the $\mathbb{G}_{\mathrm{m}}$-action on $\operatorname{End}(\mathcal{S}_0)$ agrees with the cohomological grading. But μ^d lowers the cohomological degree by $d-2$, so any $\mathbb{G}_{\mathrm{m}}$-equivariant A_∞-structure must have vanishing μ^d for $d \neq 2$.

Other examples of the above construction were subsequently studied in [63, 48], but all of these work with examples in dimension one. In this paper, we begin to explore higher dimensions.

1.3. **Application to homological mirror symmetry.** Let $\check{V}$ be a once-punctured torus viewed as a Weistein manifold, and

$$Z := \left\{ [x:y:z] \in \mathbb{P}(2,3,1) \;\middle|\; x^3 + y^2 + xyz = 0 \right\} \tag{1.18}$$

be a rational curve with a single ordinary double point. Theorem 1.1 was obtained in [47] as a tool for proving a quasi-equivalence

$$\mathcal{F}\left(\check{V}\right) \simeq \operatorname{perf} Z \tag{1.19}$$

of pretriangulated A_∞-categories over $\mathbb{Z}$ of the split-closed derived Fukaya category of compact exact Lagrangians in $\check{V}$ and the perfect derived category of Z. The strategy is first to identify generators on both sides, and then match their endomorphism algebras as A_∞-algebras. It is often difficult to explicitly compute such A_∞-algebras, but even if one does, finding a quasi-isomorphism between two different chain models is usually a hard task. The computation of cohomology level structures (and matching them) is much easier, and knowing the moduli of A_∞-structures allows one to appeal to indirect methods to conclude the proof of the existence of a chain level isomorphism. Such a strategy was applied also for proving homological mirror symmetry in a number of other cases in dimension one. Namely, in [48] a class of curve singularities $C_{1,n}$ for $n \geq 1$ were considered, where $C_{1,1}$ is the cuspidal curve, $C_{1,2}$ is tacnodal curve given by the equation $y^2 = yx^2$, and $C_{1,n}$ is the elliptic n-fold singularity given by n lines in $\mathbb{A}^{n-1}$. These are all the Gorenstein singularities of arithmetic genus one [73, Appendix A]. Carrying out the above strategy has led to a proof of homological mirror symmetry for n-punctured tori [49].

The equivalence (1.19) is an instance of homological mirror symmetry at the large volume limit. The equivalence is known to extend to a formal neighborhood of this limit to give an equivalence

$$\mathcal{F}\left(\check{Y}\right) \simeq \operatorname{perf} \hat{\mathcal{Y}} \tag{1.20}$$

over $\mathbb{Z}[\![q]\!]$ where $\check{Y}$ is the compactification of $\check{V}$ and $\hat{\mathcal{Y}}$ is the Tate elliptic curve, a formal neighborhood of the nodal curve Z (see [47] for a proof). A general strategy for proving homological mirror symmetry as in (1.20) introduced in [69] is to view the categories in (1.20) as deformations of the categories given in (1.19). Hence, in this context deducing homological mirror symmetry for the compact manifold $\check{Y}$ from homological mirror symmetry for the Weinstein manifold $\check{V}$ ultimately reduces to a problem in deformation theory.

1.4. **New results and a general conjectural picture.** In this paper, we lay out a program that aims to extend the above results to higher dimensions, leading to new homological mirror symmetry conjectures for higher-dimensional Calabi–Yau manifolds at the large volume limit and in its formal neighborhood. It is based on the relation between homological mirror symmetry for Calabi–Yau manifolds and homological mirror symmetry for singularities, which goes back to [44, 56, 78].

A weighted homogeneous polynomial $\mathbf{w} \in \mathbb{C}[x_1, \ldots, x_n]$ with an isolated critical point at the origin is *invertible* if there is an integer matrix $A = (a_{ij})_{i,j=1}^n$ with non-zero determinant such that

$$\mathbf{w} = \sum_{i=1}^n \prod_{j=1}^n x_j^{a_{ij}}. \tag{1.21}$$

The corresponding weight system $(d_1, \ldots, d_n; h)$ satisfying $\gcd(d_1, \ldots, d_n, h) = 1$ is determined uniquely.

The *transpose* of $\mathbf{w}$ is defined in [9] as

$$\check{\mathbf{w}} = \sum_{i=1}^n \prod_{j=1}^n x_j^{a_{ji}}, \tag{1.22}$$

whose exponent matrix $\check{A}$ is the transpose matrix of A. We write $\left(\check{d}_1, \check{d}_2, \ldots, \check{d}_n; \check{h}\right)$ for the weight system associated with $\check{\mathbf{w}}$.

The group

$$\Gamma_{\mathbf{w}} := \{(t_1, \ldots, t_n) \in (\mathbb{G}_\mathrm{m})^n \mid t_1^{a_{11}} \cdots t_n^{a_{1n}} = \cdots = t_1^{a_{n1}} \cdots t_n^{a_{nn}}\} \tag{1.23}$$

acts naturally on $\mathbb{A}^n$. One has a homomorphism $\phi\colon \mathbb{G}_\mathrm{m} \to \Gamma_{\mathbf{w}}$ sending $t \in \mathbb{G}_\mathrm{m}$ to $\left(t^{d_1}, \ldots, t^{d_n}\right) \in \Gamma_{\mathbf{w}}$. Let $\operatorname{mf}(\mathbb{A}^n, \Gamma_{\mathbf{w}}, \mathbf{w})$ be the idempotent completion of the dg category of $\Gamma_{\mathbf{w}}$-equivariant matrix factorizations of $\mathbf{w}$.

Homological mirror symmetry conjecture for invertible polynomials is the following:

Conjecture 1.2. For any invertible polynomial $\mathbf{w}$, one has a quasi-equivalence

$$\operatorname{mf}(\mathbb{A}^n, \Gamma_{\mathbf{w}}, \mathbf{w}) \simeq \mathcal{W}(\check{\mathbf{w}}). \tag{1.24}$$

Here $\mathcal{W}(\check{\mathbf{w}})$ is the partially wrapped Fukaya category of $\check{\mathbf{w}}$, which is quasi-equivalent to the Fukaya–Seidel category of (a Morsification of) $\check{\mathbf{w}}$. Conjecture 1.2 is stated for Brieskorn–Pham singularities in 3 variables in [77], for polynomials in 3 variables associated with a regular system of weights of dual type in the sense of Saito in [75] (with a prototype appearing earlier in [74]), and for invertible polynomials in 3 variables in [20]. It is proved for $n = 2$ in [35], and for Sebastiani–Thom sums of polynomials of type A and D in [26, 27].

The conjecture that $\operatorname{mf}(\mathbb{A}^n, \mathbf{w}, \Gamma_{\mathbf{w}})$ has a full exceptional collection, which is implied by Conjecture 1.2, is stated in [37, Conjecture 1.4], and proved in [24].

The following conjecture is stated for $n = 3$ in [20]:

Conjecture 1.3. For any invertible polynomial $\mathbf{w}$, the category $\operatorname{mf}(\mathbb{A}^n, \mathbf{w}, \Gamma_{\mathbf{w}})$ has a tilting object.

A slightly stronger conjecture that $\operatorname{mf}(\mathbb{A}^n, \mathbf{w}, \Gamma_{\mathbf{w}})$ has a full strong exceptional collection, stated in [37, Conjecture 1.2], is known for $n \le 3$ by [45], and for a class of invertible polynomials called of chain type by [37].

In a similar vein, one may ask whether for an invertible polynomial $\mathbf{w}$, the derived category of coherent sheaves on the stack

$$\mathfrak{X}_{\mathbf{w}} := [((\operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]/(\mathbf{w})) \setminus \mathbf{0})/\Gamma_{\mathbf{w}}] \tag{1.25}$$

has a tilting object. If $\mathbf{w}$ is of Brieskorn–Pham type, then $\mathfrak{X}_{\mathbf{w}}$ has a full strong exceptional collection of line bundles [38]. Note that $\mathfrak{X}_{\mathbf{w}}$ is always a smooth proper rational stack of Picard number one. It is known that for a smooth proper toric Deligne–Mumford stack of Picard number at most two, there exists a full strong exceptional collection of line bundles [11]. On the other hand, the stack $\mathfrak{X}_{\mathbf{w}}$ does not have a full strong exceptional collection of line bundles in general — a counterexample was given in [25].

We write (the Liouville completion of) the Milnor fiber of $\check{\mathbf{w}}$ as

$$\check{V}_{\check{\mathbf{w}}} := \check{\mathbf{w}}^{-1}(1) = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid \check{\mathbf{w}} = 1\}. \tag{1.26}$$

The main conjecture that we introduce in this paper is the following:

Conjecture 1.4. For any invertible polynomial $\mathbf{w}$, one has a quasi-equivalence

$$\operatorname{mf}\left(\mathbb{A}^{n+1}, \mathbf{w} + x_0 \cdots x_n, \Gamma_{\mathbf{w}}\right) \simeq \mathcal{W}\left(\check{V}_{\check{\mathbf{w}}}\right). \tag{1.27}$$

The affine variety $\check{V}_{\check{\mathbf{w}}}$ is log Fano, log Calabi–Yau, or of log general type depending on whether $\check{d}_0 := \check{h} - \sum_{i=1}^n \check{d}_i$ is negative, zero, or positive respectively. In dimension 2, the log Fano case corresponds to simple singularities which have a well-known ADE classification. Fukaya categories of their Milnor fiber are identified in [22, 23] with module categories of the corresponding (derived) preprojective algebras, and Conjecture 1.4 is proved in [51]. The log Calabi–Yau case follows from homological mirror symmetry for the wrapped Fukaya categories of the Milnor fibers of hypersurface cusp singularites proved in [41] by a variation of Orlov's theorem. In this paper, we almost exclusively concentrate on the case of log general type. See e.g. [79, Section 2] for more on this trichotomy in dimension 2.

In the log general type case, Orlov's theorem gives an equivalence of the left hand side of (1.27) with the derived category $\operatorname{coh} \mathcal{Z}_{\mathbf{w}}$ of coherent sheaves on

$$\mathcal{Z}_{\mathbf{w}} := [(\operatorname{Spec} \mathbb{C}[x_0, \dots, x_n]/(\mathbf{w} + x_0 x_1 \cdots x_n) \setminus \mathbf{0})/\Gamma_{\mathbf{w}}], \tag{1.28}$$

where the action of $\Gamma_{\mathbf{w}}$ comes from the identification

$$\Gamma_{\mathbf{w}} \cong \left\{ (t_0, t_1, \dots, t_n) \in (\mathbb{G}_{\mathrm{m}})^{n+1} \;\middle|\; t_1^{a_{11}} \cdots t_n^{a_{1n}} = \cdots = t_1^{a_{n1}} \cdots t_n^{a_{nn}} = t_0 t_1 \dots t_n \right\}. \tag{1.29}$$

Recall that an object X of $\operatorname{coh} \mathcal{Z}$ on a proper stack $\mathcal{Z}$ is perfect if and only if it is Ext-*finite*, i.e., the dimension of $\bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}^i(X, Y)$ is finite for any object Y. It is reasonable to expect that the full subcategory of the wrapped Fukaya category $\mathcal{W}\left(\check{V}_{\check{\mathbf{w}}}\right)$ consisting of Ext-finite objects is equivalent to the compact Fukaya category $\mathcal{F}\left(\check{V}_{\check{\mathbf{w}}}\right)$, so that Conjecture 1.4 would imply

$$\operatorname{perf} \mathcal{Z}_{\mathbf{w}} \simeq \mathcal{F}\left(\check{V}_{\check{\mathbf{w}}}\right). \tag{1.30}$$

The first instance of an equivalence of this form was obtained in [46] for $\mathbf{w} = x_1^3 + x_2^2$ and recently Habermann proved this equivalence when $\mathbf{w}$ is an arbitrary invertible polynomial of two variables [36].

The way that the wrapped Floer cohomology can be infinite depends on the sign of $\check{d}_0$; it can be infinite in the negative cohomological degrees with finite graded pieces in the log Fano case, infinite in finite cohomological degrees in the log Calabi–Yau case, and infinite in the positive cohomological degrees with finite graded pieces in the log general type case. In the log Fano and log Calabi–Yau case, the quotient $\mathcal{W}(\check{V}_{\check{\mathbf{w}}})/\mathcal{F}(\check{V}_{\check{\mathbf{w}}})$ are generalized cluster categories (see e.g. [43, Section 9] and references therein). In the log general type case, we make the following conjecture, which is a compact analog of [5, Conjecture 1.2]:

Conjecture 1.5. Let $\check{X}$ be a smooth ample divisor in a Calabi–Yau manifold $\check{Y}$ and $\check{V} := \check{Y} \setminus \check{X}$ be the complement. Then one has a quasi-equivalence

$$\mathcal{W}\left(\check{V}\right)/\mathcal{F}\left(\check{V}\right) \simeq \mathcal{F}(\check{X}). \tag{1.31}$$

Conjecture 1.5 reduces homological mirror symmetry for the manifold $\check{X}$ of general type to that for the affine manifold $\check{V}$. If $\check{d}_0 = 1$, then $\check{V}_{\check{\mathbf{w}}}$ admits a compactification to a Calabi–Yau orbifold $\check{Y}_{\check{\mathbf{w}}}$ such that $\check{X}_{\check{\mathbf{w}}} := \check{Y}_{\check{\mathbf{w}}} \setminus \check{V}_{\check{\mathbf{w}}}$ is a smooth ample divisor, and Conjecture 1.4 together with Conjecture 1.5 implies

$$D^b_{\mathrm{sing}}(\mathcal{Z}_{\mathbf{w}}) \simeq \mathcal{F}\left(\check{X}_{\check{\mathbf{w}}}\right). \tag{1.32}$$

Recall that the *degree d trivial extension algebra* (also known as the *Frobenius completion of degree d*) of a finite-dimensional $\mathbf{k}$-algebra A^0 has $A^0 \oplus \operatorname{Hom}_{\mathbf{k}}(A^0, \mathbf{k})[-d]$ as the underlying graded vector space, and the multiplication is given by

$$(a, f) \cdot (b, g) = (ab, ag + fb). \tag{1.33}$$

Theorem 1.6. *Let* $\mathbf{w} \in \mathbf{k}[x_1, \ldots, x_n]$ *be a weighted homogeneous polynomial and* Γ *be a subgroup of* $\Gamma_{\mathbf{w}}$ *containing* $\phi(\mathbb{G}_{\mathrm{m}})$ *as a subgroup of finite index. Assume that*

(1) $\mathbf{w}$ *has an isolated critical point at the origin,*
(2) d_0 *defined by* (2.13) *is positive,*
(3) $\mathrm{mf}(\mathbb{A}^n, \mathbf{w}, \Gamma)$ *has a tilting object* E*, and*
(4) the pair $(\mathbf{w}, \Gamma)$ *does not have twisted deformations in the sense of Definition 3.3.*

Let A^0 *be the endomorphism algebra of the tilting object* E *and* A *be the degree* $n-1$ *trivial extension algebra of* A^0*. Then there is a* $\mathbb{G}_{\mathrm{m}}$*-equivariant isomorphism*

$$U \xrightarrow{\sim} \mathcal{U}_\infty(A) \tag{1.34}$$

of affine schemes from the positive part of the base space of the semiuniversal unfolding of $\mathbf{w}$ *to the moduli space of* A_∞*-structures on* A *sending the origin* $0 \in U$ *to the formal* A_∞*-structure on* A*.*

Although the existence of a tilting object and the non-existence of twisted deformations are restrictive assumptions on a pair $(\mathbf{w}, \Gamma)$, there are many interesting examples where both of them holds. Conjecture 1.3 states that the former holds when $\mathbf{w}$ is an invertible polynomial and $\Gamma = \Gamma_{\mathbf{w}}$. We will see examples where the latter holds in Sections 3.2–3.5.

To apply Theorem 1.6 to homological mirror symmetry, one needs to find a generator of the Fukaya category whose Yoneda algebra is isomorphic to A. When $\mathbf{w}$ is a Sebastiani–Thom sum of polynomials of type A or D, i.e., a decoupled sum of polynomials of the form x^{n+1} or $x^2y + y^{n-1}$, homological mirror symmetry for singularities [26, 27] gives a collection $(S_i)_{i=1}^{\mu}$ of Lagrangian spheres in $\check{V}_{\check{\mathbf{w}}}$ such that the Yoneda algebra of their direct sum $S = \bigoplus_{i=1}^{\mu} S_i$ in the Fukaya category $\mathcal{F}\left(\check{V}_{\check{\mathbf{w}}}\right)$ is isomorphic to the trivial extension algebra of the tensor product of the path algebras of the Dynkin quivers of the corresponding types. For example, the algebra A^0 in the case of $x^4 + y^4 + z^4$ is the path algebra of the quiver in Figure 1.2, with the relations that the composition of arrows along the sides of each small square commutes.

By combining the proof of a special case of [69, Conjecture 4] which states, under assumptions satisfied for $\check{V}_{\check{\mathbf{w}}}$, an isomorphism

$$\mathrm{SH}^*\left(\check{V}_{\check{\mathbf{w}}}\right) \simeq \mathrm{HH}^*\left(\mathcal{F}\left(\check{V}_{\check{\mathbf{w}}}\right)\right) \tag{1.35}$$

of the symplectic cohomology and the Hochschild cohomology of the Fukaya category, with the computation of the symplectic cohomology $\mathrm{SH}^*\left(\check{V}_{\check{\mathbf{w}}}\right)$ using a spectral sequence, originally due to McLean [57] and full detail of which was written later by Ganatra and Pomerleano [33] (who in addition proved that this spectral sequence is multiplicative),

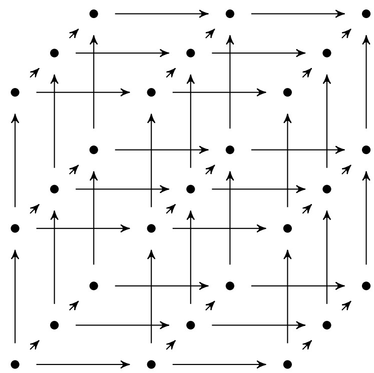

FIGURE 1.2. A quiver for $\mathbf{w} = x^4 + y^4 + z^4$

we show that the Yoneda A_∞-algebra $\mathcal{A}$ of the generator of the Fukaya category is not formal. Hence $\mathcal{A}$ can be identified with a point in the moduli space

$$\mathcal{M}_\infty(A) \coloneqq [(\mathcal{U}_\infty(A) \setminus \mathbf{0})/\mathbb{G}_{\mathrm{m}}] \tag{1.36}$$

of non-formal A_∞-structures. Conjecture 1.4 identifies exactly which point this is, and in order to prove it, one has to distinguish points on $\mathcal{M}_\infty(A)$ by computable invariants of $\mathcal{F}\left(\check{V}_{\check{\mathbf{w}}}\right)$. For $\mathbf{w} = x_1^{n+1} + x_2^{n+1} + \cdots + x_n^{n+1}$ and $\mathbf{w} = x_1^2 + x_2^{2n} + \cdots + x_n^{2n}$, this space is one-dimensional, and we can prove Conjecture 1.4 in this case:

Theorem 1.7. *(i) Let*

$$\check{V} \coloneqq \left\{(x_1, x_2, \dots, x_n) \in \mathbb{C}^n \;\middle|\; x_1^{n+1} + x_2^{n+1} + \cdots + x_n^{n+1} = 1\right\} \tag{1.37}$$

be the Milnor fiber considered as an exact symplectic manifold, and

$$K \coloneqq \left\{[\operatorname{diag}(t_0, t_1, \dots, t_n)] \in \mathrm{PGL}_{n+1}(\mathbb{C}) \;\middle|\; t_1^{n+1} = \cdots = t_n^{n+1} = t_0 t_1 \cdots t_n = 1\right\} \tag{1.38}$$

be a finite group acting on the projective hypersurface

$$Z \coloneqq \left\{[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n \;\middle|\; x_1^{n+1} + x_2^{n+1} + \cdots + x_n^{n+1} + x_0 x_1 \cdots x_n = 0\right\}. \tag{1.39}$$

Then we have a quasi-equivalence

$$\mathcal{F}\left(\check{V}\right) \simeq \operatorname{perf}[Z/K] \tag{1.40}$$

of pretriangulated A_∞-categories over $\mathbb{C}$.

(ii) Let

$$\check{V} \coloneqq \left\{(x_1, x_2, \dots, x_n) \in \mathbb{C}^n \;\middle|\; x_1^2 + x_2^{2n} + \cdots + x_n^{2n} = 1\right\} \tag{1.41}$$

be the Milnor fiber considered as an exact symplectic manifold, and

$$K \coloneqq \left\{[\operatorname{diag}(t_0, \dots, t_n)] \in \operatorname{Aut}\mathbb{P} \;\middle|\; t_1^2 = t_2^{2n} = \cdots = t_n^{2n} = t_0 t_1 \cdots t_n = 1\right\} \tag{1.42}$$

be a finite group acting on the weighted projective hypersurface

$$Z \coloneqq \left\{[x_0 : x_1 : \cdots : x_n] \in \mathbb{P} \;\middle|\; x_1^2 + x_2^{2n} + \cdots + x_n^{2n} + x_0 x_1 \cdots x_n = 0\right\}, \tag{1.43}$$

where $\mathbb{P} = \mathbb{P}(1, n, 1, \dots, 1)$ *is a weighted projective space considered as a smooth stack. Then we have a quasi-equivalence*

$$\mathcal{F}\left(\check{V}\right) \simeq \operatorname{perf}[Z/K] \tag{1.44}$$

of pretriangulated A_∞*-categories over* $\mathbb{C}$*.*

1.5. **The relation with results of Seidel and Sheridan.** The large complex structure limits in Theorem 1.7 are different from those appearing in [66] and its generalizations [70, 71]. In his construction, Seidel removes the divisor $\{x_1x_2x_3 = 0\}$ from the Milnor fiber $\check{V}$ on the A-side and considers the reducible singular variety $\{x_0x_1x_2x_3 = 0\}$ instead of Z on the B-side (cf. [50, Section 5]).

The generator S of $\mathcal{F}\left(\check{V}\right)$ used in the proof of Theorem 1.7.(i) is the direct sum of vanishing cycles of the Lefschetz fibration $\check{\mathbf{w}} = x_1^{n+1} + \cdots + x_n^{n+1} \colon \mathbb{C}^{n+1} \to \mathbb{C}$, which is also an object of $\mathcal{F}\left(\check{Y}\right)$. The Yoneda algebra computed in $\mathcal{F}\left(\check{Y}\right)$ is a deformation [69] of the Yoneda algebra A computed in $\mathcal{F}(\check{V})$, and hence isomorphic to it since $\operatorname{HH}^2(A)_0 \cong 0$, so that the Yoneda A_∞-algebra computed in $\mathcal{F}\left(\check{Y}\right)$ is described by a Novikov ring-valued point of $\mathcal{M}_\infty(A)$, which is the open-string mirror map.

The generator used by Seidel in [66] is the direct sum of the vanishing cycles of the Lefschetz fibration $\check{\mathbf{w}}' \coloneqq (\check{\mathbf{w}} + 1)/(x_1x_2x_3) \colon (\mathbb{C}^\times)^3 \to \mathbb{C}$ mirror to the toric variety whose fan polytope is polar dual to that of $\mathbb{P}^3$. The generator used by Sheridan in [70] is the cover of an immersed Lagrangian sphere in a pair of pants, which is shown to be the direct sum of vanishing cycles of the Lefschetz fibration $\check{\mathbf{w}}' \coloneqq (\check{\mathbf{w}} + 1)/(x_1 \cdots x_n) \colon (\mathbb{C}^\times)^n \to \mathbb{C}$ in [60]. One has $\check{D} \coloneqq \check{V} \setminus (\check{\mathbf{w}}')^{-1}(0) = \check{D}_1 \cup \cdots \cup \check{D}_n$ where $\check{D}_i \coloneqq \check{V} \cap \{x_i = 0\}$. Let $\mathcal{F}(\check{V}, \check{D})$ be the relative Fukaya category, which is an A_∞-category over $\mathbb{C}[q_1, \dots, q_n]$ whose objects are Lagrangian submanifolds of $\check{V} \setminus \check{D}$ and compositions are counted with intersection numbers with $\check{D}_i$. Since $\check{V}$ is Stein, the definition of $\mathcal{F}(\check{V}, \check{D})$ involves only the classical theory of pseudo-holomorphic maps, and the coefficient ring is a polynomial ring. The argument of Seidel and Sheridan shows that the idempotent-complete pretriangulated A_∞-category generated by the full subcategory of $\mathcal{F}(\check{V}, \check{D})$ consisting of the cover of the immersed Lagrangian sphere is equivalent to $\operatorname{perf}[\mathcal{Z}/K]$ where $\mathcal{Z} \coloneqq \operatorname{Proj} \mathbb{C}[q_1, \dots, q_n][x_1, \dots, x_n]/(q_1x_1^{n+1} + \cdots + q_nx_n^{n+1} + x_0 \cdots x_n)$. This suggests generalizations of Conjecture 1.4 to more general partial compactifications of covers of a pair of pants.

Even if one's goal is to prove homological mirror symmetry for a compact Calabi–Yau manifold over the Novikov field, it is useful not to go directly from a cover of a pair of pants to the compact Calabi–Yau, but to divide it into two steps, first to the Milnor fiber and then to the compact Calabi–Yau: The Fukaya category of a cover of a pair of pants has many deformations, but it is easy to control the deformation to the Milnor fiber, essentially because the Milnor fiber is Stein and the deformation is locally constant along a stratification of the base space. Once one comes to the Milnor fiber, and take the direct sum of vanishing cycles as a generator, then we can understand not only formal deformations but the global moduli space of A_∞-structures. It is an interesting problem to obtain the same level of understanding for deformations of the Fukaya category of a cover of a pair of pants, which would have non-smoothing components in general.

1.6. **Moduli of lattice polarized K3 surfaces.** Special cases of the moduli space (1.36) give modular compactifications of moduli spaces of a certain class of lattice polarized K3 surfaces. The point is that the choice of a generator $\mathcal{S}$ and an isomorphism $\psi\colon \operatorname{End}\mathcal{S} \xrightarrow{\sim} A$ with a fixed graded algebra A is a derived category analog of a choice of a lattice polarization. Similar identification of a choice of a full strong exceptional collection as an analog of a choice of a marking (an isomorphism of the Picard lattice with a fixed lattice) of a del Pezzo surface was a starting point of [1, 58].

Let P be a *lattice*, i.e., a free abelian group equipped with a symmetric bilinear form. A *P-polarized K3 surface* is a pair (Y, j) of a K3 surface and a primitive lattice embedding $j\colon P \hookrightarrow \operatorname{Pic} Y$. It follows from the global Torelli theorem and the surjectivity of the period map that the coarse moduli space of P-polarized K3 surfaces is the quotient of a symmetric domain of type IV by a discrete group. As an example, consider the case $P = E_8 \perp U$. This is the complement of U of the 'half' of the extended K3 lattice $E_8 \perp E_8 \perp U \perp U \perp U \perp U$, and as such is self-mirror, since mirror symmetry for lattice polarized K3 surfaces interchanges the algebraic lattice and the transcendental lattice inside the extended K3 lattice [17]. The Satake–Baily–Borel compactification of the coarse moduli space of $E_8 \perp U$-polarized K3 surfaces is known to be the 10-dimensional weighted projective space $\mathbf{P}(\boldsymbol{w})$ of weight $\boldsymbol{w} = (4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42)$ [12]. Similar descriptions exist for lattices coming from exceptional unimodal singularities by [53], which lead to a 'striking' ([54, page 586]) conclusion that certain rings of meromorphic automorphic forms are polynomial rings. Theorem 1.6 together with the discussion in Section 3.5 gives an interpretation of the spectrum of all of these polynomial rings as moduli spaces of A_∞-structures. This is a K3 analog of the description of $\overline{\mathcal{M}}_{1,1}$ as moduli of A_∞-structures recalled in Section 1.1. Similarly, the coarse moduli space of (1.36) for the $n = 3$ case of Theorem 1.7.(i) can be identified with the coarse moduli space of $E_8 \perp E_8 \perp U \perp \langle -4 \rangle$-polarized K3 surfaces. This is a K3 analog of the Hesse pencil of cubic curves, which are elliptic curves with level 3 structures. These examples are the first of infinite series, discussed in Section 3.4 and Section 3.2 respectively, where Theorem 1.6 applies.

1.7. **Sebastiani–Thom summation.** Yet another motivation for Conjecture 1.4, besides moduli of A_∞-structures and partial compactifications of covers of a pair of pants, comes from a conjectural compatibility of Conjecture 1.2 and Conjecture 1.4 under the Sebastiani–Thom summation. Let $\check{\mathbf{w}}_i\colon \mathbb{C}^{n_i} \to \mathbb{C}^1$ for $i = 1, 2$ be Lefschetz fibrations coming from transpositions of invertible polynomials $\mathbf{w}_i\colon X_i \coloneqq \mathbb{A}^{n_i} \to \mathbb{A}^1$ and

$$Y_i \coloneqq \{(x_{i,1}, \ldots, x_{i,n_i}) \in \mathbb{A}^{n_i} \mid x_{i,1} \cdots x_{i,n_i} = 0\} \tag{1.45}$$

be the unions of coordinate hyperplanes. We also write the union of coordinate hyperplanes in $X \coloneqq X_1 \times X_2$ as Y. Let $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2\colon \mathbb{A}^{n_1+n_2} \to \mathbb{A}^1$ be the Sebastiani–Thom sum of $\mathbf{w}_1$ and $\mathbf{w}_2$, and set

$$\Gamma \coloneqq \{((t_{1,0}, \dots, t_{1,n_1}), (t_{2,0}, \dots, t_{2,n_2})) \in \Gamma_1 \oplus \Gamma_2 \mid t_{1,0} = t_{2,0}\} \tag{1.46}$$

where $\Gamma_i \coloneqq \Gamma_{\mathbf{w}_i}$. It follows from [62] that

$$\operatorname{mf}\left(\mathbb{A}^{n_i+1}, \Gamma_i, \mathbf{w}_i + x_{i,0} \cdots x_{i,n_i}\right) \simeq \operatorname{mf}\left(Y_i, \Gamma_i, \mathbf{w}_i\right). \tag{1.47}$$

The push-out diagram

$$\begin{tikzcd} Y_1 \times Y_2 \arrow[r] \arrow[d] & X_1 \times Y_2 \arrow[d] \\ Y_1 \times X_2 \arrow[r] & Y \end{tikzcd} \tag{1.48}$$

should induce the push-out diagram

$$\begin{tikzcd} \operatorname{mf}\left(Y_1 \times Y_2, \Gamma, \mathbf{w}\right) \arrow[r] \arrow[d] & \operatorname{mf}\left(Y_1 \times X_2, \Gamma, \mathbf{w}\right) \arrow[d] \\ \operatorname{mf}\left(X_1 \times Y_2, \Gamma, \mathbf{w}\right) \arrow[r] & \operatorname{mf}\left(Y, \Gamma, \mathbf{w}\right), \end{tikzcd} \tag{1.49}$$

which gives

$$\begin{tikzcd} \operatorname{mf}\left(Y_1, \Gamma_1, \mathbf{w}_1\right) \otimes \operatorname{mf}\left(Y_2, \Gamma_2, \mathbf{w}_2\right) \arrow[r] \arrow[d] & \operatorname{mf}\left(Y_1, \Gamma_1, \mathbf{w}_1\right) \otimes \operatorname{mf}\left(X_2, \Gamma_2, \mathbf{w}_2\right) \arrow[d] \\ \operatorname{mf}\left(X_1, \Gamma_1, \mathbf{w}_1\right) \otimes \operatorname{mf}\left(Y_2, \Gamma_2, \mathbf{w}_2\right) \arrow[r] & \operatorname{mf}\left(Y, \Gamma, \mathbf{w}\right) \end{tikzcd} \tag{1.50}$$

by the Sebastiani–Thom theorem for matrix factorizations [64]. This matches the push-out diagram

$$\begin{tikzcd} \mathcal{W}(\check{\mathbf{w}}_1^{-1}(0)) \otimes \mathcal{W}(\check{\mathbf{w}}_2^{-1}(0)) \arrow[r] \arrow[d] & \mathcal{W}(\check{\mathbf{w}}_1) \otimes \mathcal{W}(\check{\mathbf{w}}_2^{-1}(0)) \arrow[d] \\ \mathcal{W}(\check{\mathbf{w}}_1^{-1}(0)) \otimes \mathcal{W}(\check{\mathbf{w}}_2) \arrow[r] & \mathcal{W}\left((\check{\mathbf{w}}_1 + \check{\mathbf{w}}_2)^{-1}(0)\right) \end{tikzcd} \tag{1.51}$$

coming from the cosheaf property of the wrapped Fukaya categories [31].

Remark 1.8. Similar compatibility exists for homological mirror symmetry for toric Fano manifolds and that for their toric boundaries giving large complex structure limits of their anti-canonical Calabi–Yau hypersurfaces. If $\check{\mathbf{w}}_i \colon (\mathbb{C}^\times)^{n_i} \to \mathbb{C}$ for $i = 1, 2$ are mirror to toric Fano manifolds X_i with toric boundaries Y_i and $\check{\mathbf{w}} \coloneqq \check{\mathbf{w}}_1 + \check{\mathbf{w}}_2 \colon (\mathbb{C}^\times)^{n_1+n_2} \to \mathbb{C}$ is mirror to $X \coloneqq X_1 \times X_2$ with its toric boundary Y, then one has the push-out diagram (1.48) inducing the push-out diagram

$$\begin{tikzcd} \operatorname{coh} Y_1 \otimes \operatorname{coh} Y_2 \arrow[r] \arrow[d] & \operatorname{coh} X_1 \otimes \operatorname{coh} Y_2 \arrow[d] \\ \operatorname{coh} Y_1 \otimes \operatorname{coh} X_2 \arrow[r] & \operatorname{coh} Y \end{tikzcd} \tag{1.52}$$

obtained from [28, Theorem 8.A.1.2] as explained in [34, Section 1.1.2] (see also [59, Section 2.4]).

1.8. This paper is organized as follows: In Section 2, we set up basic notations for weighted homogeneous polynomials and their semiuniversal unfoldings. In Section 3, we compute Hochschild cohomologies of (not necessarily smooth) proper algebraic stacks associated with weighted homogeneous polynomials using matrix factorizations. In Section 4, we give a generator $\mathcal{S}$ of perf $\mathscr{Y}$, and prove the formality of end $\mathcal{S}_0$. We prove Theorem 1.6 in Section 5. In Section 6, we prove that $\mathrm{HH}^*\left(\mathcal{F}\left(\check{V}\right)\right)$ is isomorphic to the symplectic cohomology of $\check{V}$. In Section 7, we give computations of symplectic cohomology of $\check{V}$ and deduce the non-formality result in $\mathcal{F}\left(\check{V}\right)$. Theorem 1.7 is proved in Section 8.

Through the rest of the paper, we will work over an algebraically closed field $\mathbf{k}$ of characteristic 0. The bounded derived category of coherent sheaves, its full subcategory consisting of perfect complexes, and the unbounded derived category of quasi-coherent sheaves on an algebraic stack $\mathcal{Y}$, considered as pretriangulated dg categories, will be denoted by coh $\mathcal{Y}$, perf $\mathcal{Y}$, and Qcoh $\mathcal{Y}$ respectively. All Fukaya categories are completed with respect to cones and direct summands.

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2. Weighted hypersurface singularities

A *weight system* is a sequence $(d_1, \ldots, d_n; h)$ of positive integers satisfying

$$h > \max\{d_1, \ldots, d_n\}. \tag{2.1}$$

We will always assume

$$\gcd(d_1, \ldots, d_n, h) = 1 \tag{2.2}$$

in this paper. Let $\mathbf{w}(x_1, \ldots, x_n) \in \mathbf{k}[x_1, \ldots, x_n]$ be a polynomial in n variables, which is weighted homogeneous of weight $(d_1, \ldots, d_n; h)$;

$$\mathbf{w}\left(t^{d_1}x_1, \ldots, t^{d_n}x_n\right) = t^h \mathbf{w}(x_1, \ldots, x_n), \quad t \in \mathbb{G}_\mathrm{m}. \tag{2.3}$$

It is written as the sum of monomials

$$\mathbf{w}(x_1, \ldots, x_n) = \sum_{\boldsymbol{i}=(i_1,\ldots,i_n)\in I_\mathbf{w}} c_{\boldsymbol{i}} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}, \quad c_{\boldsymbol{i}} \in \mathbb{G}_\mathrm{m}, \tag{2.4}$$

where the index set $I_\mathbf{w}$ is a subset of the set of non-negative integers satisfying

$$d_1 i_1 + d_2 i_2 + \cdots + d_n i_n = h. \tag{2.5}$$

We will always assume that $\mathbf{w}$ determines the weight system satisfying (2.2) uniquely.

Let $\Gamma_{\mathbf{w}}$ be the commutative algebraic group defined by

$$\Gamma_{\mathbf{w}} := \left\{ (t_1, \dots, t_{n+1}) \in \mathbb{G}_{\mathrm{m}}^{n+1} \;\middle|\; t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n} = t_{n+1} \text{ for all } (i_1, \dots, i_n) \in I_{\mathbf{w}} \right\}. \tag{2.6}$$

The group $\widehat{\Gamma}_{\mathbf{w}} := \operatorname{Hom}(\Gamma_{\mathbf{w}}, \mathbb{G}_{\mathrm{m}})$ of characters of $\Gamma_{\mathbf{w}}$ is written as

$$\widehat{\Gamma}_{\mathbf{w}} = \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_{n+1} \big/ (i_1\chi_1 + \cdots + i_n\chi_n - \chi_{n+1})_{\boldsymbol{i} \in I_{\mathbf{w}}}, \tag{2.7}$$

where $\chi_i \in \widehat{\Gamma}_{\mathbf{w}}$ for $1 \le i \le n+1$ is defined by $(t_1, \dots, t_{n+1}) \mapsto t_i$. Since the composition $\Gamma_{\mathbf{w}} \hookrightarrow \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}^n$ with the first projection is injective, we will think of $\Gamma_{\mathbf{w}}$ as a subgroup of $\mathbb{G}_{\mathrm{m}}^n$, and set $\chi_{\mathbf{w}} := \chi_{n+1}$. The group $\Gamma_{\mathbf{w}}$ consists of diagonal transformations of $\mathbb{A}^n$ which keeps $\mathbf{w}$ semi-invariant;

$$\mathbf{w}(t \cdot (x_1, \dots x_n)) = \chi_{\mathbf{w}}(t)\mathbf{w}(x_1, \dots, x_n), \quad t \in \Gamma_{\mathbf{w}}. \tag{2.8}$$

The injective homomorphism

$$\phi \colon \mathbb{G}_m \to \Gamma_{\mathbf{w}}, \quad t \mapsto \left(t^{d_1}, \dots, t^{d_n}\right) \tag{2.9}$$

fits into the exact sequence

$$1 \to \mathbb{G}_{\mathrm{m}} \xrightarrow{\phi} \Gamma_{\mathbf{w}} \to \ker \chi_{\mathbf{w}} / \langle j_{\mathbf{w}} \rangle \to 1, \tag{2.10}$$

where $j_{\mathbf{w}} := \left(e^{2\pi\sqrt{-1}d_1/h}, \dots, e^{2\pi\sqrt{-1}d_n/h}\right)$ is the *grading element* generating the cyclic group $\ker \chi_{\mathbf{w}} \cap \phi(\mathbb{G}_{\mathrm{m}})$ of order h.

Let Γ be a subgroup of $\Gamma_{\mathbf{w}}$ containing $\phi(\mathbb{G}_m)$ as a subgroup of finite index. For such Γ, the kernel of $\chi := \chi_{\mathbf{w}}|_{\Gamma}$ is a finite group, and such subgroups Γ are in bijection with finite subgroups of $\ker \chi_{\mathbf{w}}$ containing the grading element $j_{\mathbf{w}}$.

The group Γ acts naturally on the spectrum of $\overline{R} := \mathbf{k}[x_1, \dots, x_n]/(\mathbf{w})$, and we write the quotient stack of the complement of the origin $\mathbf{0}$ as

$$\mathcal{X} := \left[\left(\operatorname{Spec} \overline{R} \setminus \mathbf{0}\right)/\Gamma\right]. \tag{2.11}$$

We let Γ act on $\mathbb{A}^{n+1} := \operatorname{Spec} \mathbf{k}[x_0, \dots, x_n]$ diagonally via $\chi_0 \oplus \cdots \oplus \chi_n$ where

$$\chi_0 := \chi - \chi_1 - \cdots - \chi_n. \tag{2.12}$$

By abuse of notation, we write the image of $\mathbf{w}$ by the inclusion of $\mathbf{k}[x_1, \dots, x_n]$ to $\mathbf{k}[x_0, \dots, x_n]$ by the same symbol, and set $R := \mathbf{k}[x_0, \dots, x_n]/(\mathbf{w})$.

If

$$d_0 := h - d_1 - \cdots - d_n \tag{2.13}$$

is positive, then $[(\mathbb{A}^{n+1} \setminus \mathbf{0})/\Gamma]$ is proper, and hence so is its closed substack

$$\mathcal{Y}_0 := [(\operatorname{Spec} R \setminus \mathbf{0})/\Gamma]. \tag{2.14}$$

Here, the subscript “0” is placed in anticipation of the deformation that we will study later on. It is a projective cone over $\mathcal{X}$, which is obtained from $\mathcal{V}_0 := \left[\operatorname{Spec} \overline{R} \big/ \ker \chi_0\right]$ by adding $\mathcal{X}$ at infinity. The character of the Γ-action on the x_0 variable in (2.12) is chosen so that the dualizing sheaf of $\mathcal{Y}_0$ is trivial.

Assume that $\mathbf{w} \colon \mathbb{A}^n \to \mathbb{A}$ has an isolated critical point at the origin. This is equivalent to the finiteness of the dimension μ, called the *Milnor number* of $\mathbf{w}$, of the Jacobi algebra

$$\mathrm{Jac}_{\mathbf{w}} := \mathbf{k}[x_1, \ldots, x_n]/(\partial_1 \mathbf{w}, \ldots, \partial_n \mathbf{w}). \tag{2.15}$$

Let $J_{\mathbf{w}}$ be the set of exponents of monomials representing a basis of $\mathrm{Jac}_{\mathbf{w}}$, and

$$\widetilde{\mathbf{w}} := \mathbf{w}(x_1, \ldots, x_n) + \sum_{\boldsymbol{j}=(j_1,\ldots,j_n) \in J_{\mathbf{w}}} u_{\boldsymbol{j}} x_1^{j_1} \ldots x_n^{j_n} \colon \mathbb{A}^n \times \widetilde{U} \to \mathbb{A}^1 \tag{2.16}$$

be a semiuniversal unfolding of $\mathbf{w}$. The base space $\widetilde{U} := \operatorname{Spec} \mathbf{k}[u_1, \ldots, u_\mu]$ is an affine space of dimension μ. Let U be the affine subspace of $\widetilde{U}$ defined by the condition that $u_{\boldsymbol{j}}$ may be non-zero only if there exists a positive integer $w_{\boldsymbol{j}}$ satisfying

$$\chi = w_{\boldsymbol{j}} \chi_0 + j_1 \chi_1 + \cdots + j_n \chi_n. \tag{2.17}$$

Let J be the set of $\boldsymbol{j} \in J_{\mathbf{w}}$ satisfying this condition. Then we have the family

$$\pi_{\mathscr{Y}} \colon \mathscr{Y} := \left[\left(\mathbf{W}^{-1}(0) \setminus (\mathbf{0} \times U) \right) / \Gamma \right] \to U \tag{2.18}$$

of stacks over U defined by

$$\mathbf{W} := \mathbf{w}(x_1, \ldots, x_n) + \sum_{\boldsymbol{j} \in J} u_{\boldsymbol{j}} x_0^{w_{\boldsymbol{j}}} x_1^{j_1} \ldots x_n^{j_n} \colon \mathbb{A}^{n+1} \times U \to \mathbb{A}^1, \tag{2.19}$$

whose fiber over $u \in U$ will be denoted by $\mathcal{Y}_u := \pi^{-1}(u)$. Here the action of Γ on $\mathbb{A}^{n+1} \times U$ in such a way that $\deg x_i = \chi_i$ for $i = 0, 1, \ldots, n$ and $\deg u_{\boldsymbol{j}} = 0$ for all $\boldsymbol{j} \in J_{\mathbf{w}}$. The divisor at infinity defined by x_0 is isomorphic to $\mathcal{X} \times U$. The relative dualizing sheaf $\omega_{\mathscr{Y}/U}$ is identified with $\omega_{(\mathbf{W}^{-1}(0) \setminus (\mathbf{0} \times U))/U}$ considered as a Γ-equivariant coherent sheaf, which in turn is isomorphic to the restriction of $\omega_{(\mathbb{A}^{n+1} \times U)/U}(\chi)$ to $\mathbf{W}^{-1}(0) \setminus (\mathbf{0} \times U)$ since $\mathbf{W}$ is a section of $\mathcal{O}_{\mathbb{A}^{n+1} \times U}$ of degree χ. This sheaf is Γ-equivariantly trivial, and we fix its trivialization, which is unique up to scaling if $d_0 > 0$. In addition, there is a $\mathbb{G}_{\mathrm{m}}$-action on $\mathbb{A}^{n+1} \times U$ given by

$$((x_0, x_1, \ldots, x_n), (u_{\boldsymbol{j}})_{\boldsymbol{j} \in J}) \mapsto \left((t^{-1} x_0, x_1, \ldots, x_n), (t^{w_{\boldsymbol{j}}} u_{\boldsymbol{j}})_{\boldsymbol{j} \in J} \right), \tag{2.20}$$

which induces actions on $\mathscr{Y}$ and U which makes $\pi_{\mathscr{Y}}$ equivariant.

Example 2.1 (tacnode)**.** When $n = 2$ and $\mathbf{w} = x^2 + y^4$, one has $(d_1, d_2; h) = (2, 1; 4)$ and

$$\Gamma_{\mathbf{w}} := \left\{ (t_1, t_2) \in \mathbb{G}_{\mathrm{m}}^2 \;\middle|\; t_1^2 = t_2^4 \right\} \xrightarrow{\sim} \mathbb{G}_{\mathrm{m}} \times \boldsymbol{\mu}_2, \quad (t_1, t_2) \mapsto (t_2, t_1 t_2^{-2}). \tag{2.21}$$

The image of the injective homomorphism

$$\phi \colon \mathbb{G}_{\mathrm{m}} \to \Gamma_{\mathbf{w}}, \quad t \mapsto (t^2, t) \tag{2.22}$$

is an index 2 subgroup isomorphic to $\mathbb{G}_{\mathrm{m}}$, so that there are two choices of Γ. By construction, we have the semi-invariance property

$$\mathbf{w}(t_1 x, t_2 y) = \chi(t_1, t_2) \mathbf{w}(x, y), \tag{2.23}$$

where $\chi \colon \Gamma \to \mathbb{G}_\mathrm{m}$ is the character sending (t_1, t_2) to $t_1^2 = t_2^4$. A semiuniversal unfolding of $\mathbf{w}$ is given by

$$\widetilde{\mathbf{w}}(x, y; u_2, u_3, u_4) = x^2 + y^4 + u_2 y^2 + u_3 y + u_4, \tag{2.24}$$

and one has

$$\mathbf{W}(x, y, z; u_2, u_3, u_4) = x^2 + y^4 + u_2 y^2 z^2 + u_3 y z^3 + u_4 z^4 \tag{2.25}$$

if $\Gamma = \phi(\mathbb{G}_\mathrm{m})$, and

$$\mathbf{W}(x, y, z; u_2, u_4) = x^2 + y^4 + u_2 y^2 z^2 + u_4 z^4. \tag{2.26}$$

if $\Gamma = \Gamma_\mathbf{w}$.

Example 2.2 (E_{12}-singularity)**.** When $n = 3$ and $\mathbf{w}(x, y, z) = x^2 + y^3 + z^7$, one has $(d_1, d_2, d_3; h) = (21, 14, 6; 42)$, $\Gamma_\mathbf{w} \cong \mathbb{G}_\mathrm{m}$, $\mathrm{Jac}_\mathbf{w} = \mathbb{A}[x, y, z]/(2x, 3y^2, 7y^6)$, and $\mu = 12$. One can take

$$J_\mathbf{w} = \left\{ (i, j, k) \in \mathbb{N}^3 \;\middle|\; i = 0,\ j \le 1,\ k \le 5 \right\}, \tag{2.27}$$

so that a semiuniversal unfolding $\widetilde{\mathbf{w}} \colon \mathbb{A}^3 \times \widetilde{U} \to \mathbb{A}^1$ of $\mathbf{w}$ is given by

$$\widetilde{\mathbf{w}} = x^2 + y^3 + z^7 + \sum_{\substack{j=0,1,\\ k=0,1,2,3,4,5}} u_{jk} y^j z^k. \tag{2.28}$$

Since $\phi(\mathbb{G}_\mathrm{m}) = \Gamma_\mathbf{w}$, the choice of Γ is unique in this case. The integer

$$w_{jk} = 42 - 14j - 6k \tag{2.29}$$

is positive unless $(j, k) = (1, 5)$, so that $U \subset U$ is the 11-dimensional subspace defined by $u_{15} = 0$, and $\mathbf{W} \colon \mathbb{A}^4 \times U \to \mathbb{A}^1$ is given by

$$\mathbf{W} = x^2 + y^3 + z^7 + \sum_{(j,k) \ne (1,5)} u_{jk} y^j z^k v^{w_{jk}}. \tag{2.30}$$

3. Hochschild cohomology via matrix factorizations

The Hochschild cohomology of a scheme Y (or more generally a perfect derived stack [8]) is defined as

$$\mathrm{HH}^*(Y) \coloneqq \mathrm{Ext}^*_{Y \times Y}\left(\mathcal{O}_\Delta, \mathcal{O}_\Delta\right), \tag{3.1}$$

where $\mathcal{O}_\Delta \coloneqq \Delta_* \mathcal{O}_Y$ and $\Delta \colon Y \to Y \times Y$ is the diagonal embedding. The right hand side of (3.1) is isomorphic to the endomorphism

$$\mathrm{HH}^*(\operatorname{Qcoh} Y) \coloneqq \mathrm{Hom}^*_{\mathrm{Fun}^\mathrm{L}(\operatorname{Qcoh} Y, \operatorname{Qcoh} Y)}\left(\mathrm{id}_{\operatorname{Qcoh} Y}, \mathrm{id}_{\operatorname{Qcoh} Y}\right) \tag{3.2}$$

of the identity in the ∞-category of colimit-preserving endofunctors of $\operatorname{Qcoh} Y$ [76, 8].

When Y is a smooth variety over $\mathbf{k}$ (see [3] for a partial extension to positive characteristics), one can compute the Hochschild cohomology by appealing to Hochschild–Kostant–Rosenberg isomorphism

$$\mathrm{HH}^n(Y) \cong \bigoplus_{p+q=n} H^p\left(Y, \Lambda^q T_Y\right). \tag{3.3}$$

However, our main interest is in the case when Y is a singular stack. A generalization of the above decomposition to singular varieties is given by Buchweitz–Flenner [13] which states

$$\mathrm{HH}^n(Y) \cong \bigoplus_{p+q=n} \mathrm{Ext}^p\left(\wedge^q \mathbb{L}_Y, \mathcal{O}_Y\right) \tag{3.4}$$

where $\mathbb{L}_Y$ is the cotangent complex over $\mathbf{k}$ and $\wedge^q$ is the derived exterior product. However, it is not always straightforward to compute with this, even when Y is a variety. We will instead use another strategy which uses the function $\mathbf{w}$ more directly.

Let $S \coloneqq \operatorname{Sym} V$ be the symmetric algebra over the vector space $V \coloneqq \operatorname{span}\{x_0, x_1, \dots, x_n\}$ of dimension $n+1$, and $\mathbb{A}^{n+1} = \operatorname{Spec} S$ be the affine space. Let further Γ be a finite extension of $\mathbb{G}_\mathrm{m}$ acting linearly on V, $\chi \in \widehat{\Gamma} \coloneqq \operatorname{Hom}(\Gamma, \mathbb{G}_\mathrm{m})$ be a character of Γ, and $\mathbf{W} \in H^0\left(\mathcal{O}_{[\mathbb{A}^{n+1}/\Gamma]}(\chi)\right) \cong (S \otimes \chi)^\Gamma$ be a non-zero element of weight χ. The quotient ring $R \coloneqq S/(\mathbf{W})$ inherits a Γ-action.

When χ is isomorphic to the top exterior power of the dual $V^\vee$ as a Γ-module, the bounded derived category $\operatorname{coh} Y$ of coherent sheaves on the quotient stack $\mathcal{Y} \coloneqq [(\operatorname{Spec} R \setminus \mathbf{0})/\Gamma]$ is quasi-equivalent to the idempotent-complete dg category $\operatorname{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{W})$ of Γ-equivariant matrix factorizations;

$$\operatorname{coh} \mathcal{Y} \cong \operatorname{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{W}). \tag{3.5}$$

This is first proved by Orlov [61, Theorem 3.11] when $\Gamma \cong \mathbb{G}_\mathrm{m}$ in the context of triangulated categories. The generalization to a finite extension of $\mathbb{G}_\mathrm{m}$ is straightforward. The quasi-equivalence of dg categories can be found in [6, 14, 39, 72]. Note also that by [61, Theorem 3.10], $\operatorname{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{W})$ is equivalent to the bounded stable derived category of the graded ring R, denoted by $D^b_{\mathrm{sing}}(\operatorname{gr} R)$. The equivalence (3.5) implies the isomorphism

$$\mathrm{HH}^*(\mathcal{Y}) \cong \mathrm{HH}^*(\mathbb{A}^{n+1}, \Gamma, \mathbf{W}), \tag{3.6}$$

where the right hand side is the Hochschild cohomology of the dg category $\operatorname{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{W})$, which can be computed as follows:

Theorem 3.1 ([19, 14, 65, 6])**.** *Let Γ be an abelian finite extension of $\mathbb{G}_\mathrm{m}$ acting linearly on $\mathbb{A}^{n+1} = \operatorname{Spec} S$, and $\mathbf{W} \in S$ be a non-zero element of degree $\chi \in \widehat{\Gamma} \coloneqq \operatorname{Hom}(\Gamma, \mathbb{G}_\mathrm{m})$. Assume that the singular locus of the zero set $Z_{(-\mathbf{W}) \boxplus \mathbf{W}}$ of the Sebastiani–Thom sum $(-\mathbf{W}) \boxplus \mathbf{W}$ is contained in the product of the zero sets $Z_\mathbf{W} \times Z_\mathbf{W}$. Then* $\mathrm{HH}^t\left(\mathbb{A}^{n+1}, \Gamma, \mathbf{W}\right)$

is isomorphic to

$$\left(\bigoplus_{\substack{\gamma \in \ker \chi,\ l \ge 0 \\ t - \dim N_\gamma = 2u}} H^{-2l}(d\mathbf{W}_\gamma) \otimes \chi^{\otimes(u+l)} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right.$$

$$\left. \oplus \bigoplus_{\substack{\gamma \in \ker \chi,\ l \ge 0 \\ t - \dim N_\gamma = 2u+1}} H^{-2l-1}(d\mathbf{W}_\gamma) \otimes \chi^{\otimes(u+l+1)} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)^\Gamma. \tag{3.7}$$

Here $H^i(d\mathbf{W}_\gamma)$ is the i-th cohomology of the Koszul complex

$$C^*(d\mathbf{W}_\gamma) := \left\{ \cdots \to \Lambda^2 V_\gamma^\vee \otimes \chi^{\otimes(-2)} \otimes S_\gamma \to V_\gamma^\vee \otimes \chi^\vee \otimes S_\gamma \to S_\gamma \right\}, \tag{3.8}$$

where the rightmost term S_γ sits in cohomological degree 0, and the differential is the contraction with

$$d\mathbf{W}_\gamma \in (V_\gamma \otimes \chi \otimes S_\gamma)^\Gamma. \tag{3.9}$$

The vector space V_γ is the subspace of γ-invariant elements in V, S_γ is the symmetric algebra of V_γ, $\mathbf{W}_\gamma$ is the restriction of $\mathbf{W}$ to $\operatorname{Spec} S_\gamma$, and N_γ is the complement of V_γ in V so that $V \cong V_\gamma \oplus N_\gamma$ as a Γ-module. The zero-th cohomology of the Koszul complex (3.8) is isomorphic to the Jacobi algebra $\operatorname{Jac}_{\mathbf{W}_\gamma}$. If $\mathbf{W}_\gamma$ has an isolated critical point at the origin, then the cohomology of (3.8) is concentrated in degree 0, so that only the summand

$$\left(\operatorname{Jac}_{\mathbf{W}_\gamma} \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)^\Gamma \tag{3.10}$$

with $l = 0$ contributes in (3.7).

The formula (3.7) is an adaptation of [6, Theorem 1.2], to which we refer the reader for a proof. The slight difference between [6, Theorem 1.2] and (3.7) comes from the convention for the Koszul complex; the latter is convenient in that when V has an additional $\mathbb{G}_{\mathrm{m}}$-action, (3.7) is equivariant with respect to it.

If the Γ-action on V satisfies $\dim (S \otimes \rho)^\Gamma < \infty$ for any $\rho \in \widehat{\Gamma}$, then one has

$$\dim \operatorname{HH}^t(\mathbb{A}^{n+1}, \mathbf{W}, \Gamma) < \infty \tag{3.11}$$

for any $t \in \mathbb{Z}$, since the Koszul complex (3.8) is bounded, the group $\ker \chi$ is finite, each direct summand in (3.7) is finite-dimensional, and there are only finitely many u contributing to a fixed t.

3.1. **Cones over isolated hypersurface singularities.** Let $\mathbf{w} \in \mathbf{k}[x_1, \dots, x_n]$ be a weighted homogeneous polynomial of weight $(d_1, \dots, d_n; h)$ satisfying $d_0 > 0$ and Γ be a subgroup of $\Gamma_{\mathbf{w}}$ containing $\phi(\mathbb{G}_\mathrm{m})$ as a subgroup of finite index as in Section 2. Assume that $\mathbf{w}$ has an isolated critical point at the origin and let $\mathbf{W}$ be the image of $\mathbf{w}$ by the inclusion $\mathbf{k}[x_1, \dots, x_n] \hookrightarrow \mathbf{k}[x_0, \dots, x_n]$. Then $\mathcal{Y} := [(\mathbf{W}^{-1}(0) \setminus \mathbf{0})/\Gamma]$ has a $\mathbb{G}_\mathrm{m}$-action given by $t \cdot [x_0 : x_1 : \cdots : x_n] = [t x_0 : x_1 : \cdots : x_n]$, which induces a $\mathbb{G}_\mathrm{m}$-action on $\mathrm{HH}^*(\mathcal{Y})$. Let $\mathrm{HH}^*(\mathcal{Y})_{<0}$ be the negative weight part of this $\mathbb{G}_\mathrm{m}$-action.

Since $\mathbf{W}$ does not contain the variable x_0, the Koszul complex $C^*(d\mathbf{W}_\gamma)$ is isomorphic to the tensor product of $C^*(d\mathbf{w}_\gamma)$ and the complex $\{\mathbf{k}x_0^\vee \otimes \chi^\vee \otimes \mathbf{k}[x_0] \to \mathbf{k}[x_0]\}$ concentrated in cohomological degree $[-1, 0]$ with the zero differential if V_γ contains $\mathbf{k}x_0 \subset V$, and to $C^*(d\mathbf{w}_\gamma)$ otherwise. Only direct summands coming from $H^k(d\mathbf{W}_\gamma)$ with $k = 0, -1$ contribute to (3.7) in the former case, and those with $k = 0$ in the latter case. Summands with $k = 0$ contribute

$$\left(\mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \mathbf{k}[x_0] \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)^\Gamma \tag{3.12}$$

to $\mathrm{HH}^{2u+\dim N_\gamma}(\mathcal{Y})$, and those with $k = -1$ contribute

$$\left(\mathbf{k}x_0^\vee \otimes \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \mathbf{k}[x_0] \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)^\Gamma \tag{3.13}$$

to $\mathrm{HH}^{2u+\dim N_\gamma+1}(\mathcal{Y})$ since

$$H^{-1}(d\mathbf{W}_\gamma) \cong \mathbf{k}x_0^\vee \otimes \chi^\vee \otimes \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \mathbf{k}[x_0]. \tag{3.14}$$

Corollary 3.2. *Under the above assumptions, one has* $\mathrm{HH}^0(\mathcal{Y}) \cong \mathbf{k}$, $\mathrm{HH}^1(\mathcal{Y})_0 \ncong 0$, *and* $\mathrm{HH}^1(\mathcal{Y})_{<0} \cong 0$.

Proof. If $u \le -1$, then (3.12) vanishes, and if $u = 0$, then (3.12) contribute to $\mathrm{HH}^0(\mathcal{Y})$ only if $N_\gamma = 0$, where it is $\mathbf{k}$. (3.12) cannot contribute to $\mathrm{HH}^1(\mathcal{Y})$, since $\dim N_\gamma = 1$ is impossible for $\gamma = (t_0, t_1, \dots, t_n) \in \Gamma$ because of the condition $t_0 \cdots t_n = 1$. One always has $u \ge -1$ in (3.13), and one can have $u = -1$ only if $N_\gamma = \mathrm{span}\{x_1, \dots, x_n\}$. Each such γ contribute $\mathbf{k}(-1)$ to $\mathrm{HH}^{n-1}(\mathcal{Y})$. The summand with $u = 0$ and $\gamma = 0$ contributes $(\mathbf{k}x_0^\vee \otimes \mathrm{Jac}_\mathbf{w} \otimes \mathbf{k}[x_0])^\Gamma$ to $\mathrm{HH}^1(\mathcal{Y})$, which has non-negative $\mathbb{G}_\mathrm{m}$-weights. In particular, the element $x_0^\vee \otimes x_0$ gives a non-zero contribution to $\mathrm{HH}^1(\mathcal{Y})_0$. Summands with $u = 0$ and $\gamma \neq 0$ or $u \ge 1$ contribute to $\mathrm{HH}^{\ge 2}(\mathcal{Y})$. $\square$

Definition 3.3. We say that the pair $(\mathbf{w}, \Gamma)$ *does not have twisted deformations* if $\mathrm{HH}^2(\mathcal{Y})_{<0}$ comes only from the direct summand $(\mathrm{Jac}_\mathbf{w} \otimes \mathbf{k}[x_0] \otimes \chi)^\Gamma$ corresponding to $u = 1$ and $\gamma = 0$ in (3.12).

This condition means that direct summands with $\gamma \neq 0$, called *twisted sectors* in string theory, do not contribute to $\mathrm{HH}^2(\mathcal{Y})_{<0}$, so that all deformations corresponding to $\mathrm{HH}^2(\mathcal{Y})_{<0}$ comes from deformations of the defining polynomial $\mathbf{w}$, and one has $\dim \mathrm{HH}^2(\mathcal{Y})_{<0} = \dim U$.

3.2. **Projective hypersurfaces.** Consider the case

$$\mathbf{w}(x_1, \dots, x_n) = x_1^{n+1} + \cdots + x_n^{n+1} \tag{3.15}$$

with

$$(d_1, \dots, d_n; h) = (1, \dots, 1; n+1) \tag{3.16}$$

and

$$\Gamma = \left\{ (t_0, \dots, t_n) \in (\mathbb{G}_{\mathrm{m}})^{n+1} \;\middle|\; t_1^{n+1} = \cdots = t_n^{n+1} = t_0 \cdots t_n \right\}. \tag{3.17}$$

This case appears in mirror symmetry for the Calabi–Yau hypersurface of degree $n+1$ in $\mathbb{P}^n$, and gives the D_4-singularity $x^3 + y^3$ for $n = 2$. The group $\widehat{\Gamma}$ of characters of Γ is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1}$, and we write the character $(t_0, \dots, t_n) \mapsto t_1^{i_1 + \cdots + i_n} t_2^{-i_2} \cdots t_n^{-i_n}$ for $(i_1, \dots, i_n) \in \mathbb{Z} \times (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1}$ as $\rho_{i_1, \dots, i_n}$. One has $\mathbf{k} x_0^\vee \cong \rho_{1,\dots,1}, \mathbf{k} x_1^\vee \cong \rho_{1,0,\dots,0}, \mathbf{k} x_2^\vee \cong \rho_{1,n,0,\dots,0}, \dots, \mathbf{k} x_n^\vee \cong \rho_{1,0,\dots,0,n}, \chi \cong \rho_{n+1,0,\dots,0}$, and $\ker \chi \cong (\mathbb{Z}/(n+1)\mathbb{Z})^n$.

When γ is the identity element, one has $V_\gamma = V$, $N_\gamma = 0$, $\mathbf{W}_\gamma = \mathbf{w}$ and

$$\operatorname{Jac}_{\mathbf{w}} \cong \mathbf{k}[x_1, \dots, x_n] / ((n+1) x_1^n, \dots, (n+1) x_n^n). \tag{3.18}$$

The element

$$x_0^{(n+1)(u-i)+i} x_1^i \cdots x_n^i \in \left(\operatorname{Jac}_{\mathbf{w}} \otimes \mathbf{k}[x_0] \otimes \chi^{\otimes u} \right)^\Gamma \tag{3.19}$$

for $i = 0, \dots, \min\{u, n-1\}$ contributes $\mathbf{k}((n+1)(u-i)+i)$ to HH^{2u}, and the element

$$x_0^\vee \otimes x_0^{(n+1)(u-i)+i+1} x_1^i \cdots x_n^i \in \left(x_0^\vee \otimes \operatorname{Jac}_{\mathbf{w}} \otimes \mathbf{k}[x_0] \otimes \chi^{\otimes u} \right)^\Gamma \tag{3.20}$$

for $i = 0, \dots, \min\{u, n-1\}$ contributes $\mathbf{k}((n+1)(u-i)+i)$ to HH^{2u+1}.

When $V_\gamma = 0$ and $N_\gamma = V$, one has $\mathbf{W}_\gamma = 0$ and the summand

$$\left(\chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)^\Gamma \cong \mathbf{k} x_0^\vee \wedge \cdots \wedge x_n^\vee, \tag{3.21}$$

contributes $\mathbf{k}(-1)$ to $\mathrm{HH}^{2u + \dim N_\gamma} = \mathrm{HH}^{-2+n+1} = \mathrm{HH}^{n-1}$. The number $\upsilon_1(n)$ of such γ is $2, 21, 204, \dots$ for $n = 2, 3, 4, \dots$ respectively.

When $V_\gamma = \mathbf{k} x_0$ and $N_\gamma = \mathbf{k} x_1 \oplus \cdots \oplus \mathbf{k} x_n$, one has $\mathbf{W}_\gamma = 0$ and the summand

$$\left(\operatorname{Jac}_{\mathbf{W}_\gamma} \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)^\Gamma \cong \mathbf{k} x_0^{(n+1)u+n} \otimes x_1^\vee \wedge \cdots \wedge x_n^\vee \tag{3.22}$$

in $\mathrm{HH}^{2u + \dim N_\gamma}$ contributes $\mathbf{k}((n+1)u+n)$ to HH^{2u+n} for $u \ge 0$, and the summand

$$\left(x_0^\vee \otimes \operatorname{Jac}_{\mathbf{W}_\gamma} \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)^\Gamma \cong \mathbf{k} x_0^\vee \otimes x_0^{(n+1)u+n+1} \otimes x_1^\vee \wedge \cdots \wedge x_n^\vee \tag{3.23}$$

in $\mathrm{HH}^{2u + \dim N_\gamma + 1}$ contributes $\mathbf{k}((n+1)u+n)$ to HH^{2u+n+1} for $u \ge -1$. The number $\upsilon_2(n)$ of such γ is $2, 6, 52, \dots$ for $n = 2, 3, 4, \dots$ respectively.

Note that one has

$$\upsilon_1(n) + \upsilon_2(n) = n^n, \tag{3.24}$$

since the left hand side is equal to is the number of elements of the set

$$\left\{(t_1,\dots,t_n)\in(\mathbb{G}_\mathrm{m}\setminus\{1\})^n \mid \gamma_1^{n+1}=\cdots=\gamma_n^{n+1}=1\right\}. \tag{3.25}$$

When $V_\gamma=\mathbf{k}x_0\oplus\cdots\oplus\mathbf{k}x_i$ and $\Lambda^{\dim N_\gamma}N_\gamma^\vee=\mathbf{k}x_{i+1}^\vee\wedge\cdots\wedge x_n^\vee$ for $0<i<n$, one has $\mathbf{W}_\gamma=x_1^{n+1}+\cdots+x_i^{n+1}$ and

$$\mathrm{Jac}_{\mathbf{W}_\gamma}=\mathbf{k}[x_0]\otimes\operatorname{span}\left\{1,x_1,\dots,x_1^{n-1}\right\}\otimes\cdots\otimes\operatorname{span}\left\{1,x_i,\dots,x_i^{n-1}\right\}. \tag{3.26}$$

Since the weight of

$$x_0^{k_0}\cdots x_i^{k_i}\otimes x_{i+1}^\vee\wedge\cdots\wedge x_n^\vee\in\mathrm{Jac}_{\mathbf{W}_\gamma}\otimes\Lambda^{\dim N_\gamma}N_\gamma^\vee \tag{3.27}$$

for $(k_0,\dots,k_i)\in\mathbb{N}\times\{0,\dots,n-1\}^i$ can never be proportional to χ, one has

$$\left(\mathrm{Jac}_{\mathbf{W}_\gamma}\otimes\chi^{\otimes u}\otimes\Lambda^{\dim N_\gamma}N_\gamma^\vee\right)^\Gamma\cong 0 \tag{3.28}$$

for any $u\in\mathbb{Z}$ and similarly for $\left(x_0^\vee\otimes\mathrm{Jac}_{\mathbf{W}_\gamma}\otimes\chi^{\otimes u}\otimes\Lambda^{\dim N_\gamma}N_\gamma^\vee\right)^\Gamma$, so that such γ does not contribute to HH^*. In total, one has

$$\mathrm{HH}^0(\mathcal{Y})\cong\mathbf{k}, \tag{3.29}$$

$$\mathrm{HH}^1(\mathcal{Y})\cong\mathbf{k}\oplus\mathbf{k}(-1)^{\oplus 4}, \tag{3.30}$$

$$\mathrm{HH}^{2i+2}(\mathcal{Y})\cong\mathrm{HH}^{2i+3}(\mathcal{Y})\cong\mathbf{k}(3i+1)\oplus\mathbf{k}(3i+2)^{\oplus 2}\oplus\mathbf{k}(3i+3)\quad\text{for } i\geq 0 \tag{3.31}$$

for $n=2$,

$$\mathrm{HH}^0(\mathcal{Y})\cong\mathrm{HH}^1(\mathcal{Y})\cong\mathbf{k}, \tag{3.32}$$

$$\mathrm{HH}^2(\mathcal{Y})\cong\mathbf{k}(-1)^{\oplus 27}\oplus\mathbf{k}(1)\oplus\mathbf{k}(4), \tag{3.33}$$

$$\mathrm{HH}^3(\mathcal{Y})\cong\mathbf{k}(1)\oplus\mathbf{k}(3)^{\oplus 6}\oplus\mathbf{k}(4), \tag{3.34}$$

$$\mathrm{HH}^{2i+4}(\mathcal{Y})\cong\mathbf{k}(4i+2)\oplus\mathbf{k}(4i+3)^{\oplus 6}\oplus\mathbf{k}(4i+5)\oplus\mathbf{k}(4i+8)\quad\text{for } i\geq 0, \tag{3.35}$$

$$\mathrm{HH}^{2i+5}(\mathcal{Y})\cong\mathbf{k}(4i+2)\oplus\mathbf{k}(4i+5)\oplus\mathbf{k}(4i+7)^{\oplus 6}\oplus\mathbf{k}(4i+8)\quad\text{for } i\geq 0 \tag{3.36}$$

for $n=3$,

$$\mathrm{HH}^0(\mathcal{Y})\cong\mathrm{HH}^1(\mathcal{Y})\cong\mathbf{k}, \tag{3.37}$$

$$\mathrm{HH}^2(\mathcal{Y})\cong\mathbf{k}(1)\oplus\mathbf{k}(5), \tag{3.38}$$

$$\mathrm{HH}^3(\mathcal{Y})\cong\mathbf{k}(-1)^{\oplus 256}\oplus\mathbf{k}(1)\oplus\mathbf{k}(5), \tag{3.39}$$

$$\mathrm{HH}^4(\mathcal{Y})\cong\mathrm{HH}^5(\mathcal{Y})\cong\mathbf{k}(2)\oplus\mathbf{k}(4)^{\oplus 52}\oplus\mathbf{k}(6)\oplus\mathbf{k}(10), \tag{3.40}$$

$$\mathrm{HH}^{2i+6}(\mathcal{Y})\cong\mathrm{HH}^{2i+7}(\mathcal{Y}) \tag{3.41}$$

$$\cong\mathbf{k}(5i+3)\oplus\mathbf{k}(5i+7)\oplus\mathbf{k}(5i+9)^{\oplus 52}\oplus\mathbf{k}(5i+11)\oplus\mathbf{k}(5i+15)\quad\text{for } i\geq 0 \tag{3.42}$$

for $n=4$, and so on.

For $n=2$ there are twisted deformations where $\mathrm{HH}^2(\mathcal{Y})_{-2}\cong\mathbf{k}^{\oplus 2}$ comes from $\gamma\neq 0$, but there are no twisted deformations for all $n\geq 3$.

3.3. **Double covers of projective spaces.** Consider the case

$$\mathbf{w}(x_1, \dots, x_n) = x_1^2 + x_2^{2n} + \dots + x_n^{2n} \tag{3.43}$$

with

$$(d_1, \dots, d_n; h) = (n, 1, \dots, 1; 2n) \tag{3.44}$$

and

$$\Gamma = \left\{ (t_0, \dots, t_n) \in (\mathbb{G}_\mathrm{m})^{n+1} \;\middle|\; t_1^2 = t_2^{2n} = \dots = t_n^{2n} = t_0 \cdots t_n \right\}. \tag{3.45}$$

This case appears in mirror symmetry for the double cover of $\mathbb{P}^{n-1}$ branched over a hypersurface of degree $2n$, and gives the tacnode singularity $x^2 + y^4$ for $n = 2$. One has $\widehat{\Gamma} \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2n\mathbb{Z})^{n-2}$ and $\ker \chi \cong \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2n\mathbb{Z})^{n-1}$.

When γ is the identity element, one has $V_\gamma = V$, $N_\gamma = 0$, $\mathbf{W}_\gamma = \mathbf{w}$ and

$$\mathrm{Jac}_\mathbf{w} \cong \mathbf{k}[x_1, \dots, x_n]/(2x_1, 2n x_2^{2n-1}, \dots, 2n x_n^{2n-1}) \tag{3.46}$$

The element

$$x_0^{2(u-i)n+2i} x_2^{2i} \cdots x_n^{2i} \in \left(\mathrm{Jac}_\mathbf{w} \otimes \mathbf{k}[x_0] \otimes \chi^{\otimes u} \right)^\Gamma \tag{3.47}$$

for $i = 0, \dots, \min\{u, n-1\}$ contributes $\mathbf{k}(2(u-i)n + 2i)$ to HH^{2u}, and the element

$$x_0^\vee \otimes x_0^{2(u-i)n+2i+1} x_2^{2i} \cdots x_n^{2i} \in \left(x_0^\vee \otimes \mathrm{Jac}_\mathbf{w} \otimes \mathbf{k}[x_0] \otimes \chi^{\otimes u} \right)^\Gamma \tag{3.48}$$

for $i = 0, \dots, \min\{u, n-1\}$ contributes $\mathbf{k}(2(u-i)n + 2i)$ to HH^{2u+1}.

When $V_\gamma = 0$ and $N_\gamma = V$, one has $\mathbf{W}_\gamma = 0$ and the summand

$$\left(\chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)^\Gamma \cong \mathbf{k} x_0^\vee \wedge \dots \wedge x_n^\vee, \tag{3.49}$$

contributes $\mathbf{k}(-1)$ to $\mathrm{HH}^{2u + \dim N_\gamma} = \mathrm{HH}^{-2+n+1} = \mathrm{HH}^{n-1}$. The set of such γ is bijective with the set of $(i_0, i_2, \dots, i_{n-1}) \in \{0, \dots, 2n-1\}^{n-1}$ satisfying $i_0 + n + i_2 + \dots + i_n \equiv 0$ modulo $2n$. The number $\upsilon_3(n)$ of such γ is $2, 21, 300, \dots$ for $n = 2, 3, 4, \dots$ respectively.

When $V_\gamma = \mathbf{k} x_0$ and $N_\gamma = \mathbf{k} x_1 \oplus \dots \oplus \mathbf{k} x_n$, one has $\mathbf{W}_\gamma = 0$ and the summand

$$\left(\mathrm{Jac}_{\mathbf{W}_\gamma} \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)^\Gamma \cong \mathbf{k} x_0^{2nu+2n-1} \otimes x_1^\vee \wedge \dots \wedge x_n^\vee \tag{3.50}$$

in $\mathrm{HH}^{2u + \dim N_\gamma}$ contributes $\mathbf{k}(2nu + 2n - 1)$ to HH^{2u+n} for $u \ge 0$, and the summand

$$\left(x_0^\vee \otimes \mathrm{Jac}_{\mathbf{W}_\gamma} \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)^\Gamma \cong \mathbf{k} x_0^\vee \otimes x_0^{2nu+2n} \otimes x_1^\vee \wedge \dots \wedge x_n^\vee \tag{3.51}$$

in $\mathrm{HH}^{2u + \dim N_\gamma + 1}$ contributes $\mathbf{k}(2nu + 2n - 1)$ to HH^{2u+n+1} for $u \ge -1$. The number $\upsilon_4(n)$ of such γ is $1, 4, 43, \dots$ for $n = 2, 3, 4, \dots$ respectively. One has

$$\upsilon_3(n) + \upsilon_4(n) = (2n-1)^{n-1} \tag{3.52}$$

just as in the case of $\upsilon_1(n) + \upsilon_2(n)$.

Other γ do not contribute, and the result is summarized as

$$\mathrm{HH}^0(\mathcal{Y}) \cong \mathbf{k}, \tag{3.53}$$
$$\mathrm{HH}^1(\mathcal{Y}) \cong \mathbf{k} \oplus \mathbf{k}(-1)^{\oplus 3}, \tag{3.54}$$
$$\mathrm{HH}^{2i+2}(\mathcal{Y}) \cong \mathrm{HH}^{2i+3}(\mathcal{Y}) \cong \mathbf{k}(4i+2) \oplus \mathbf{k}(4i+3) \oplus \mathbf{k}(4i+4) \quad \text{for } i \geq 0 \tag{3.55}$$

for $n = 2$,

$$\mathrm{HH}^0(\mathcal{Y}) \cong \mathrm{HH}^1(\mathcal{Y}) \cong \mathbf{k}, \tag{3.56}$$
$$\mathrm{HH}^2(\mathcal{Y}) \cong \mathbf{k}(-1)^{\oplus 25} \oplus \mathbf{k}(2) \oplus \mathbf{k}(6), \tag{3.57}$$
$$\mathrm{HH}^3(\mathcal{Y}) \cong \mathbf{k}(2) \oplus \mathbf{k}(5)^{\oplus 4} \oplus \mathbf{k}(6), \tag{3.58}$$
$$\mathrm{HH}^{2i+4}(\mathcal{Y}) \cong \mathbf{k}(6i+4) \oplus \mathbf{k}(6i+5)^{\oplus 4} \oplus \mathbf{k}(6i+8) \oplus \mathbf{k}(6i+12) \quad \text{for } i \geq 0, \tag{3.59}$$
$$\mathrm{HH}^{2i+5}(\mathcal{Y}) \cong \mathbf{k}(6i+4) \oplus \mathbf{k}(6i+8) \oplus \mathbf{k}(6i+11)^{\oplus 4} \oplus \mathbf{k}(6i+12) \quad \text{for } i \geq 0 \tag{3.60}$$

for $n = 3$,

$$\mathrm{HH}^0(\mathcal{Y}) \cong \mathrm{HH}^1(\mathcal{Y}) \cong \mathbf{k}, \tag{3.61}$$
$$\mathrm{HH}^2(\mathcal{Y}) \cong \mathbf{k}(2) \oplus \mathbf{k}(8), \tag{3.62}$$
$$\mathrm{HH}^3(\mathcal{Y}) \cong \mathbf{k}(-1)^{\oplus 256} \oplus \mathbf{k}(1) \oplus \mathbf{k}(5), \tag{3.63}$$
$$\mathrm{HH}^4(\mathcal{Y}) \cong \mathrm{HH}^5(\mathcal{Y}) \cong \mathbf{k}(4) \oplus \mathbf{k}(7)^{\oplus 43} \oplus \mathbf{k}(10) \oplus \mathbf{k}(16), \tag{3.64}$$
$$\mathrm{HH}^{2i+6}(\mathcal{Y}) \cong \mathrm{HH}^{2i+7}(\mathcal{Y}) \tag{3.65}$$
$$\cong \mathbf{k}(8i+6) \oplus \mathbf{k}(8i+12) \oplus \mathbf{k}(8i+15)^{\oplus 43} \oplus \mathbf{k}(8i+18) \oplus \mathbf{k}(8i+24) \quad \text{for } i \geq 0 \tag{3.66}$$

for $n = 4$, and so on. There are twisted deformations for $n = 2$, but there are no twisted deformations for all $n \geq 3$.

3.4. **Sylvester's sequence.** Consider the case $\mathbf{w}(x_1, \ldots, x_n) = x_1^{s_1} + \cdots + x_n^{s_n}$ where $(s_i)_{i=1}^\infty = (2, 3, 7, 43, 1807, \ldots)$ is the Sylvester's sequence defined by $s_i = 1 + s_1 \cdots s_{i-1}$. This case appears in mirror symmetry for the Calabi–Yau hypersurface in $\mathbb{P}(1, s_1, \ldots, s_n)$, and gives the cusp singularity $x^2 + y^3$ for $n = 2$. One has

$$(d_0, d_1, \ldots, d_n; h) = (1, h/s_1, \ldots, h/s_n; s_{n+1} - 1) \tag{3.67}$$

and $\phi \colon \mathbb{G}_\mathrm{m} \to \Gamma$ is an isomorphism.

When γ is the identity element, one has $V_\gamma = V$, $N_\gamma = 0$, $\mathbf{W}_\gamma = \mathbf{w}$ and

$$\mathrm{Jac}_\mathbf{w} \cong \mathbf{k}[x_1, \ldots, x_n]/(s_1 x_1^{s_1 - 1}, \ldots, s_n x_n^{s_n - 1}). \tag{3.68}$$

The monomial $x_0^{w_j + (u-1)h} x_1^{j_1} \cdots x_n^{j_n}$ from the summand

$$\left(\mathrm{Jac}_\mathbf{w} \otimes \mathbf{k}[x_0] \otimes \chi^{\otimes u}\right)^\Gamma \tag{3.69}$$

contributes $\mathbf{k}(w_{\boldsymbol{j}} + (u-1)h)$ to HH^{2u} for each $\boldsymbol{j} = (j_1, \ldots, j_n)$ satisfying $0 \le j_i \le s_i - 1$ for $i = 1, \ldots, n$ and $w_{\boldsymbol{j}} := h - d_1 j_1 - \cdots - d_n j_n \ge -(u-1)h$. Such $\boldsymbol{j}$ also contributes $\mathbf{k}(w_{\boldsymbol{j}} + (u-1)h)$ to HH^{2u+1} just as in Section 3.2.

Each γ with $V_\gamma = 0$ contributes $\mathbf{k}(-1)$ to HH^{n-1}. The set of such γ can be identified with the set of integers from 0 to $h-1$ prime to all s_i for $i = 1, \ldots, n$. The cardinality of this set is given by $2, 12, 504, \ldots$ for $n = 2, 3, 4, \ldots$ respectively.

One never has $V_\gamma = \mathbf{k} x_0$ in this case. For any γ with $V_\gamma \neq 0, V$ does not contribute to HH^* just as in Section 3.2.

The result is summarized as

$$\mathrm{HH}^0(\mathcal{Y}) \cong \mathbf{k}, \tag{3.70}$$

$$\mathrm{HH}^1(\mathcal{Y}) \cong \mathbf{k} \oplus \mathbf{k}(-1)^{\oplus 2}, \tag{3.71}$$

$$\mathrm{HH}^{2i+2}(\mathcal{Y}) \cong \mathrm{HH}^{2i+3}(\mathcal{Y}) \cong \mathbf{k}(6i+4) \oplus \mathbf{k}(6i+6) \quad \text{for } i \ge 0 \tag{3.72}$$

for $n = 2$,

$$\mathrm{HH}^0(\mathcal{Y}) \cong \mathbf{k}, \tag{3.73}$$

$$\mathrm{HH}^1(\mathcal{Y}) \cong \mathbf{k}, \tag{3.74}$$

$$\mathrm{HH}^2(\mathcal{Y}) \cong \mathbf{k}(-1)^{\oplus 12} \oplus \mathbf{k}(\boldsymbol{w}), \tag{3.75}$$

$$\mathrm{HH}^3(\mathcal{Y}) \cong \mathbf{k}(\boldsymbol{w}), \tag{3.76}$$

$$\mathrm{HH}^{2i+4}(\mathcal{Y}) \cong \mathrm{HH}^{2i+5}(\mathcal{Y}) \cong \mathbf{k}(\widetilde{\boldsymbol{w}} + 42(i+1)) \quad \text{for } i \ge 0 \tag{3.77}$$

where $\boldsymbol{w} = (4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42)$ and $\widetilde{\boldsymbol{w}} = (-2, \boldsymbol{w})$ for $n = 3$, and so on. There are no twisted deformations for all $n \ge 2$.

3.5. **Exceptional unimodal singularities.** Consider the weighted homogeneous polynomials given in Table 3.1, which define Arnold's 14 exceptional unimodal singularities [4, Table 14]. We take $\Gamma = \phi(\mathbb{G}_{\mathrm{m}})$. The Hilbert polynomial for the Jacobi ring

$$\mathrm{Jac}_{\mathbf{w}} := \mathbf{k}[x_1, x_2, x_3]/(\partial_1 \mathbf{w}, \partial_2 \mathbf{w}, \partial_3 \mathbf{w}) \tag{3.78}$$

is given by

$$\frac{(1 - T^{h-d_1})(1 - T^{h-d_2})(1 - T^{h-d_3})}{(1 - T^{d_1})(1 - T^{d_2})(1 - T^{d_3})}. \tag{3.79}$$

We define a non-decreasing sequence $\widetilde{\boldsymbol{w}} = (w_0 \le \cdots \le w_{\mu-1})$ of integers in such a way that (3.79) is equal to $\sum_{i=0}^{\mu-1} T^{h-w_i}$. Then one always has $w_0 = -2$, and $\boldsymbol{w} := (w_i)_{i=1}^{\mu-1}$ is as in Table 3.1. The identity element $\gamma = \mathrm{id}_V$ contributes $\mathbf{k}$ to HH^0 and HH^1, $\mathbf{k}(\boldsymbol{w})$ to HH^2 and HH^3, and $\mathbf{k}(\widetilde{\boldsymbol{w}} + (i+1)h)$ to HH^{2i+4} and HH^{2i+5} for $i \ge 0$. By adding the term x_0^h, one obtains a smooth Deligne–Mumford stack $\mathcal{Y}_1$ derived-equivalent to a K3 surface. Since V^γ for $\gamma \neq \mathrm{id}_V$ does not contain the x_0-axis, contributions from $\gamma \neq \mathrm{id}_V$ is the same for $\mathcal{Y}$ and $\mathcal{Y}_1$. On the other hand, the rank of the total Hochschild cohomology of $\mathcal{Y}_1$ is 24, and $\gamma = \mathrm{id}_V$ contributes $\mathbf{k}$ to $\mathrm{HH}^0(\mathcal{Y}_1)$ via the element $1 \in \mathrm{Jac}_{\mathbf{w}}$ of degree 0, $\mathbf{k}^{\oplus(\mu-2)}$

Name	Normal form	$(d_1, d_2, d_3; h)$	μ	$\boldsymbol{w}$
Q_{10}	$x^2z + y^3 + z^4$	$(9, 8, 6; 24)$	10	$(4, 6, 7, 10, 12, 15, 16, 18, 24)$
Q_{11}	$x^2z + y^3 + yz^3$	$(7, 6, 4; 18)$	11	$(2, 4, 5, 6, 8, 10, 11, 12, 14, 18)$
Q_{12}	$x^2z + y^3 + z^5$	$(6, 5, 3; 15)$	12	$(1, 3, 4, 4, 6, 7, 9, 9, 10, 12, 15)$
Z_{11}	$x^2 + y^3z + z^5$	$(15, 8, 6; 30)$	11	$(4, 6, 10, 12, 14, 16, 18, 22, 24, 30)$
Z_{12}	$x^2 + y^3z + yz^4$	$(11, 6, 4; 22)$	12	$(2, 4, 6, 8, 10, 10, 12, 14, 16, 18, 22)$
Z_{13}	$x^2 + y^3z + z^6$	$(9, 5, 3; 18)$	13	$(1, 3, 4, 6, 7, 8, 9, 10, 12, 13, 15, 18)$
S_{11}	$x^2z + xy^2 + z^4$	$(6, 5, 4; 16)$	11	$(2, 3, 4, 6, 7, 8, 10, 11, 12, 16)$
S_{12}	$x^2z + xy^2 + yz^3$	$(5, 4, 3; 13)$	12	$(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13)$
W_{12}	$x^2 + y^4 + z^5$	$(10, 5, 4; 20)$	12	$(2, 3, 6, 7, 8, 10, 11, 12, 15, 16, 20)$
W_{13}	$x^2 + y^4 + yz^4$	$(8, 4, 3; 16)$	13	$(1, 2, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16)$
E_{12}	$x^2 + y^3 + z^7$	$(21, 14, 6; 42)$	12	$(4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42)$
E_{13}	$x^2 + y^3 + yz^5$	$(15, 10, 4; 30)$	13	$(2, 6, 8, 10, 12, 14, 16, 18, 20, 22, 26, 30)$
E_{14}	$x^2 + y^3 + z^8$	$(12, 8, 3; 24)$	14	$(1, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 21, 24)$
U_{12}	$x^3 + y^3 + z^4$	$(4, 4, 3; 12)$	12	$(1, 2, 2, 4, 5, 5, 6, 8, 8, 9, 12)$

(3.85)

TABLE 3.1. 14 exceptional unimodal singularities

to $\mathrm{HH}^2(\mathcal{Y}_1)$ via elements of degrees between 1 and $h+1$, and $\mathbf{k}$ to HH^4 via the element of degree $h+2$. It follows that $\gamma \neq \mathrm{id}_V$ contribute $\mathbf{k}^{\oplus(24-\mu)}$ to $\mathrm{HH}^2(\mathcal{Y}_1)$. Since V_γ does not contain the x_0-axis, each of these contributions contains $x_0^\vee$ from $\Lambda^{\dim N_\gamma} N_\gamma$, and hence the $\mathbb{G}_\mathrm{m}$-weight for the contribution to $\mathrm{HH}^2(\mathcal{Y})$ is 1. This shows

$$\mathrm{HH}^0(\mathcal{Y}) \cong \mathbf{k}, \tag{3.80}$$

$$\mathrm{HH}^1(\mathcal{Y}) \cong \mathbf{k}, \tag{3.81}$$

$$\mathrm{HH}^2(\mathcal{Y}) \cong \mathbf{k}(-1)^{\oplus(24-\mu)} \oplus \mathbf{k}(\boldsymbol{w}), \tag{3.82}$$

$$\mathrm{HH}^3(\mathcal{Y}) \cong \mathbf{k}(\boldsymbol{w}), \tag{3.83}$$

$$\mathrm{HH}^{2i+4}(\mathcal{Y}) \cong \mathrm{HH}^{2i+5}(\mathcal{Y}) \cong \mathbf{k}(\widetilde{\boldsymbol{w}} + (i+1)h) \quad \text{for } i \geq 0. \tag{3.84}$$

There are no twisted deformations in all these cases.

3.6. **Cusp singularities.** Consider the case

$$\mathbf{W}(x_0, \ldots, x_n) = x_1^{n+1} + \cdots + x_n^{n+1} + x_0 \cdots x_n \tag{3.86}$$

with the same weight (3.16) and the group (3.17) as in Section 3.2.

When γ is the identity element, one has $V_\gamma = V$, $N_\gamma = 0$, and $\mathbf{W}_\gamma = \mathbf{W}$. The subring of S consisting of semi-invariants with respect to χ is equal to the invariant ring with respect to $\ker \chi \cong (\boldsymbol{\mu}_{n+1})^n$. This ring is generated by $n+2$ monomials $x_0^{n+1}, \ldots, x_n^{n+1}, x_0 \cdots x_n$ with one relation $x_0^{n+1} \cdots x_n^{n+1} = (x_0 \cdots x_n)^{n+1}$. The $n+1$ monomials $x_1^{n+1}, \dots, x_n^{n+1}, x_0 \cdots x_n$ are zero in $\operatorname{Jac}_{\mathbf{W}}$, so that

$$\dim \left(\operatorname{Jac}_{\mathbf{W}} \otimes \chi^{\otimes u} \right)^{\Gamma} = \begin{cases} 0 & u \le -1, \\ 1 & u \ge 0. \end{cases} \tag{3.87}$$

The Grothendieck ring $\operatorname{rep}_\Gamma$ of finite-dimensional Γ-vector spaces can be identified with the group ring of $\widehat{\Gamma}$, generated by $[x_0], \dots, [x_n]$ and their inverses with relations $[x_0]^{n+1} = \cdots = [x_n]^{n+1} = [x_0] \cdots [x_n]$. The ring S is a $\widehat{\Gamma}$-graded ring, and the class $[C^*(d\mathbf{W})]$ of the Koszul complex is an element of a suitable completion of $\operatorname{rep}_\Gamma$ given by

$$[C^*(d\mathbf{W})] = (1 + [x_0] + \cdots + [x_0]^{n-1}) \cdots (1 + [x_n] + \cdots + [x_n]^{n-1}). \tag{3.88}$$

Among n^{n+1} monomials in (3.88), only $[x_0]^i \cdots [x_n]^i$ for $i = 0, \dots, n-1$ are proportional to a power of $[\chi]$. By projecting to the subring generated by $T \coloneqq [x_0] \cdots [x_n]$, one obtains

$$\left[\left(C^*(d\mathbf{W}) \right)^{\Gamma} \right] = 1 + T + \cdots + T^{n-1}. \tag{3.89}$$

Since $(\partial_i \mathbf{W})_{i=0}^{n-1}$ is a regular sequence in S, the cohomology of the Koszul complex is concentrated in degree -1 and 0. It follows that

$$[\operatorname{Jac}_{\mathbf{W}}] - \left[H^{-1}(d\mathbf{W}) \right] = 1 + T + \cdots + T^{n-1}, \tag{3.90}$$

so that

$$\dim \left(H^{-1}(d\mathbf{W}) \otimes \chi^{\otimes(u+1)} \right)^{\Gamma} = \begin{cases} 0 & u \le n-2, \\ 1 & u \ge n-1. \end{cases} \tag{3.91}$$

Hence $\gamma = 0$ contributes $\mathbf{k}$ to HH^{2u} for $u \ge 0$ and HH^{2u+1} for $u \ge n-1$.

Contributions from non-trivial γ is the same as in Section 3.2. The result is summarized as

$$\operatorname{HH}^0(\mathcal{Y}) \cong \mathbf{k}, \tag{3.92}$$
$$\operatorname{HH}^1(\mathcal{Y}) \cong \mathbf{k}^{\oplus 4}, \tag{3.93}$$
$$\operatorname{HH}^{i+2}(\mathcal{Y}) \cong \mathbf{k}^{\oplus 3} \quad \text{for } i \ge 0 \tag{3.94}$$

for $n = 2$,

$$\operatorname{HH}^0(\mathcal{Y}) \cong \mathbf{k}, \tag{3.95}$$
$$\operatorname{HH}^1(\mathcal{Y}) \cong 0, \tag{3.96}$$
$$\operatorname{HH}^2(\mathcal{Y}) \cong \mathbf{k}^{\oplus 28}, \tag{3.97}$$
$$\operatorname{HH}^3(\mathcal{Y}) \cong \mathbf{k}^{\oplus 6}, \tag{3.98}$$
$$\operatorname{HH}^{4+i}(\mathcal{Y}) \cong \mathbf{k}^{\oplus 7} \quad \text{for } i \ge 0 \tag{3.99}$$

for $n = 3$,

$$\mathrm{HH}^0(\mathcal{Y}) \cong \mathbf{k}, \tag{3.100}$$
$$\mathrm{HH}^1(\mathcal{Y}) \cong 0, \tag{3.101}$$
$$\mathrm{HH}^2(\mathcal{Y}) \cong \mathbf{k}, \tag{3.102}$$
$$\mathrm{HH}^3(\mathcal{Y}) \cong \mathbf{k}^{\oplus 256}, \tag{3.103}$$
$$\mathrm{HH}^4(\mathcal{Y}) \cong \mathbf{k}^{\oplus 53}, \tag{3.104}$$
$$\mathrm{HH}^5(\mathcal{Y}) \cong \mathbf{k}^{\oplus 52}, \tag{3.105}$$
$$\mathrm{HH}^{6+i}(\mathcal{Y}) \cong \mathbf{k}^{\oplus 53} \quad \text{for } i \geq 0 \tag{3.106}$$

for $n = 4$, and so on.

Similarly, the case

$$\mathbf{W}(x_0, \dots, x_n) = x_1^2 + x_2^{2n} + \cdots + x_n^{2n} + x_0 \cdots x_n \tag{3.107}$$

with the same weight (3.44) and the group (3.45) as in Section 3.3 gives

$$\mathrm{HH}^0(\mathcal{Y}) \cong \mathbf{k}, \tag{3.108}$$
$$\mathrm{HH}^1(\mathcal{Y}) \cong \mathbf{k}^{\oplus 3}, \tag{3.109}$$
$$\mathrm{HH}^{i+2}(\mathcal{Y}) \cong \mathbf{k}^{\oplus 2} \quad \text{for } i \geq 0 \tag{3.110}$$

for $n = 2$,

$$\mathrm{HH}^0(\mathcal{Y}) \cong \mathbf{k}, \tag{3.111}$$
$$\mathrm{HH}^1(\mathcal{Y}) \cong 0, \tag{3.112}$$
$$\mathrm{HH}^2(\mathcal{Y}) \cong \mathbf{k}^{\oplus 26}, \tag{3.113}$$
$$\mathrm{HH}^3(\mathcal{Y}) \cong \mathbf{k}^{\oplus 4}, \tag{3.114}$$
$$\mathrm{HH}^{4+i}(\mathcal{Y}) \cong \mathbf{k}^{\oplus 5} \quad \text{for } i \geq 0 \tag{3.115}$$

for $n = 3$, and so on.

3.7. **Ordinary double points.** Consider the case $\mathbf{W}(x_0, x_1, \dots, x_{n+1}) = x_0^{n+1} + \cdots + x_n^{n+1} - (n+1) x_0 \cdots x_n$ with the same weight (3.16) and the group (3.17) as in Section 3.2.

When γ is the identity element, one has $V_\gamma = V$, $N_\gamma = 0$, and $\mathbf{W}_\gamma = \mathbf{W}$. The generators $x_0^{n+1}, \dots, x_n^{n+1}, x_0 \cdots x_n$ of the invariant ring $S^{\ker \chi}$ belongs to the same class in $\operatorname{Jac}_{\mathbf{W}}$, so that

$$\dim \left(H^0(d\mathbf{W}) \otimes \chi^{\otimes k} \right)^\Gamma = \begin{cases} 0 & k \leq -1, \\ 1 & k \geq 0. \end{cases} \tag{3.116}$$

The same reasoning as in Section 3.6 shows that $\gamma = 0$ contributes $\mathbf{k}$ to HH^{2i} for $i \geq 0$ and HH^{2i+1} for $i \geq 2$.

Contributions from non-trivial γ is the same as in Section 3.6, except that the coordinate x_0 behaves exactly the same way as other coordinates. The result is summarized as

$$\mathrm{HH}^0(\mathcal{Y}) \cong \mathbf{k}, \tag{3.117}$$
$$\mathrm{HH}^1(\mathcal{Y}) \cong \mathbf{k}^{\oplus 2}, \tag{3.118}$$
$$\mathrm{HH}^{i+2}(\mathcal{Y}) \cong \mathbf{k} \quad \text{for } i \geq 0 \tag{3.119}$$

for $n = 2$,

$$\mathrm{HH}^0(\mathcal{Y}) \cong \mathbf{k}, \tag{3.120}$$
$$\mathrm{HH}^1(\mathcal{Y}) \cong 0, \tag{3.121}$$
$$\mathrm{HH}^2(\mathcal{Y}) \cong \mathbf{k}^{\oplus 22}, \tag{3.122}$$
$$\mathrm{HH}^3(\mathcal{Y}) \cong 0, \tag{3.123}$$
$$\mathrm{HH}^{4+i}(\mathcal{Y}) \cong \mathbf{k} \quad \text{for } i \geq 0 \tag{3.124}$$

for $n = 3$,

$$\mathrm{HH}^0(\mathcal{Y}) \cong \mathbf{k}, \tag{3.125}$$
$$\mathrm{HH}^1(\mathcal{Y}) \cong 0, \tag{3.126}$$
$$\mathrm{HH}^2(\mathcal{Y}) \cong \mathbf{k}, \tag{3.127}$$
$$\mathrm{HH}^3(\mathcal{Y}) \cong \mathbf{k}^{\oplus 204}, \tag{3.128}$$
$$\mathrm{HH}^4(\mathcal{Y}) \cong \mathbf{k}, \tag{3.129}$$
$$\mathrm{HH}^5(\mathcal{Y}) \cong 0, \tag{3.130}$$
$$\mathrm{HH}^{6+i}(\mathcal{Y}) \cong \mathbf{k} \quad \text{for } i \geq 0 \tag{3.131}$$

for $n = 4$, and so on.

Similarly, the case

$$\mathbf{W}(x_0, \ldots, x_n) = x_1^2 + x_2^{2n} + \cdots + x_n^{2n} + x_0^{2n} + x_0 \cdots x_n \tag{3.132}$$

with the same weight (3.44) and the group (3.45) as in Section 3.3 gives

$$\mathrm{HH}^0(\mathcal{Y}) \cong \mathbf{k}, \tag{3.133}$$
$$\mathrm{HH}^1(\mathcal{Y}) \cong \mathbf{k}^{\oplus 2}, \tag{3.134}$$
$$\mathrm{HH}^{i+2}(\mathcal{Y}) \cong \mathbf{k} \quad \text{for } i \geq 0 \tag{3.135}$$

for $n = 2$,

$$\mathrm{HH}^0(\mathcal{Y}) \cong \mathbf{k}, \tag{3.136}$$
$$\mathrm{HH}^1(\mathcal{Y}) \cong 0, \tag{3.137}$$
$$\mathrm{HH}^2(\mathcal{Y}) \cong \mathbf{k}^{\oplus 22}, \tag{3.138}$$
$$\mathrm{HH}^3(\mathcal{Y}) \cong 0, \tag{3.139}$$
$$\mathrm{HH}^{4+i}(\mathcal{Y}) \cong \mathbf{k} \quad \text{for } i \geq 0 \tag{3.140}$$

for $n = 3$, and so on.

4. Generators and formality

We use the same notation as in Section 2, and assume the existence of a tilting object E of $\operatorname{mf}(\mathbb{A}^n, \mathbf{w}, \Gamma)$. Let $\mathcal{E}$ be the pull-back of E to $\operatorname{mf}(\mathbb{A}^n_U, \Gamma, \mathbf{w})$, so that one has $\operatorname{End}(\mathcal{E}) \cong A^0 \otimes \boldsymbol{k}$ where $\boldsymbol{k} := \mathbf{k}[U]$ is the coordinate ring of U and $A^0 := \operatorname{End} E$. Let further $\mathcal{S}$ be the push-forward of $\mathcal{E}$ to $\operatorname{mf}(\mathbb{A}^{n+1}_U, \Gamma, \mathbf{W})$, considered as an object of $\operatorname{coh} \mathscr{Y}$ via a variation

$$\operatorname{mf}(\mathbb{A}^{n+1}_U, \Gamma, \mathbf{W}) \simeq \operatorname{coh} \mathscr{Y} \tag{4.1}$$

of [61, Theorem 16], which can be proved by a straightforward adaptation of the original proof (see the proof of Theorem 4.1 below).

Theorem 4.1. *The object* $\mathcal{S}$ *split-generates* $\operatorname{perf} \mathscr{Y}$.

Proof. For the simplicity of notation, we assume $\Gamma \cong \mathbb{G}_\mathrm{m}$, so that $\mathscr{Y}$ is an anti-canonical hypersurface in $\mathbb{P} := \mathbb{P}_U(d_0, \ldots, d_n)$; the extension to the general case is straightforward (cf. e.g., [77, Section 3]). We write $\boldsymbol{R} := \boldsymbol{k}[x_0, \ldots, x_n]/(\mathbf{W})$ and $\overline{\boldsymbol{R}} := \boldsymbol{k}[x_1, \ldots, x_n]/(\mathbf{w}) \cong \boldsymbol{R}/(x_0) \cong \overline{R} \otimes \boldsymbol{k}$. We will work with $D^b_{\mathrm{sing}}\left(\operatorname{gr} \overline{\boldsymbol{R}}\right)$ and $D^b_{\mathrm{sing}}(\operatorname{gr} \boldsymbol{R})$ instead of $\operatorname{mf}(\mathbb{A}^n_U, \Gamma, \mathbf{w})$ and $\operatorname{mf}(\mathbb{A}^{n+1}_U, \Gamma, \mathbf{W})$, which are equivalent by [61, Theorem 39]. Since the object $\overline{\boldsymbol{R}}/(x_1, \ldots, x_n)$ of $D^b_{\mathrm{sing}}\left(\operatorname{gr} \overline{\boldsymbol{R}}\right)$ can be described as a cone constructed out of $\mathcal{E}$, and its push-forward to $D^b_{\mathrm{sing}}(\operatorname{gr} \boldsymbol{R})$ is $\boldsymbol{R}/\mathfrak{m}$ where $\mathfrak{m} := (x_0, \ldots, x_n)$, it suffices to show that the images of $\boldsymbol{R}/\mathfrak{m}(i)$ for $i \in \mathbb{Z}$ under the equivalence

$$D^b_{\mathrm{sing}}(\operatorname{gr} \boldsymbol{R}) \cong \operatorname{coh} \mathscr{Y} \tag{4.2}$$

split-generate $\operatorname{perf} \mathscr{Y}$. One has

$$\hom_{\boldsymbol{R}}(\boldsymbol{R}/\mathfrak{m}(-i), \boldsymbol{R}(j)) = \begin{cases} \boldsymbol{k}[-n] & i = -j, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3}$$

and the proof of [61, Theorem 16] gives semiorthogonal decompositions

$$D^b(\operatorname{gr} \boldsymbol{R}_{\geq 0}) = \langle \mathcal{D}_0, \mathcal{S}_{\geq 0} \rangle = \langle \mathcal{P}_{\geq 0}, \mathcal{T}_0 \rangle, \tag{4.4}$$

equivalences

$$\mathcal{D}_0 \cong \operatorname{coh} \mathscr{Y}, \quad \mathcal{T}_0 \cong D^b_{\mathrm{sing}}(\operatorname{gr} \boldsymbol{R}), \tag{4.5}$$

and an equality

$$\mathcal{D}_0 = \mathcal{T}_0, \tag{4.6}$$

where $D^b(\operatorname{gr} \boldsymbol{R}_{\geq 0})$ is the derived category of finitely-generated non-negatively graded $\boldsymbol{R}$-modules, and $\mathcal{S}_{\geq 0}$ and $\mathcal{P}_{\geq 0}$ are its full subcategories generated by torsion modules (i.e., modules M such that $\mathfrak{m}^k M = 0$ for some $k \in \mathbb{N}$ which may depend on M) and free modules respectively. In order to send an object $\overline{Z} \in D^b_{\mathrm{sing}}(\operatorname{gr} \boldsymbol{R})$ by the equivalence

$$D^b_{\mathrm{sing}}(\operatorname{gr} \boldsymbol{R}) \cong \mathcal{T}_0 = \mathcal{D}_0 \cong \operatorname{coh} \mathscr{Y}, \tag{4.7}$$

we

(1) find an object $Z \in D^b(\operatorname{gr} \boldsymbol{R}_{\geq 0})$ which goes to $\overline{Z}$ by the localization functor $D^b(\operatorname{gr} \boldsymbol{R}_{\geq 0}) \to D^b_{\mathrm{sing}}(\operatorname{gr} \boldsymbol{R})$,
(2) take the semiorthogonal component M of Z, i.e., find a distinguished triangle
$$M \to Z \to N \to M[1] \tag{4.8}$$
such that $M \in \mathcal{T}_0 = {}^\perp\mathcal{P}_{\geq 0}$ and $N \in \mathcal{P}_{\geq 0}$, and
(3) take the image $\mathcal{M}$ of M by the localization functor $\pi \colon D^b(\operatorname{gr} \boldsymbol{R}_{\geq 0}) \to \operatorname{coh} \mathscr{Y}$.

If we start with $Z_i = (\boldsymbol{R}/\mathfrak{m})(-i)[-n+1]$ for $0 \leq i < h$, then

$$\operatorname{Cone}\left((\boldsymbol{R}/\mathfrak{m})(-i)[-n] \to \boldsymbol{R}(-i)\right) \tag{4.9}$$

belongs to $\mathcal{S}^\perp_{\geq i+1}$, which is equal to ${}^\perp\mathcal{P}_{\geq i+1}$ in the semiorthogonal decomposition

$$D^b(\operatorname{gr} \boldsymbol{R}_{\geq 0}) = \langle \mathcal{P}_{\geq 0}, \mathcal{T}_0 \rangle = \langle \mathcal{P}_{\geq i+1}, \boldsymbol{R}(-i), \boldsymbol{R}(-i+1), \dots, \boldsymbol{R}, \mathcal{T}_0 \rangle. \tag{4.10}$$

Since $(\boldsymbol{R}/\mathfrak{m})(-i)$ is orthogonal to $\boldsymbol{R}(-i+1), \dots, \boldsymbol{R}$ and its image in $\operatorname{coh} \mathscr{Y}$ is zero, the image $\mathcal{M}_i \in D^b \operatorname{coh} \mathscr{Y}$ of the semiorthogonal component $M_i \in \mathcal{T}_0 = \mathcal{D}_0$ of Z_i is isomorphic to the image of the semiorthogonal component of $\boldsymbol{R}(-i)$.

Since $i < h$, the operation of taking the semiorthogonal component of $\boldsymbol{R}(-i)$ is the same as that for the polynomial ring T, and the resulting object $\mathcal{M}_i$ is the restriction to $\mathscr{Y}$ of the object $\mathcal{F}_i$ in $\operatorname{coh} \mathbb{P}$ obtained by mutating $\mathcal{O}_{\mathbb{P}}(-i)$ across $\mathcal{O}_{\mathbb{P}}(-i+1), \dots, \mathcal{O}_{\mathbb{P}}$. The collection $(\mathcal{F}_i)_{i=0}^{h-1}$ is left dual to the full exceptional collection $(\mathcal{O}_{\mathbb{P}}(-i))_{i=0}^{h-1}$ by construction, and hence is full again. Now [66, Lemma 5.4] shows $\bigoplus_{i=0}^{h-1} \mathcal{M}_i$ split-generates $\operatorname{perf} \mathscr{Y}$. $\square$

It follows from [78, Theorem 1.1] that a choice of a section of $\omega_{\mathbb{A}^{n+1}_U/U}(\chi)$ gives an isomorphism $\operatorname{End}(\mathcal{S}) \cong A \otimes \mathcal{O}_U$, where A is the degree $n-1$ trivial extension algebra of A^0. Let $\mathcal{A}$ be the minimal model of the Yoneda dg algebra $\operatorname{end}(\mathcal{S})$, so that one has a quasi-equivalence

$$\operatorname{Qcoh} \mathscr{Y} \simeq \operatorname{Mod}(\mathcal{A}) \tag{4.11}$$

of $\boldsymbol{k}$-linear pretriangulated A_∞-categories.

Let $\mathcal{A}_0 \coloneqq \mathcal{A} \otimes_{\boldsymbol{k}} \mathbf{k}$ be the A_∞-algebra over $\mathbf{k}$ obtained by restricting $\mathcal{A}$ to the origin $0 \in U$. By using a $\mathbb{G}_{\mathrm{m}}$-action, we can prove the following:

Theorem 4.2. *$\mathcal{A}_0$ is formal.*

Proof. We fix a $\mathbb{G}_\mathrm{m}$-equivariant structure on $\mathcal{S}_0$ with respect to the $\mathbb{G}_\mathrm{m}$-action $(x_0, x_1, \dots, x_n) \mapsto (\alpha x_0, x_1, \dots, x_n)$ on $\mathbb{A}^{n+1}$ in such a way that $\operatorname{End}^0(\mathcal{S}_0) \cong \operatorname{End}^0(E)$ is $\mathbb{G}_\mathrm{m}$-invariant (this is possible since $\mathcal{S}_0$ is obtained by push-forward from an object on the $\mathbb{G}_\mathrm{m}$-invariant subspace). Note that $\omega_{\mathbb{A}^{n+1}}(\chi)$ is isomorphic to $\mathcal{O}_{\mathbb{A}^{n+1}}$ as a Γ-module, but has weight 1 with respect to the $\mathbb{G}_\mathrm{m}$-action. It follows that the weight for the $\mathbb{G}_\mathrm{m}$-action on $\operatorname{End}^{n-1}(\mathcal{S}_0) \cong \left(\operatorname{End}^0(E)\right)^\vee$ is one. This shows that the cohomological degree on the $\mathbb{N}$-graded algebra $\operatorname{End}^*(\mathcal{S}_0)$ is $(n-1)$ times the $\mathbb{G}_\mathrm{m}$-weight. Since the group $\mathbb{G}_\mathrm{m}$ is reductive, the chain homotopy to transfer the dg structure on $\operatorname{end}(\mathcal{S}_0)$ to the minimal model $\mathcal{A}_0$ can be chosen to be $\mathbb{G}_\mathrm{m}$-equivariant, so that the resulting A_∞-operations are $\mathbb{G}_\mathrm{m}$-equivariant. Since the A_∞-operation μ^d has the cohomological degree $2-d$ and the cohomological degree is proportional to the $\mathbb{G}_\mathrm{m}$-weight, one must have $\mu^d = 0$ for $d \neq 2$. □

As a result, we have an isomorphism

$$\mathrm{HH}^*(A) \cong \mathrm{HH}^*(\mathcal{Y}_0) \tag{4.12}$$

of graded vector spaces. Moreover, the proof of Theorem 4.2 shows that the length grading on the left hand side is mapped to $(n-1)$ times the weight of the $\mathbb{G}_\mathrm{m}$-action.

5. Moduli of A_∞-structures

We prove Theorem 1.6 in this section

Proof of Theorem 1.6. We use the same notations as in Section 4. Corollary 3.2 and (4.12) together with [63, Corollary 3.2.5] shows that the moduli functor of A_∞-structures on A is represented by an affine scheme $\mathcal{U}_\infty(A)$. We define the morphism (1.34) as the classifying morphism for the family $\mathcal{A}$ of minimal A_∞-structures on A over U. We consider the $\mathbb{G}_\mathrm{m}$-action on $\mathscr{Y}$ as in (2.20), and equip $\mathcal{S}$ with the $\mathbb{G}_\mathrm{m}$-equivariant structure such that $\operatorname{End}(\mathcal{S})$ is $\mathbb{G}_\mathrm{m}$-equivariantly isomorphic to $A \otimes \boldsymbol{k}$, where the $\mathbb{G}_\mathrm{m}$-weight on A is proportional to the cohomological grading as in the proof of Theorem 4.2. Then the dg algebra $\operatorname{end}(\mathcal{S})$ is also $\mathbb{G}_\mathrm{m}$-equivariant, and so is the A_∞-algebra $\mathcal{A}$. This means that the morphism (1.34) is $\mathbb{G}_\mathrm{m}$-equivariant.

In order to prove that φ is an isomorphism, first assume that $d_0 = 1$ and $G := \Gamma/\phi(\mathbb{G}_\mathrm{m})$ is the trivial group. Recall from [53, Section (A.5)] that an *$\overline{R}$-polarized scheme* consists of a projective scheme Y, an ample Weil divisor $X \subset Y$, and an isomorphism $R/tR \cong \overline{R}$ of graded $\mathbf{k}$-algebras, where $R := \bigoplus_{i=0}^\infty H^0(\mathcal{O}_Y(iX))$ and $t \in R_1$ is the element corresponding to 1. It is shown in [53, Proposition A.6] that U is the fine moduli space of $\overline{R}$-polarized schemes, and the universal family is given by the coarse moduli scheme $\mathcal{Y}$ of $\mathscr{Y}$. We will show that one can reconstruct the family $\mathcal{Y}$ of $\overline{R}$-polarized schemes from the family $\mathcal{A}$ of A_∞-algebras. Then the fine moduli interpretation of U gives a morphism ψ from the image of φ to U such that $\psi \circ \varphi = \mathrm{id}_U$. This implies that the map on tangent spaces induced by

φ is an injection, and hence an isomorphism since $\dim U = \dim \mathrm{HH}^2(A)_{<0} \geq \dim \mathcal{U}_\infty(A)$. Since φ is a $\mathbb{G}_\mathrm{m}$-equivariant morphism from an affine space to an affine scheme with good $\mathbb{G}_\mathrm{m}$-actions inducing an isomorphism on tangent spaces, it is an isomorphism of schemes.

In order to reconstruct the family $\mathcal{Y} \to U$ of schemes from the family $\mathcal{A}$ of A_∞-algebras, first note from Theorem 4.1 that $\mathcal{O}_{\mathscr{Y}}(i)$ for any $i \in \mathbb{Z}$ can be described as a particular object obtained from the generator $\mathcal{S}$ by taking shifts, cones, and direct summands. This allows one to reconstruct the $\mathbb{Z}$-algebra $\left(\mathrm{Hom}^0(\mathcal{O}_{\mathscr{Y}}(i), \mathcal{O}_{\mathscr{Y}}(j))\right)_{i,j\in\mathbb{Z}}$ up to isomorphism from $\mathcal{A}$. Recall that

- a $\mathbb{Z}$-*algebra* as defined in [10] is a category whose set of objects is identified with $\mathbb{Z}$,
- a $\mathbb{Z}$-graded algebra $B = \bigoplus_{i\in\mathbb{Z}} B_i$ gives a $\mathbb{Z}$-algebra $\check{B} = \bigoplus_{i,j\in\mathbb{Z}} \check{B}_{ij}$ by $\check{B}_{ij} = B_{i-j}$, and
- the category $\operatorname{Qgr} C$ for a $\mathbb{Z}$-algebra C is defined just as in the case of $\mathbb{Z}$-graded algebras, so that one has $\operatorname{Qgr} B \cong \operatorname{Qgr} \check{B}$ for any $\mathbb{Z}$-graded algebra B (see e.g. [80, Section 2]).

Note that $\mathrm{Hom}^0(\mathcal{O}_{\mathscr{Y}}(i), \mathcal{O}_{\mathscr{Y}}(j)) \cong \mathrm{Hom}^0(\mathcal{O}_{\mathcal{Y}}(i), \mathcal{O}_{\mathcal{Y}}(j))$ for any $i, j \in \mathbb{Z}$. The abelian category $\operatorname{Qcoh} \mathcal{Y}$ can be reconstructed from the $\mathbb{Z}$-algebra $\left(\mathrm{Hom}^0(\mathcal{O}_{\mathcal{Y}}(i), \mathcal{O}_{\mathcal{Y}}(j))\right)_{i,j\in\mathbb{Z}}$ (since $\operatorname{Qcoh} \mathcal{Y}$ is the Qgr of the graded ring $\bigoplus_{i\in\mathbb{Z}} H^0(\mathcal{O}_{\mathcal{Y}}(i))$, and $\left(\mathrm{Hom}^0(\mathcal{O}_{\mathcal{Y}}(i), \mathcal{O}_{\mathcal{Y}}(j))\right)_{i,j\in\mathbb{Z}}$ is isomorphic to the $\mathbb{Z}$-algebra associated with this graded ring), which in turn allows the reconstruction of $\mathcal{Y}$ by the Gabriel–Rosenberg reconstruction theorem. This allows us to recover the monoidal structure on $\operatorname{Qcoh} \mathcal{Y}$, and hence the $\mathbb{Z}$-graded ring $\bigoplus_{i\in\mathbb{Z}} H^0(\mathcal{O}_{\mathcal{Y}}(i))$, from the A_∞-algebra $\mathcal{A}$.

Since $\operatorname{coh} \mathcal{X}$ is a semiorthogonal summand of $\operatorname{mf}(\mathbb{A}^n, \Gamma, \mathbf{w})$ and the isomorphism $\operatorname{End} E \cong A^0$ is given, one has a fixed isomorphism of the homogeneous coordinate ring of the divisor $X \times U$ at infinity with $\overline{R} \otimes \mathcal{O}_U$. This concludes the reconstruction of the family of $\overline{R}$-polarized schemes from the family of A_∞-algebras in the case when $d_0 = 1$ and $\Gamma = \phi(\mathbb{G}_\mathrm{m})$.

When $\Gamma \supsetneq \phi(\mathbb{G}_\mathrm{m})$, then $G \coloneqq \Gamma/\phi(\mathbb{G}_\mathrm{m})$ acts on $\overline{R}$, and hence on X. The affine space U, defined in Section 2 as the fixed locus of the natural G-action on the positive part of $\widetilde{U}$, is the fine moduli scheme of $\overline{R}$-polarized schemes equipped with a G-action extending that on X by [53, Theorem A.2]. Now one can run exactly the same argument as above to show that φ is an isomorphism.

The generalization to the case where $d_0 \neq 1$ is completely parallel to the generalization to the case where $\Gamma \supsetneq \phi(\mathbb{G}_\mathrm{m})$ given above; if one introduces a variable t of degree 1 and set $x_0 = t^{d_0}$, then U is the fixed locus of the $\boldsymbol{\mu}_{d_0}$-action on the positive part of $\widetilde{U}$ induced by $\boldsymbol{\mu}_{d_0} \ni \zeta \colon (x_1, \ldots, x_n) \mapsto (\zeta^{d_1} x_1, \ldots, \zeta^{d_n} x_n)$. $\square$

6. Hochschild cohomology of the Fukaya category of the Milnor fiber

For an object X of an A_∞-category $\mathscr{A}$, the *left Yoneda module* $\mathscr{Y}_X^{\mathrm{l}}$ is defined on objects by

$$\mathscr{Y}_X^{\mathrm{l}}(Y) = \hom_{\mathscr{A}}(X, Y). \tag{6.1}$$

The *right Yoneda module* $\mathscr{Y}_X^{\mathrm{r}}$ is defined similarly by

$$\mathscr{Y}_X^{\mathrm{r}}(Y) = \hom_{\mathscr{A}}(Y, X). \tag{6.2}$$

There exists a full and faithful functor $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \operatorname{Bimod} \mathscr{A}$ sending $X \otimes Y$ to $\mathscr{Y}_X^{\mathrm{l}} \otimes \mathscr{Y}_Y^{\mathrm{r}}$.

For a functor $F \colon \mathscr{A} \to \mathscr{B}$, the *graph bimodule* Γ_F is the $\mathscr{B}$-$\mathscr{A}$-bimodule defined on objects by

$$\Gamma_F(b, a) = \hom_{\mathscr{B}}(F(a), b) \tag{6.3}$$

for $a \in \mathscr{A}$ and $b \in \mathscr{B}$. The composition of A_∞-functors is compatible with the tensor product of bimodules. For $F \colon \mathscr{A} \to \mathscr{B}$ and $G \colon \mathscr{B} \to \mathscr{C}$, one has

$$\Gamma_G \otimes_{\mathscr{B}} \Gamma_F \coloneqq \Gamma_G \otimes T\mathscr{B} \otimes \Gamma_F \simeq \Gamma_{G \circ F}, \tag{6.4}$$

where $T\mathscr{B}$ is the bar complex of $\mathscr{B}$.

The Hochschild cohomology of an A_∞-category $\mathscr{A}$ is defined as the endomorphism of the *diagonal bimodule*, which in turn is defined as the graph bimodule $\Delta_{\mathscr{A}} \coloneqq \Gamma_{\mathrm{id}_{\mathscr{A}}}$ of the identity functor $\mathrm{id}_{\mathscr{A}}$.

Although Hochschild cohomology is less functorial than Hochschild homology, it has the restriction morphism $F^* \colon \operatorname{HH}^*(\mathscr{B}) \to \operatorname{HH}^*(\mathscr{A})$ with respect to a full and faithful functor $F \colon \mathscr{A} \to \mathscr{B}$.

Theorem 6.1 ([76, Corollary 8.2], cf. also [55] and references therein)**.** *The restriction morphism with respect to the Yoneda embedding $\mathscr{A} \to \operatorname{Mod}(\mathscr{A})$ is an isomorphism.*

Corollary 6.2. *If $\mathscr{A}$ is a full subcategory of $\mathscr{B}$ and $\mathscr{B}$ is a full subcategory of $\operatorname{Mod} \mathscr{A}$, then $\operatorname{HH}^*(\mathscr{A})$ is isomorphic to $\operatorname{HH}^*(\mathscr{B})$.*

Proof. Let $F \colon \mathscr{A} \to \mathscr{B}$ and $G \colon \mathscr{B} \to \operatorname{Mod} \mathscr{A}$ be the full embedding, and

$$H \colon \operatorname{Mod} \mathscr{A} \to \operatorname{Mod} \operatorname{Mod} \mathscr{A} \to \operatorname{Mod} \mathscr{B} \tag{6.5}$$

be the composition of the Yoneda embedding and the pull-back along G. The sequence

$$\mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \operatorname{Mod} \mathscr{A} \xrightarrow{H} \operatorname{Mod} \mathscr{B} \tag{6.6}$$

of full and faithful functors gives a sequence

$$\operatorname{HH}^*(\operatorname{Mod} \mathscr{B}) \xrightarrow{H^*} \operatorname{HH}^*(\operatorname{Mod} \mathscr{A}) \xrightarrow{G^*} \operatorname{HH}^*(\mathscr{B}) \xrightarrow{F^*} \operatorname{HH}^*(\mathscr{A}) \tag{6.7}$$

of restriction morphisms. Then G^* is surjective since $G^* \circ H^*$ is an isomorphism, and G^* is injective since $F^* \circ G^*$ is an isomorphism. $\square$

The diagonal argument [7] shows the following:

Lemma 6.3. *If $\mathscr{A}$ is a full subcategory of $\mathscr{B}$ and the diagonal bimodule $\Delta_{\mathscr{B}}$ is a colimit of objects in the image of the composition $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \mathscr{B}^{\mathrm{op}} \otimes \mathscr{B} \to \operatorname{Bimod} \mathscr{B}$, then $\mathscr{B}$ is a full subcategory of $\operatorname{Mod} \mathscr{A}$.*

Let $\check{V}$ be the Milnor fiber of a weighted homogeneous polynomial $\check{\mathbf{w}} \colon \mathbb{C}^n \to \mathbb{C}$ with an isolated critical point at the origin. The Fukaya category $\mathcal{F}\left(\check{V}\right)$ is a full subcategory of the wrapped Fukaya category $\mathcal{W}\left(\check{V}\right)$. Let $(S_i)_{i=1}^{\mu}$ be a distinguished basis of vanishing cycles, and $\mathcal{S}$ be the full subcategory of $\mathcal{F}\left(\check{V}\right)$ consisting of $(S_i)_{i=1}^{\mu}$. The total morphism A_∞-algebra of $\mathcal{S}$ will be denoted by

$$\mathcal{A} \coloneqq \bigoplus_{i,j=1}^{\mu} \hom_{\mathcal{F}\left(\check{V}\right)}(S_i, S_j). \tag{6.8}$$

We assume

$$\check{d}_0 \coloneqq \check{h} - \check{d}_1 - \cdots - \check{d}_n \neq 0. \tag{6.9}$$

It is shown in [68, 4.c] that

$$\left(T_{S_1} \circ \cdots \circ T_{S_\mu}\right)^{\check{h}} = [2\check{d}_0]. \tag{6.10}$$

It follows by [66, Lemma 5.4] that $\mathcal{S}$ split-generates $\mathcal{F}\left(\check{V}\right)$, so that

$$\mathcal{F}\left(\check{V}\right) \cong \operatorname{perf} \mathcal{S} \tag{6.11}$$

and hence

$$\mathrm{HH}^*\left(\mathcal{F}\left(\check{V}\right)\right) \cong \mathrm{HH}^*\left(\mathcal{S}\right). \tag{6.12}$$

Theorem 6.4. *Under the assumption* (6.9)*, one has an isomorphism*

$$\mathrm{HH}^*\left(\mathcal{W}\left(\check{V}\right)\right) \cong \mathrm{HH}^*\left(\mathcal{S}\right). \tag{6.13}$$

Theorem 6.4 fails without (6.9); one can take $\check{\mathbf{w}} = x^2 + y^2$ as a counter-example. Theorem 6.4 should be understood as a consequence of Koszul duality between $\mathcal{F}\left(\check{V}\right)$ and $\mathcal{W}\left(\check{V}\right)$ (see [22, 21, 52]), although we do not give a proof of this here.

Recall that a Liouville manifold is said to be *non-degenerate* if there is a finite collection of Lagrangians such that the open-closed map from the Hochschild homology of the full subcategory of the wrapped Fukaya category consisting of them to the symplectic cohomology hits the identity element [2]. Any Weinstein manifold is known to be non-degenerate [15] [30].

Theorem 6.5 ([29])**.** *There is an A_∞-category $\mathcal{W}^2\left(\check{V}\right)$, containing product Lagrangians $L \times L'$ and the diagonal $\Delta_{\check{V}}$ in $\check{V}^- \times \check{V}$, and an A_∞-functor*

$$\mathbf{M} \colon \mathcal{W}^2\left(\check{V}\right) \to \operatorname{Bimod} \mathcal{W}\left(\check{V}\right) \tag{6.14}$$

which is full on the full subcategory of $\mathcal{W}^2\left(\check{V}\right)$ *consisting of product Lagrangians, and* $\Delta_{\check{V}}$ *is sent to the diagonal bimodule* $\Delta_{\mathcal{W}(\check{V})}$*. One has*

$$\mathrm{Hom}^*_{\mathcal{W}^2(\check{V})}\left(\Delta_{\check{V}}, \Delta_{\check{V}}\right) \cong \mathrm{SH}^*\left(\check{V}\right). \tag{6.15}$$

If $\check{V}$ *is non-degenerate, then* $\Delta_{\check{V}}$ *is split-generated by product Lagrangians, and* $\mathbf{M}$ *induces an isomorphism*

$$\mathrm{Hom}^*_{\mathcal{W}^2(\check{V})}(\Delta_{\check{V}}, \Delta_{\check{V}}) \cong \mathrm{HH}^*\left(\mathcal{W}\left(\check{V}\right)\right). \tag{6.16}$$

Theorem 6.4 combined with Theorem 6.5 gives a proof of [69, Conjecture 4] in our case:

Corollary 6.6. *Under the assumption* (6.9)*, one has an isomorphism*

$$\mathrm{SH}^*\left(\check{V}\right) \cong \mathrm{HH}^*\left(\mathcal{F}\left(\check{V}\right)\right). \tag{6.17}$$

To prove Theorem 6.4, it suffices to show the following:

Proposition 6.7. $\Delta_{\mathcal{W}(\check{V})}$ *is a colimit of objects of the image of* $\mathcal{S}^{\mathrm{op}} \otimes \mathcal{S}$ *in* $\mathrm{Bimod}\,\mathcal{W}\left(\check{V}\right)$*.*

Proof. We write $\mathcal{W} = \mathcal{W}\left(\check{V}\right)$ and $\Delta = \Delta_{\mathcal{W}}$. It suffices to find a sequence

$$\begin{tikzcd} \Delta \arrow[rr] & & \Delta[a] \arrow[rr] \arrow[dl] & & \Delta[2a] \arrow[r] \arrow[dl] & \cdots \\ & U_1' \arrow[ul, dotted] & & U_2' \arrow[ul, dotted] & & \end{tikzcd} \tag{6.18}$$

with $a \neq 0$ and $U_m' \in \mathcal{S}^{\mathrm{op}} \otimes \mathcal{S}$. Then the octahedral axiom gives distinguished triangles

$$U_m \to U_{m+1} \to U_{m+1}' \xrightarrow{[1]} \tag{6.19}$$

and

$$\Delta \to \Delta[ma] \to U_m \xrightarrow{[1]} \tag{6.20}$$

with $U_m \in \mathcal{S}^{\mathrm{op}} \otimes \mathcal{S}$. Then for any $X, Y \in \mathcal{W}$, one has $\mathrm{colim}_m\, U_m[-1](X,Y) \cong \Delta(X,Y)$ in $D(\mathbb{C})$, and hence $\mathrm{colim}_m\, U_m[-1] \cong \Delta$ in $\mathrm{Bimod}\,\mathcal{W}$.

For $S \in \mathcal{S}$, the *dual twist functor* is defined on objects by the distinguished triangle

$$T_S^\vee(X) \to X \xrightarrow{\mathrm{ev}^\vee} \hom(X,S)^\vee \otimes S \xrightarrow{[1]}. \tag{6.21}$$

This gives

$$\hom(S,Y) \otimes \hom(X,S) \to \hom(X,Y) \to \hom(T_S^\vee(X), Y) \xrightarrow{[1]}, \tag{6.22}$$

so that

$$\Delta_S \to \Delta \to \Gamma_{T_S^\vee} \xrightarrow{[1]} \tag{6.23}$$

where $\Delta_S := \mathscr{Y}_S^{\mathrm{l}} \otimes \mathscr{Y}_S^{\mathrm{r}}$.

For a pair (S_1, S_2) of spherical objects, one has the distinguished triangle

$$\hom(S_2, Y) \otimes \hom(T_{S_1}^\vee(X), S_2) \to \hom(T_{S_1}^\vee(X), Y) \to \hom(T_{S_2}^\vee \circ T_{S_1}^\vee(X), Y) \xrightarrow{[1]} . \tag{6.24}$$

Since the dual twist functor is inverse to the *twist functor* defined on objects by

$$X \mapsto T_S(X) \coloneqq \{\hom(S, X) \otimes S \to X\}, \tag{6.25}$$

the right $\mathcal{W}$-module

$$X \mapsto \hom(T_{S_1}^\vee(X), S_2) \tag{6.26}$$

is isomorphic to the right $\mathcal{W}$-module

$$X \mapsto \hom(X, T_{S_1}(S_2)), \tag{6.27}$$

so that the above distinguished triangle gives a distinguished triangle

$$\mathscr{Y}_{S_2}^{\mathrm{l}} \otimes \mathscr{Y}_{T_{S_1}(S_2)}^{\mathrm{r}} \to \Gamma_{T_{S_1}^\vee} \to \Gamma_{T_{S_2}^\vee \circ T_{S_1}^\vee} \xrightarrow{[1]} \tag{6.28}$$

of $\mathcal{W}$-bimodules. Similarly, one obtains

$$\mathscr{Y}_{S_3}^{\mathrm{l}} \otimes \mathscr{Y}_{T_{S_1} \circ T_{S_2}(S_3)}^{\mathrm{r}} \to \Gamma_{T_{S_2}^\vee \circ T_{S_1}^\vee} \to \Gamma_{T_{S_3}^\vee \circ T_{S_2}^\vee \circ T_{S_1}^\vee} \xrightarrow{[1]} \tag{6.29}$$

and so on. This gives a diagram of the form

$$\begin{tikzcd}
\Delta \arrow[rr] & & \Gamma_{T_{S_1}^\vee} \arrow[rr] \arrow[dl] & & \Gamma_{T_{S_2}^\vee \circ T_{S_1}^\vee} \arrow[r] \arrow[dl] & \cdots \\
& \mathscr{Y}_{S_1}^{\mathrm{l}} \otimes \mathscr{Y}_{S_1}^{\mathrm{r}}[1] \arrow[ul, dotted] & & \mathscr{Y}_{S_2}^{\mathrm{l}} \otimes \mathscr{Y}_{T_{S_1}(S_2)}^{\mathrm{r}}[1] \arrow[ul, dotted] & & \cdots \arrow[ul, dotted]
\end{tikzcd} . \tag{6.30}$$

Together with (6.10), this concludes the proof of Proposition 6.7. □

7. Symplectic cohomology of the Milnor fiber

In this section, we recall a spectral sequence converging to $\mathrm{SH}^*\left(\check{V}\right)$ associated to a normal crossings compactification of $\check{V}$ due to [57, 33]. It is based on a standard model of the Reeb flow in a neighborhood of compactification divisor and can be perceived as an elaborate version of the standard Morse-Bott model discussed in [67] when the compactification divisor is smooth. See also [32] and [16] for related results.

Let $\check{Y}$ be a smooth projective variety containing an affine variety with $c_1\left(\check{V}\right) = 0$ in such a way that $\check{D} \coloneqq \check{Y} \setminus \check{V}$ is a normal crossing divisor;

$$\check{D} = \bigcup_{i \in I} \check{D}_i. \tag{7.1}$$

For $J \subset I$, we set $\check{D}_J = \bigcap_{i \in J} \check{D}_i$, and also set $\check{D}_\emptyset = \check{V}$.

Choose a sequence $\kappa = (\kappa_i)_{i\in I}$ of positive integers such that the divisor $\sum_{i\in I} \kappa_i \check{D}_i$ on $\tilde{Y}$ is ample. Let $(c_i)_{i\in I}$ be another sequence of integers such that $\sum_{i\in I} c_i \check{D}_i$ is linearly equivalent to the canonical divisor of $\tilde{Y}$. When $\tilde{Y}$ is a Calabi–Yau manifold, one can set $c_i = 0$ for all $i \in I$.

Still following [57, 33], for each $J \subset I$, we let $N\check{D}_J$ be a small tubular neighborhood of $\check{D}_J$ such that $N\check{D}_J \cap \check{D}_{J'}$ is a tubular neighborhood of $\check{D}_{J\cup J'}$ for all $J' \cap S$. Moreover, we require that the boundary $\partial N\check{D}_J$ intersects $\check{D}_{J'}$ for all $J' \subset I$. Next, we let

$$\mathring{N}\check{D}_J = N\check{D}_J \setminus \cup_{i\in I}\check{D}_i \tag{7.2}$$

be the punctured tubular neighborhood.

Theorem 7.1 ([57, 33] (see also [32, Remark 3.9]))**.** *There is a cohomological spectral sequence converging to* $\mathrm{SH}^*(\check{V})$ *with* E_1*-page given by*

$$E_1^{p,q} = \bigoplus_{\left\{(k_i)\in\mathbb{Z}^I_{\geq 0}\middle|\sum k_i\kappa_i=-p\right\}} H^{p+q-2\sum_i k_i(c_i+1)}\left(\mathring{N}\check{D}_{J_{(k_i)}}\right) \tag{7.3}$$

where $J_{(k_i)} = \{i \in I \mid k_i \neq 0\}$.

Since κ_i is positive for all i, for each p, we have $E_1^{p,q} \neq 0$ only for finitely many q, and is a finite sum of finite-dimensional vector spaces. Moreover, if $c_i > -1$ for all i, then the spectral sequence is regular.

We will apply this spectral sequence to deduce $\mathrm{SH}^1\left(\check{V}\right) = 0$, where $\check{V}$ is the Milnor fiber of a weighted homogeneous singularity.

Corollary 7.2. *Let* $\check{V}$ *be the Milnor fiber of a weighted homogeneous polynomial with an isolated critical point at the origin,* $d_0 > 0$ *and* $\dim\check{V} \geq 2$*, admitting a compactification to a Calabi–Yau manifold by adding a normal crossing divisor. One has* $\mathrm{SH}^i\left(\check{V}\right) = 0$ *for* $i < 0$*,* $\mathrm{SH}^0\left(\check{V}\right) = \mathbb{C}$*, and* $\mathrm{SH}^1\left(\check{V}\right) = 0$*.*

Proof. Since $\check{V}$ is simply connected, we do not get any contribution from $H^1(\check{V}) = 0$. The vanishing of c_i and the positivity of κ_i imply that the orbits coming from the normal crossing divisor contribute to $\mathrm{SH}^i\left(\check{V}\right)$ for $i \geq 2$. □

Now we can prove a generalization of the non-formality result in [46], which corresponds to the case $\mathbf{w} = x^2 + y^3$.

Theorem 7.3. *Under the same assumption as Corollary 7.2,* $\mathcal{A}$ *is not formal.*

Proof. By Corollary 3.2, we have $\mathrm{HH}^1(A) \neq 0$. On the other hand, we know by Corollary 6.6 that $\mathrm{HH}^1(\mathcal{A}, \mathcal{A})$ is isomorphic to $\mathrm{SH}^1(\check{V})$, which is zero by Corollary 7.2. Hence we conclude that $\mathcal{A}$ is not formal. □

A non-zero element of $\mathrm{HH}^1(A)$ is given by the Euler derivation defined by

$$\mathrm{eu}(x) = \deg(x)x. \tag{7.4}$$

Recall that for any A_∞-algebra $\mathscr{A}$ with $H^*(\mathscr{A}) = A$, there exists a length spectral sequence converging to $\mathrm{HH}^*(\mathscr{A})$ with E_2-page given by $E_2^{p,q} = \mathrm{HH}^{p+q}(A)_q$. It is shown in [66, Equation 3.14] that the class of the Euler vector field is killed by the differential on E_2 if $\mathscr{A}$ is non-formal.

In dimension 2, Theorem 7.3 can also be proved as follows: If $\mathcal{A}$ is formal, then $\mathrm{HH}^*(\mathcal{A}) \cong \mathrm{HH}^*(Y_0)$ has a dilation since the BV operator on $\mathrm{HH}^*(Y_0)$ induced by the holomorphic volume form sends $\mathrm{eu}/2 \in \mathrm{HH}^1$ to $1 \in \mathrm{HH}^0$. On the other hand, $\mathrm{SH}^*\left(\check{V}\right)$ cannot have a dilation due to the existence of an exact Lagrangian torus in $\check{V}$ proved in [42]. Note that this argument uses that the BV operator on $\mathrm{SH}^*\left(\check{V}\right)$ agrees with BV operator on $\mathrm{HH}^*(\mathcal{A})$, which holds since any two BV operators differ by an invertible element in HH^0, which is of rank 1 in our case.

We give computations of the spectral sequence in a few examples.

7.1. **The affine quartic surface.** Let $\check{V} = \mathbf{w}^{-1}(-1)$ be the Milnor fiber of the quartic polynomial $\mathbf{w}(x,y,z) = x^4 + y^4 + z^4$, which can be compactified to a quartic K3 surface $\check{Y}$ in $\mathbb{P}^3$ by adding a smooth curve $\check{D}$ of genus 3. We can take $\kappa = 1$ and $c = 0$, so that the E_1-page of the resulting spectral sequence is given in Table 7.1.

					q
	$\mathbb{C}^6$	0	0	0	$\vdots$
	$\mathbb{C}$	$\mathbb{C}$	0	0	9
	0	$\mathbb{C}^6$	0	0	8
	0	$\mathbb{C}^6$	0	0	7
	0	$\mathbb{C}$	$\mathbb{C}$	0	6
	0	0	$\mathbb{C}^6$	0	5
	0	0	$\mathbb{C}^6$	0	4
	0	0	$\mathbb{C}$	0	3
	0	0	0	$\mathbb{C}^{27}$	2
	0	0	0	0	1
	0	0	0	$\mathbb{C}$	0
p	$\dots$	-2	-1	0	

TABLE 7.1. E_1 page of the spectral sequence for $x^4 + y^4 + z^4$.

We immediately conclude that $\mathrm{SH}^0(\check{V}) = \mathbb{C}$, $\mathrm{SH}^1(\check{V}) = 0$, $\mathrm{SH}^2(\check{V}) = \mathbb{C}^{28}$, $\mathrm{SH}^3(\check{V}) = \mathbb{C}^6$, and $\mathrm{SH}^i(\check{V}) = \mathbb{C}^6$ or $\mathbb{C}^7$ for $i > 3$.

More generally, let $\check{V} = \mathbf{w}^{-1}(-1)$ for the polynomial

$$\mathbf{w}(x_1, \ldots, x_n) = x_1^{n+1} + \ldots + x_n^{n+1}$$

which compactifies to a Calabi-Yau hypersurface of degree $n+1$ in $\mathbb{P}^n$ by looking at the zero set of $\mathbf{W}(x_0, x_1, \ldots, x_n) = x_0^{n+1} + \ldots + x_n^{n+1}$ in $\mathbb{P}^n$. The smooth divisor at infinity $\check{D}$ is defined by $\mathbf{w} = 0$ in $\mathbb{P}^{n-1} = \{x_0 = 0\}$. By standard arguments (cf. [18]) we can compute the cohomology of $\check{D}$ as follows:

$$H^*(\check{D}) = \begin{cases} \mathbb{C} & * = 2k, \text{ for } 0 \leq 2k < (n-2) \\ \mathbb{C}^{\left\lfloor \frac{n^n}{n+1} \right\rfloor + (-1)^n + 1} & * = n-2, \\ \mathbb{C} & * = 2k \text{ for } (n-2) < 2k \leq 2(n-2). \end{cases} \tag{7.5}$$

In view of the Lefschetz hyperplane theorem, the only non-trivial part of the computation is the Betti number $b_{n-2}(\check{D})$. This can be computed via the formula $b_{n-2}(\check{D}) = (-1)^n(\chi(\check{D}) - 2\left\lfloor \frac{n-1}{2} \right\rfloor)$ and the Euler characteristic can in turn be computed via Chern classes to be $\frac{1}{n+1}((-1)^n n^n + n(n+1) - 1)$.

The circle bundle $N\check{D}$ has Euler class $(n+1)$ times the hyperplane class. This implies via the Leray-Serre spectral sequence that the cohomology of $N\check{D}$ is given by

$$H^*(N\check{D}) = \begin{cases} \mathbb{C} & * = 0 \\ \mathbb{C}^{\left\lfloor \frac{n^n}{n+1} \right\rfloor + \frac{(-1)^n+1}{2}} & * = n-2, n-1 \\ \mathbb{C} & * = 2n. \end{cases} \tag{7.6}$$

As for the Milnor fiber, the homotopy type is given as a wedge of μ spheres where Milnor number $\mu = n^n$ for $\mathbf{w}$. Thus, we have

$$H^*(\check{V}) = \begin{cases} \mathbb{C} & * = 0, \\ \mathbb{C}^{n^n} & * = n-1. \end{cases} \tag{7.7}$$

In constructing the spectral sequence we can, as before, take $\kappa = 1$ and $c = 0$. From the spectral sequence, we can immediately conclude that for $n > 3$, we have $SH^0(\check{V}) = \mathbb{C}, SH^1(\check{V}) = 0, SH^2(\check{V}) = \mathbb{C}$ and $SH^{n-1}(\check{V}) = \mathbb{C}^{n^n}$ or $\mathbb{C}^{n^n-1}$.

7.2. **The double cover of the plane branched along a sextic.** Let $\check{V} = \mathbf{w}^{-1}(-1)$ be the Milnor fiber of the polynomial $\mathbf{w}(x,y,z) = x^2 + y^6 + z^6$, which can be compactified to the double cover $\tilde{Y}$ of $\mathbb{P}^2$ branched along a smooth sextic curve by adding a smooth curve $\check{D}$ of genus 2. We can take $\kappa = 1$ and $c = 0$, so that the E_1-page of the resulting spectral sequence is given in Table 7.2.

					q
	$\mathbb{C}^4$	0	0	0	$\vdots$
	$\mathbb{C}$	$\mathbb{C}$	0	0	9
	0	$\mathbb{C}^4$	0	0	8
	0	$\mathbb{C}^4$	0	0	7
	0	$\mathbb{C}$	$\mathbb{C}$	0	6
	0	0	$\mathbb{C}^4$	0	5
	0	0	$\mathbb{C}^4$	0	4
	0	0	$\mathbb{C}$	0	3
	0	0	0	$\mathbb{C}^{25}$	2
	0	0	0	0	1
	0	0	0	$\mathbb{C}$	0
p	$\dots$	-2	-1	0	

TABLE 7.2. E_1 page of the spectral sequence for $x^2 + y^6 + z^6$.

We immediately conclude that $\mathrm{SH}^0(\check{V}) = \mathbb{C}$, $\mathrm{SH}^1(\check{V}) = 0$, $\mathrm{SH}^2(\check{V}) = \mathbb{C}^{26}$, $\mathrm{SH}^3(\check{V}) = \mathbb{C}^4$, and $\mathrm{SH}^i(\check{V}) = \mathbb{C}^4$ or $\mathbb{C}^5$ for $i > 3$.

More generally, let $\check{V} = \mathbf{w}^{-1}(-1)$ for the polynomial

$$\mathbf{w}(x_1, \dots, x_n) = x_1^2 + x_2^{2n} + \dots + x_n^{2n}$$

which compactifies to a Calabi-Yau hypersurface in $\mathbb{P}(n, 1, 1, \dots, 1)$ by looking at the zero set of $\mathbf{W}(x_0, x_1, \dots, x_n) = x_0^{2n} + x_1^2 + x_2^{2n} + \dots + x_n^{2n}$ in $\mathbb{P}(1, n, 1, 1, \dots, 1)$. The smooth divisor at infinity $\check{D}$ is defined by $\mathbf{w} = 0$ in $\mathbb{P}(n, 1, \dots, 1) = \{x_0 = 0\}$. By standard arguments (cf. [18]) we can compute the cohomology of $\check{D}$ as follows:

$$H^*(\check{D}) = \begin{cases} \mathbb{C} & * = 2k, \text{ for } 0 \le 2k < (n-2) \\ \mathbb{C}^{\lfloor \frac{(2n-1)^{n-1}}{2n} \rfloor + (-1)^n + 1} & * = n-2, \\ \mathbb{C} & * = 2k \text{ for } (n-2) < 2k \le 2(n-2). \end{cases} \tag{7.8}$$

In view of the Lefschetz hyperplane theorem, the only non-trivial part of the computation is the Betti number $b_{n-2}(\check{D})$. This can be computed via the formula $b_{n-2}(\check{D}) = (-1)^n(\chi(\check{D}) - 2\lfloor \frac{n-1}{2} \rfloor)$ and the Euler characteristic can in turn be computed via Chern classes to be $\frac{1}{2n}((-1)^n(2n-1)^{n-1} + 2n(n-1) + 1)$.

The circle bundle $N\check{D}$ has Euler class $2n$ times the hyperplane class. This implies via the Leray-Serre spectral sequence that the cohomology of $N\check{D}$ is given by

$$H^*(N\check{D}) = \begin{cases} \mathbb{C} & * = 0 \\ \mathbb{C}^{\lfloor \frac{(2n-1)^{n-1}}{2n} \rfloor + \frac{(-1)^n + 1}{2}} & * = n-2, n-1 \\ \mathbb{C} & * = 2n. \end{cases} \tag{7.9}$$

As for the Milnor fiber, the homotopy type is given as a wedge of μ spheres where Milnor number $\mu = (2n-1)^{n-1}$ for $\mathbf{w}$. Thus, we have

$$H^*(\check{V}) = \begin{cases} \mathbb{C} & * = 0, \\ \mathbb{C}^{(2n-1)^{n-1}} & * = n-1. \end{cases} \tag{7.10}$$

In constructing the spectral sequence we can, as before, take $\kappa = 1$ and $c = 0$. From the spectral sequence, we can immediately conclude that for $n > 3$, we have $SH^0(\check{V}) = \mathbb{C}, SH^1(\check{V}) = 0, SH^2(\check{V}) = \mathbb{C}$ and $SH^{n-1}(\check{V}) = \mathbb{C}^{(2n-1)^{n-1}}$ or $\mathbb{C}^{(2n-1)^{n-1}-1}$.

8. Homological mirror symmetry for Milnor fibers

We prove Theorem 1.7 in this section.

Proof of Theorem 1.7. Let $\check{V} := \left\{(x_1, \dots, x_n) \in \mathbb{C}^n \;\middle|\; x_1^{n+1} + \cdots + x_n^{n+1} = 1\right\}$ be the Milnor fiber of $\mathbf{w} = x_1^{n+1} + \cdots + x_n^{n+1}$. A distinguished basis $(S_i)_{i=1}^{n^n}$ of vanishing cycles generates the compact Fukaya category of $\check{V}$, and the cohomology A of the total morphism A_∞-algebra $\mathcal{A} := \bigoplus_{i,j=1}^{n^n} \hom(S_i, S_j)$ is the degree $n-1$ trivial extension algebra of the tensor product $\mathfrak{A}_n^{\otimes n}$ of the Dynkin quiver $\mathfrak{A}_n$ of type A_n. The A_∞-algebra $\mathcal{A}$ is not formal by Theorem 7.3, and $\mathrm{HH}^*(\mathcal{F}(\check{V}))$ is isomorphic to $\mathrm{SH}^*(\check{V})$ computed in Section 7.1.

The graded algebra A also appears as the cohomology of the Yoneda dg algebra $\mathcal{A}_u$ of a generator $\mathcal{S}_u$ of $\operatorname{perf} \mathcal{Y}_u$ where $\mathcal{Y}_u$ for $u \in U := \operatorname{Spec} \mathbb{C}[u_1, u_{n+1}]$ is the quotient stack $[(\operatorname{Spec} S_u \setminus \mathbf{0})/\Gamma]$ for $S_u := \mathbb{C}[x_0, \dots, x_n]/(x_1^{n+1} + \cdots + x_n^{n+1} + u_1 x_0 \cdots x_n + u_{n+1} x_0^{n+1})$ and $\Gamma := \left\{(t_1, \dots, t_n) \in \mathbb{G}_\mathrm{m}^n \;\middle|\; t_1^{n+1} = \cdots = t_n^{n+1}\right\}$. The moduli space $\mathcal{U}_\infty(A)$ of minimal A_∞-structures on A is identified with U.

In order to identify $u \in U$ satisfying $\mathcal{A} \simeq \mathcal{A}_u$, we compare $\mathrm{HH}^*(\mathcal{A}_u)$ and $\mathrm{HH}^*(\mathcal{A}) \cong \mathrm{SH}^*(\check{V})$ as graded vector spaces. Since $\mathrm{SH}^*(\check{V})$ is infinite-dimensional over $\mathbf{k}$, the mirror surface $\mathcal{Y}_u$ must be singular. Up to the action of $\mathbb{G}_\mathrm{m}$ on U, there are precisely two non-zero $u \in U$ such that $\mathcal{Y}_u$ is singular, i.e., $(u_1, u_{n+1}) = (1, 0)$ and $(-n-1, 1)$. The Hochschild cohomologies of these singular surfaces are computed in Sections 3.6 and 3.7. Comparing this with $\mathrm{SH}^*(\check{V})$ computed in Section 7.1, we conclude that the mirror of the $\check{V}$ is the surface associated with $(u_1, u_{n+1}) = (1, 0)$.

The proof for $\check{V} := \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1^2 + x_2^{2n} + \cdots + x_n^{2n} = 1\}$ goes along the same lines. The cohomology A of the total morphism A_∞-algebra of a distinguished basis of vanishing cycles is given by the degree $n-1$ trivial extension algebra of $\mathfrak{A}_{2n-1}^{\otimes(n-1)}$. The moduli space $\mathcal{U}_\infty(A)$ of minimal A_∞-structures is identified with U, and there are precisely two non-zero $u \in U$ up to the action of $\mathbb{G}_\mathrm{m}$ such that $\mathcal{Y}_u = [(\operatorname{Spec} S_u \setminus \mathbf{0})/\Gamma]$ for $S_u := \mathbb{C}[x_0, \dots, x_n]/(x_1^2 + x_2^{2n} + \cdots + x_n^{2n} + u_2 x_0^2 \cdots x_n^2 + u_{2n} x_0^{2n})$ and $\Gamma := \{(t_1, \dots, t_n) \in \mathbb{G}_\mathrm{m}^n \mid t_1^2 = t_2^{2n} = \cdots = t_n^{2n}\}$ is singular. The mirror is identified with $\mathcal{Y}_u$ for $u = (u_2, u_{2n}) = (1, 0)$ by comparing $\mathrm{HH}^*(\mathcal{Y}_u)$ with $\mathrm{SH}^*(\check{V})$ given in Section 7.2. $\square$

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