

Geometric Composition in Quilted Floer Theory

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We prove that Floer cohomology of cyclic Lagrangian correspondences is invariant under transverse and embedded composition under a general set of assumptions. We also give an application of this result in the negatively monotone setting to construct an isomorphism in Floer theory of broken fibrations.

1 Introduction

1.1 Lagrangian correspondences and geometric composition

Given two symplectic manifolds (M_1, ω_1) , (M_2, ω_2) a Lagrangian correspondence is a Lagrangian submanifold $L \subset (M_1 \times M_2, -\omega_1 \oplus \omega_2)$. These are the central objects of the theory of holomorphic quilts as developed by Wehrheim and Woodward in [21]. Consider two Lagrangian correspondences $L_i \subset (M_{i-1} \times M_i, -\omega_{i-1} \oplus \omega_i)$ for $i = 1, 2$. Let

$$\Delta = \{(x, y, z, t) \in M_0 \times M_1 \times M_1 \times M_2 \mid y = z\}$$

If $L_1 \times L_2$ is transverse to Δ , we may form the fibre product $L_1 \times_{M_1} L_2 \subset M_0 \times M_1 \times M_1 \times M_2$ by intersecting Δ with $L_1 \times L_2$. If the projection $L_1 \times_{M_1} L_2 \rightarrow M_0 \times M_2$ is an embedding, we say that L_1 and L_2 are composable and $L_1 \circ L_2$ is naturally a Lagrangian submanifold of $M_0 \times M_2$ and is called the (geometric) composition of L_1 and L_2 . As a point set one has

$$L_1 \circ L_2 = \{(x, z) \in M_0 \times M_2 \mid \exists y \in M_1 \text{ such that } (x, y) \in L_1 \text{ and } (y, z) \in L_2\}$$

1.2 Floer cohomology of a cyclic set of Lagrangian correspondences

A cyclic set of Lagrangian correspondences of length k is a set of Lagrangian correspondences $L_i \subset M_{i-1} \times M_i$ for $i = 1, \dots, k$ such that $(M_0, \omega_0) = (M_k, \omega_k)$.

Given a cyclic set of Lagrangian correspondences, Wehrheim and Woodward in [21] define a Floer cohomology group $HF(L_1, \dots, L_k)$ (see Section 2.1 below for a review). This can be identified with the Floer cohomology group of the Lagrangians

$$L_{(0)} = L_1 \times L_3 \times \dots \times L_{k-1} \text{ and } L_{(1)} = L_2 \times L_4 \times \dots \times L_k$$

in the product manifold $\underline{M} = M_0^- \times M_1 \times M_2^- \times \dots \times M_{k-1}$ if k is even. If k is odd, one inserts the diagonal $\Delta_{M_0} \subset M_0^- \times M_0 = M_{k+1}^- \times M_0$ to get a cyclic set of Lagrangian correspondences with even length. (We denote by M^- the symplectic manifold $(M, -\omega)$ where ω is the given symplectic form on M). Under appropriate assumptions on the underlying Lagrangians, one expects an isomorphism

$$HF(L_0, \dots, L_r, L_{r+1}, \dots, L_{k-1}) \simeq HF(L_0, \dots, L_r \circ L_{r+1}, \dots, L_{k-1})$$

when L_r and L_{r+1} are composable. The main goal of the present work is to prove such an isomorphism under a rather general set of assumptions. Such an isomorphism should exist whenever the Floer cohomology groups on either side can be defined. For instance, let us discuss this isomorphism in the aspherical case. For this we need to introduce some notation. Namely, given two transverse Lagrangians $L, L' \subset (M, \omega)$, we consider the path space:

$$\mathcal{P} = \mathcal{P}(L, L') = \{\gamma : [0, 1] \rightarrow M \mid \gamma(0) \in L, \gamma(1) \in L'\}$$

Now pick $x_0 \in L \cap L'$ to be the constant path on a fixed component \mathcal{P}_0 of \mathcal{P} . Then given any path $\gamma \in \mathcal{P}_0$, we can pick a smooth homotopy γ_t such that $\gamma_0 = x_0$ and $\gamma_1 = \gamma$. Then consider the action functional :

$$\begin{aligned} \mathcal{A} : \mathcal{P}_0 &\rightarrow \mathbb{R} \\ \gamma &\rightarrow \int_{[0,1]^2} \gamma_t^* \omega \end{aligned}$$

This is not always well-defined, because in general it depends on the choice of the homotopy γ_t . However, under various topological assumptions, it is possible to avoid this dependence.

A simple case of the main result in this paper is the following statement:

Theorem 1 *Given a cyclic set of compact connected orientable Lagrangian correspondences L_1, \dots, L_k in compact symplectic manifolds $(M_0, \omega_0), \dots, (M_k, \omega_k)$ such that for some r , L_r and L_{r+1} can be composed, suppose that the following topological properties hold:*

$$(1) \quad \text{For any } v : S^1 \times [0, 1] \rightarrow \underline{M} \text{ such that } v|_{S^1 \times \{0\}} \subset L_{(0)} \text{ and } v|_{S^1 \times \{1\}} \subset L_{(1)} \\ \int v^* \omega_{\underline{M}} = 0$$

Then,

$$HF(L_0, \dots, L_r, L_{r+1}, \dots, L_{k-1}) \simeq HF(L_0, \dots, L_r \circ L_{r+1}, \dots, L_{k-1})$$

The assumption (1) is used to avoid bubbling in various moduli spaces and to ensure that the action functional

$$\mathcal{A} : \mathcal{P}_0(L_{(0)}, L_{(1)}) \rightarrow \mathbb{R}$$

is single valued (on any of its path components) which ensures that the Floer differential squares to zero. These assumptions are already required for the Floer cohomology groups considered above to be well-defined. One could replace them with assumptions of similar nature but not dispose of them altogether.

The analogous result under positive monotonicity assumptions was proved earlier by Wehrheim and Woodward in [21]. The difficulty in extending their proof to our setting is the fact that the strip shrinking argument in [21] might give rise to certain *figure-eight bubbles* for which no removal of singularities is known. Our proof of the theorem above does not involve strip shrinking and does not give rise to figure-eight bubbles. Applying the idea used for the proof of Theorem 1, we will give an alternative proof of the positive monotone case considered in [21].

Theorem 2 (*positively monotone case*) *Let L_1, \dots, L_k be a cyclic set of compact orientable Lagrangian correspondences in compact connected symplectic manifolds $(M_0, \omega_0), \dots, (M_k, \omega_k)$ such that for some r , L_r and L_{r+1} can be composed. Let $\tau > 0$ be a fixed real number. Suppose that the following topological properties hold:*

$$(2) \quad \begin{aligned} & \text{For any } v : S^1 \times [0, 1] \rightarrow \underline{M} \text{ such that } v|_{S^1 \times \{0\}} \subset L_{(0)} \text{ and } v|_{S^1 \times \{1\}} \subset L_{(1)} \\ & \int v^* \omega_{\underline{M}} = \tau I_{\text{Maslov}}(v^* L_{(0)}, v^* L_{(1)}) \end{aligned}$$

The minimal Maslov index for disks in $\pi_2(\underline{M}, L_{(0)})$ and $\pi_2(\underline{M}, L_{(1)})$ is ≥ 3 .

Then,

$$HF(L_0, \dots, L_r, L_{r+1}, \dots, L_{k-1}) \simeq HF(L_0, \dots, L_r \circ L_{r+1}, \dots, L_{k-1})$$

Note that we only require monotonicity for the annuli with boundary on $L_{(0)}$ and $L_{(1)}$ which makes the group on the left well-defined. However, it is easy to see that the corresponding monotonicity relation for the group on the right hand side follows from this. Furthermore, via the natural map from $\pi_2(\underline{M}) \rightarrow \pi_2(\underline{M}; L_{(0)}, L_{(1)})$ the hypotheses of the theorem implies the following monotonicity of the symplectic manifolds M_i , which determines the monotonicity constant τ .

$$[\omega_{M_i}] = \tau c_1(TM_i) \text{ for all } i.$$

Note also that when $\tau = 0$, the symplectic manifolds are exact and necessarily non-compact, thus one needs to assume convexity properties at infinity (as for example in [19]) in order to ensure compactness of various moduli spaces. The proof is simpler in the exact case. Indeed, the hypothesis of Theorem 1 are satisfied, hence this case is covered by the previous result.

Finally, we extend the argument to the (strongly) negatively monotone case which is needed for our application. Recall that for $[u] \in \pi_2(M)$, the expected dimension of the moduli space of unparametrized holomorphic spheres in class $[u]$ is given by $2\langle c_1(TM), [u] \rangle + \dim(M) - 3$ and for $[u] \in \pi_2(M, L)$, the expected dimension of the moduli space of unparametrized holomorphic disks in class $[u]$ is given by $\mu_L([u]) + \dim(M) - 3$ (where μ_L is the Maslov homomorphism). In the strongly negative case, we require these numbers to be sufficiently negative in order to avoid bubbling in 0, 1 and 2-dimensional moduli spaces.

Theorem 3 (strongly negative monotone case) *Given a cyclic set of compact connected orientable Lagrangian correspondences L_1, \dots, L_k in compact connected symplectic manifolds $(M_0, \omega_0), \dots, (M_k, \omega_k)$ such that for some r , L_r and L_{r+1} can be composed. Let $\tau < 0$ be a fixed real number. Denote $\dim M_i = 2m_i$. Suppose that the following topological properties hold for all $i = 0, \dots, k$:*

$$(3) \quad \text{For any } v : S^1 \times [0, 1] \rightarrow \underline{M} \text{ such that } v|_{S^1 \times \{0\}} \subset L_{(0)} \text{ and } v|_{S^1 \times \{1\}} \subset L_{(1)} \\ \int v^* \omega_{\underline{M}} = \tau I_{\text{Maslov}}(v^* L_{(0)}, v^* L_{(1)})$$

$$(4) \quad \text{If } \int u^*(\omega_i) > 0 \text{ for } [u] \in \pi_2(M_i), \text{ then } \langle c_1(TM_i), [u] \rangle < -m_i + 2. \\ \text{If } \int u^*(-\omega_i \oplus \omega_{i+1}) > 0 \text{ for } [u] \in \pi_2(M_i \times M_{i+1}, L_{i+1}), \\ \text{then } \mu_{L_{i+1}}([u]) < -(m_i + m_{i+1}) + 1.$$

Then,

$$HF(L_0, \dots, L_r, L_{r+1}, \dots, L_{k-1}) \simeq HF(L_0, \dots, L_r \circ L_{r+1}, \dots, L_{k-1})$$

In all of the cases, the main idea is to construct a particular homomorphism

$$\Phi : HF(L_0, \dots, L_r, L_{r+1}, \dots, L_{k-1}) \rightarrow HF(L_0, \dots, L_r \circ L_{r+1}, \dots, L_{k-1})$$

Once Φ is constructed, a simple energy argument shows that Φ is an isomorphism.

The main motivation for proving Theorem 3 is an application to an explicit example. Namely, we apply Theorem 3 to get rid of a technical assumption in the proof of an isomorphism between Lagrangian matching invariants and Heegaard Floer homology of 3-manifolds, which appeared in a previous work of the first author [10]. Our main construction was also used in [14] in order to prove the topological invariance of their symplectic construction of a Floer homology group which they conjecture to be isomorphic to a version of instanton Floer homology.

In order to avoid repetition, we will not give all the details involved in the definition of holomorphic quilts and Floer cohomology of a cyclic set of Lagrangian correspondences. The more comprehensive discussion of foundations of this theory is available in [21].

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2 Morphisms between Floer cohomology of Lagrangian correspondences

2.1 Chain complex of a cyclic set of Lagrangian correspondences

Let us recall that when $L_{(0)}$ and $L_{(1)}$ intersect transversely the chain complex $CF(\underline{L})$ associated with $\underline{L} = (L_1, \dots, L_k)$ is the freely generated group over a base ring Λ by the generalized intersection points $\mathcal{I}(\underline{L})$ where

$$\mathcal{I}(\underline{L}) = \{\underline{x} = (x_1, \dots, x_k) \mid (x_k, x_1) \in L_1, (x_1, x_2) \in L_2, \dots, (x_{k-1}, x_k) \in L_k\}$$

The role of Λ here is no different than its role in the usual Lagrangian Floer cohomology. We will mostly take Λ to be \mathbb{Z}_2 (or more generally Novikov rings over a base ring of characteristic 2) in order to avoid getting into sign considerations. The full discussion of orientations in this set-up appeared in [23], from which one expects that under assumptions on orientability of the relevant moduli spaces (say when L_i are relatively spin), our results still hold over \mathbb{Z} .

More generally, we can choose Hamiltonian functions $H_i : [0, \delta_i] \times M_i \rightarrow \mathbb{R}$ and perturb \underline{L} with a Hamiltonian isotopy on each M_i to ensure that $L_{(0)}$ and $L_{(1)}$ intersect transversely, so that $\mathcal{I}(\underline{L})$ is a finite set. It is an easy lemma to show that for a generic choice of $(H_i)_{i=1, \dots, k}$, transversality of $L_{(0)}$ and $L_{(1)}$ holds after the perturbation, in particular the set $\mathcal{I}(\underline{L})$ is finite (see [22] page 7). From now on, we will always assume that $L_{(0)}$ and $L_{(1)}$ are perturbed into general position and we will take $H_i \equiv 0$ for all i , so that $\mathcal{I}(\underline{L})$ consists of a finite set of generalized intersection points as defined in the beginning of this section.

Next, to define the differential on $CF(\underline{L})$, for each $i = 1, \dots, k$, we choose a compatible almost complex structure J_i on M_i and extend the definition of the Floer differential to our setting in the following way. Let $\underline{x}, \underline{y}$ be generalized intersection points in $\mathcal{I}(\underline{L})$. We define the moduli space of

finite energy quilted holomorphic strips connecting \underline{x} and \underline{y} by

$$\begin{aligned} \mathcal{M}(\underline{x}, \underline{y}) &= \{u_i : \mathbb{R} \times [0, \delta_i] \rightarrow M_i \mid \bar{\partial}_{J_i, H_i} u_i := \partial_s u_i + J_i(\partial_t u_i - X_{H_i}(u_i)) = 0, \\ E(u_i) &:= \int u_i^* \omega_i - d(H_i(u_i)) dt < \infty \\ \lim_{s \rightarrow -\infty} u_i(s, \cdot) &= x_i, \lim_{s \rightarrow +\infty} u_i(s, \cdot) = y_i \\ (u_i(s, \delta_i), u_{i+1}(s, 0)) &\in L_{i+1} \text{ for all } i = 1, \dots, k\} / \mathbb{R} \end{aligned}$$

Under monotonicity assumptions (as in Theorem 2), it is proven in [21] (with corrections from [24]) that, given $(\delta_i)_{i=1, \dots, k}$ and $(H_i)_{i=1, \dots, k}$, there is a Baire second category subset of almost complex structures $(J_i)_{i=1, \dots, k}$ for which these moduli spaces are cut out transversely and compactness properties of the usual Floer differential carry over. It is straightforward to check that the same result holds when we replace the monotonicity assumptions by the set of assumptions in the statement of Theorem 1 (for more details, see the proof of Theorem 5.2.3 in [21]). As we check in the proof of Theorem 3, the assumptions of Theorem 3 also gives rise to well-defined moduli spaces. Therefore, in either case one can define the Floer differential for a cyclic set of Lagrangian correspondences by :

$$\partial_{\underline{x}} = \sum_{\underline{y} \in \mathcal{I}(\underline{L})} \# \mathcal{M}(\underline{x}, \underline{y}) \underline{y}$$

where # means counting isolated points modulo 2.

Remark 4 *If one has the additional choices in place so that the moduli spaces $\mathcal{M}(\underline{x}, \underline{y})$ are oriented (cf. [19], [23]), then # would mean the signed count of isolated points. Similarly, if one uses a Novikov ring as the base ring Λ , then the above differential should be modified accordingly as usual to accommodate various other quantities of interest (homotopy class, area, . . . etc.). The same remark applies to any of the moduli spaces and corresponding counts that we use in this paper.*

The compactness and gluing properties of the above moduli spaces allow one to prove that the differential squares to zero, hence we get a well-defined Floer cohomology group. We refer the reader to Proposition 5.3.1 in [21] for a continuation argument which shows that the resulting group is independent of the choices of $(\delta_i, H_i, J_i)_{i=1, \dots, k}$.

Following [22], we will prove Theorems 1, 2 and 3 in a special case (the general case is proved in exactly the same way). Let $(M_i, \omega_i)_{i=0, 1, 2}$ be symplectic manifolds of dimension $2n_i$ and let

$$L_0 \subset M_0, L_{01} \subset M_0^- \times M_1, L_{12} \subset M_1^- \times M_2, L_2 \subset M_2^-$$

be compact Lagrangian submanifolds such that the geometric composition $L_{02} = L_{01} \circ L_{12} \subset M_0^- \times M_2$ is embedded. As discussed above, we can perturb L_0 and L_2 so that the generalized intersections of $(L_0, L_{01}, L_{12}, L_2)$ as well as (L_0, L_{02}, L_2) are transverse. Our goal is to construct a map

$$\Phi : HF(L_0, L_{01}, L_{12}, L_2) \rightarrow HF(L_0, L_{02}, L_2)$$

which we will prove to be an isomorphism. Note that there is an obvious bijection of the chain groups

$$CF(L_0, L_{01}, L_{12}, L_2) \cong CF(L_0, L_{02}, L_2)$$

The map Φ will not necessarily be induced by this bijection. As we will later demonstrate, it will differ from this bijection possibly by a nilpotent matrix.

2.2 Defining the quilt

Our construction of Φ is summarized in Figure 1 below.

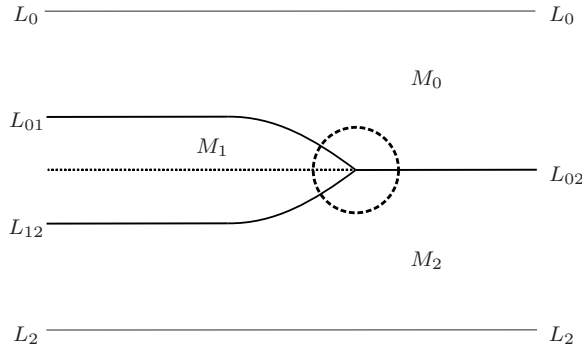


Figure 1: The geometric composition map

Let Σ be the pictured *quilt* (without the dotted line segment and the dotted circle). More precisely, let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, where topologically speaking, each Σ_i is homeomorphic to a closed unit disk with punctures at the boundary. Namely, $\Sigma_0 = D^2 \setminus \{-1, -i, 1\}$, $\Sigma_1 = D^2 \setminus \{-1, 1\}$ and $\Sigma_2 = D^2 \setminus \{-1, i, 1\}$. These are the embedded regions (patches) in Figure 1. Ignoring the dotted curves, Σ_i is embedded in the region labelled M_i and Σ is obtained by identifying the connected components of the boundary of Σ_i as indicated in Figure 1. As pictured, Σ has 3 (resp. 2) boundary punctures on the left (resp. right) which we refer to as incoming (resp. outgoing) ends. There is also the puncture in the middle, which we will refer to as the Y-end. (One can visualise the Y-end as a semi-infinite cylinder where 3 semi-infinite strips are glued together along the 3 solid (straight, parallel) lines that go to infinity. On each patch Σ_i we fix a complex structure j_i with real analytic boundary conditions as in [20]. In short, this means that the seams, the solid curve segments in Figure 1, are embedded as real analytic sets in Σ . A concrete way of arranging such j_i is as follows: Take a thin neighborhood of each seam in Σ and identify it with a small neighborhood $\mathbb{R} \times (-i\epsilon, i\epsilon)$ of \mathbb{R} in \mathbb{C} , and choose $\{j_i\}$ to be holomorphically compatible with this identification. As in Figure

1, let us label the maps from the three patches as $u_i : \Sigma_i \rightarrow M_i$ with $i = 0, 1, 2$. Along the boundary components of these patches, these have the labeled “seam conditions” as in the picture. This means first of all that there are choices of diffeomorphisms ϕ_{ij} between adjacent boundary components of each patch (the way we constructed the $\{j_i\}$ near the seams by using the embedding of Σ_i in Σ determines these choices). Now, if we consider the maps from adjacent patches, say u_i and u_j , as a map $(u_i, u_j) : \Sigma_i \times \Sigma_j \rightarrow M_i \times M_j$ of product manifolds, for each point x in the boundary of Σ_i adjacent to Σ_j , we should have that $(u_i(x), u_j(\phi_{ij}(x))) \in L_{ij}$ where L_{ij} is the labeled Lagrangian submanifold in the product $M_i \times M_j$. What we describe here is a particular example of a holomorphic quilt. We refer to Definition 3.1 in [20] for a more detailed and general definition of a holomorphic quilt.

We may identify the region inside the dotted circle, which is a neighborhood of the Y-end, with $(0, \infty) \times [0, 1]$ mapping to $M_0^- \times M_1 \times M_1^- \times M_2$. More precisely, first we split the strip corresponding to $u_1 : \Sigma_1 \rightarrow M_1$ along the dotted horizontal seam in Figure 1 and put the diagonal seam condition $\Delta \subset M_1 \times M_1^-$. This has no effect on the moduli space that we consider. However, now at the Y-end we can “fold” the strip to get the desired map. Specifically, at the Y-end instead of looking at maps from different strips to different manifolds, one can consider a single map from $(0, \infty) \times [0, 1]$ to the product $M_0^- \times M_1 \times M_1^- \times M_2$. Therefore, we choose our complex structure j_i so that near the Y-end they are identified with the standard complex structure on $(0, \infty) \times [0, 1]$. Similarly, we can choose j_i near the incoming and outgoing ends so that we can identify our strips with $(-\infty, 0] \times [0, 1]$ and $[0, \infty) \times [0, 1]$ and we fix these choices once and for all. (For a standard discussion about these choices see the discussion of strip-like ends in [19] Section (8d)). At the Y-end, let us label the map obtained by folding by

$$v : (0, \infty) \times [0, 1] \rightarrow M_0^- \times M_1 \times M_1^- \times M_2 = \underline{M}$$

This has the seam conditions $v(s, 0) \in L_{01} \times L_{12}$ and $v(s, 1) \in L_{02} \times \Delta$. (Note that at this point we do not impose any condition on the behaviour of v as $s \rightarrow \infty$. It will be seen to converge to an intersection point of $L_{01} \times L_{12}$ and $L_{02} \times \Delta$ as a consequence of holomorphicity.) Next, we would like to specify the complex structures on each M_i . Assume that we have chosen J_i on M_i such that $HF(L_0, L_{01}, L_{12}, L_2)$ and $HF(L_0, L_{02}, L_2)$ are both defined. In general, such J_i may need to be t -dependent near

$$(s, t) \in [0, \infty) \times [0, 1] \cup (-\infty, 0] \times [0, 1]$$

to ensure transversality for the moduli spaces that appear in the definition of the Floer differential. Note that this specifies $\underline{J} = J_0 \times -J_1 \times J_1 \times -J_2$ on $M_0^- \times M_1 \times M_1^- \times M_2$. To ensure transversality for the moduli space of quilted maps, we now introduce a domain dependent $J(z)$ on M . Pick a small holomorphically embedded disk $D \subset (0, \infty) \times (0, 1)$ (Note that this is an interior disk). We define $J(z)$ by letting $J(z) = \underline{J}$ outside D and letting $J(z)$ be chosen generically from the set of compatible complex structures inside D . Such a $J(z)$ need not preserve the product structure on \underline{M} . A similar

construction in quilted Floer theory already appears in [18]. We will denote such domain dependent complex structures for our quilt, simply by J .

Definition 5 Let $\underline{x}, \underline{y}$ be two generalized intersection points for (L_0, L_{02}, L_2) (or equivalently $(L_0, L_{01}, L_{12}, L_2)$). Let $\mathcal{M}_J(\underline{x}, \underline{y})$ be the set of all finite energy maps $\underline{u} = (u_i)_{i=0}^2$ that are holomorphic with respect to J , have the quilted Lagrangian boundary conditions and converge to \underline{x} on the incoming end and to \underline{y} on the outgoing end.

Note that $L_{02} \times \Delta$ and $L_{01} \times L_{12}$ intersect cleanly in

$$\tilde{L}_{02} = (L_{02} \times \Delta) \cap (L_{01} \times L_{12})$$

which is diffeomorphic to L_{02} . By definition, this means that

$$T\tilde{L}_{02} = T(L_{02} \times \Delta) \cap T(L_{01} \times L_{12})$$

The finite energy assumption guarantees that the map near the Y-end has exponential decay. More precisely, at the Y-end we have a holomorphic map

$$v : (0, \infty) \times [0, 1] \rightarrow M_0^- \times M_1 \times M_1^- \times M_2$$

for which we have the following decay estimate (see [25], lemma 2.5 or [7], appendix 3):

Lemma 6 *There exists $\epsilon_0 > 0$, such that for any holomorphic v with finite energy there exists C such that*

$$\sup_{t \in [0, 1]} |\nabla v(s, t)| \leq C e^{-\epsilon_0 s}$$

Here the gradient is taken with respect to a reference metric and the constant C only depends on this metric. This lemma combined with Gromov-Floer compactness implies that as $s \rightarrow \infty$ each v converges exponentially fast to some point $z \in \tilde{L}_{02}$ and all derivatives of $v(s, t)$ converge to 0 exponentially fast. We will denote the point of convergence by $v(\infty)$.

Note that the use of Gromov-Floer compactness is justified by the following standard argument: We cover our quilt by a finite number of domains such that each domain can be folded to yield a map of a holomorphic curve from a single Riemann surface (possibly with boundary and strip-like ends) into a product of symplectic manifolds. (Note that the folding can be carried out even when there are boundary conditions as one requires seams to be embedded as real-analytic subsets). On each domain we apply the usual Gromov-Floer compactness to deduce convergence in compact subdomains outside a finite collection of points. This implies Gromov-Floer compactness holds in any compact subdomain of a quilt. Finally, at the strip-like ends, we can again fold and reduce to the standard Gromov-Floer compactness statement. Putting this all together yields the desired Gromov-Floer compactness statement for a quilt. (See Chapter 4 of [15] for a detailed discussion of

compactness in the case the domain is compact which applies to the setting of holomorphic quilts without any change and Theorem 2.14 of [25] concerning strips with clean intersection Lagrangian boundary conditions).

2.3 Morse-Bott intersections and transversality

Given that any element $u \in \mathcal{M}_J(\underline{x}, \underline{y})$ has exponential decay for some uniform $\epsilon_0 > 0$ at the Y-end, we can view $\mathcal{M}_J(\underline{x}, \underline{y})$ as the zero set of a Fredholm section of a Banach bundle defined using a norm with exponential weights at the Y-end. We briefly review this construction with the purpose of identifying the relevant tangent spaces.

Fix some $p > 2$, intersection points $x \in CF(L_0, L_{01}, L_{12}, L_2)$ and $y \in CF(L_0, L_{02}, L_2)$. Let $(0, \infty) \times [0, 1]$ be a neighborhood of the Y-end in the quilt Σ . Let $\underline{\Sigma} = \Sigma - (1, \infty) \times [0, 1]$ be the complement of a slightly smaller end. On the open subdomain $\underline{\Sigma}$, we define the Banach manifold B_1 of all $L_{1,\epsilon}^p$ maps with quilted boundary conditions (seam conditions) that converge to \underline{x} on the incoming end and to \underline{y} on the outgoing end. For any sufficiently small $\epsilon > 0$, we may define on $(0, \infty) \times [0, 1]$ the Banach manifold B_2 of all $L_{1,\epsilon}^p$ maps

$$v : (0, \infty) \times [0, 1] \rightarrow M_0 \times M_1 \times M_1 \times M_2 = \underline{\mathbf{M}}$$

with Lagrangian boundary conditions $v(s, 0) \in L_{01} \times L_{12}$, $v(s, 1) \in L_{02} \times \Delta$ and exponential decay with coefficient ϵ . For a recent review of exponential weights (with references to older treatments) see [25]. Any element $v \in B_2$ converges to some point on the manifold $(L_{01} \times L_{12}) \cap (L_{02} \times \Delta)$. The tangent space to such v are a pair

$$v' = (v'_a, v'_b)$$

where v'_a is a section of $v^*(T\underline{\mathbf{M}})$ with totally real boundary conditions and exponential decay at infinity, while v'_b is an element of the finite dimensional space

$$T_{v(\infty)}((L_{01} \times L_{12}) \cap (L_{02} \times \Delta))$$

The norm on v' is specified by

$$\left(\int_{(0,\infty) \times [0,1]} |e^{\epsilon s} v'_a|^p + |\nabla(e^{\epsilon s} v'_a)|^p ds dt \right)^{1/p} + |v'_b|$$

Here we may use any norm $|\cdot|$ on the tangent spaces induced by smooth Riemannian metrics on the M_i 's. Since we assume that each M_i is compact, all such metrics lead to equivalent topologies on the tangent space to v .

A chart of B_2 near v is obtained by applying the exponential map to all such v' where the

norm of v' is sufficiently small. To be precise, one must use a t -dependent metric on \underline{M} which makes $L_{01} \times L_{12}$ totally geodesic for $t = 0$ and $L_{02} \times \Delta$ totally geodesic for $t = 1$. See [25] Section 2.2 for more details. Finally, we define our Banach manifold $\mathcal{B}_\epsilon(\underline{x}, \underline{y})$ over Σ as pairs $(w, v) \subset (B_1, B_2)$ which agree on the overlap $(0, 1) \times [0, 1]$.

Now, let \mathcal{V} be the Banach bundle over $\mathcal{B}_\epsilon(\underline{x}, \underline{y})$ whose fibre over \underline{u} is given by $\Omega^{0,1}(\Sigma, \underline{E})$ where \underline{E} is the pullback of the tangent bundles of M_i and $\Omega^{0,1}(\Sigma, \underline{E})$ denotes the space of $(0, 1)$ -forms with finite L^p norm and with exponential decay at the Y -end. On each of the two pieces, standard arguments (see [21] for B_1 and [25] for B_2) imply that the $\bar{\partial}$ operator is a restriction of a Fredholm operator to an open domain. Note that this does *not* mean that restriction of $\bar{\partial}$ to B_1 or B_2 is a Fredholm operator; we are saying that there exists a way to embed the open domains $\underline{\Sigma}$ and $(1, \infty) \times (0, 1)$ to bigger domains where one has extensions of the corresponding $\bar{\partial}$ operators to Fredholm operators. One can arrange this easily in the case of B_1 and B_2 . For example, one can take the double of each domain, and at the same time doubling all the structure of Banach bundles and the sections defined by $\bar{\partial}$ operators (see [6] for a discussion of such doubling in a related problem). In that way, B_1 is embedded into a standard Fredholm problem for a holomorphic quilt as studied in ([21], pg. 877) and B_2 is embedded into a standard Fredholm problem that underlies the definition of Floer differential for the Lagrangian Floer homology of $(L_{01} \times L_{12}, L_{02} \times \Delta)$ ([25], pg. 14).

Knowing that the $\bar{\partial}$ operator is a restriction of a Fredholm operator to an open domain for both B_1 and B_2 allows us to apply a standard patching argument (see for example [3] pg. 50) which implies that the $\bar{\partial}$ operator defines a Fredholm section over $\mathcal{B}_\epsilon(\underline{x}, \underline{y})$. Note that ϵ has to be chosen sufficiently small to ensure that $\bar{\partial}$ on B_2 is the restriction of a Fredholm operator to an open domain and that each element of $\mathcal{M}_J(\underline{x}, \underline{y})$ actually belongs to $\mathcal{B}_\epsilon(\underline{x}, \underline{y})$.

For future reference, note that the linearization of $\bar{\partial}$ at some v on $(0, \infty) \times [0, 1]$ has the form:

$$D\bar{\partial} \oplus K : L_{1,\epsilon}^p(\underline{E}, \underline{F}) \oplus T_{v(\infty)}((L_{01} \times L_{12}) \cap (L_{02} \times \Delta)) \rightarrow L_{0,\epsilon}^p(\Omega^{0,1}(\underline{E}))$$

Here $\underline{E} = v^*T\underline{M}$ and \underline{F} is the Lagrangian subbundle given by $v^*(T(L_{01} \times L_{12}))$ along $\{0, \infty\} \times \{0\}$ and $v^*(T(L_{02} \times \Delta))$ along $\{0, \infty\} \times \{1\}$; K is some operator with a finite dimensional linear domain $T_{v(\infty)}((L_{01} \times L_{12}) \cap (L_{02} \times \Delta))$. The specific form of K depends on the choice of the t -dependent metric g_t and will not need to be made explicit for our purposes. The special case when v is constant will be discussed below (Section 2.4). We now give a proof of the following claim:

Proposition 7 *For a generic choice of J , $\mathcal{M}_J(\underline{x}, \underline{y})$ is a smooth finite dimensional manifold.*

Proof We need to verify that, for a generic choice of J , the Fredholm section defined by $\bar{\partial}$ of the Banach bundle \mathcal{V} over $\mathcal{B}_\epsilon(\underline{x}, \underline{y})$ will be transverse to the zero section. Let \mathcal{J} be the space of almost complex structures constructed above (see page 8). Recall that we consider domain

dependent almost complex structures where this dependence is at the Y-end and only on a small disk $D \subset (0, \infty) \times (0, 1)$. Thus, given that $u_i \in \mathcal{M}_J(x, y)$, we need to show that the linearized operator $D\bar{\partial}(\underline{u}, J)$ is surjective. Note that as we use a domain dependent J , the linearized operator is the sum of two terms - one term coming from $D\bar{\partial}(\underline{u})$ corresponding to variations of \underline{u} and the second term corresponds to variations of J . First assume that the map v defined at the Y-end is non-constant. We will show that any $L_{0, -\epsilon}^q$ -section ($1/p + 1/q = 1$) η of the dual bundle $\Omega^{0,1}(\underline{E}, \underline{F})$ orthogonal to the image of the linearization must vanish around some point in D . Then the unique continuation principle will yield that η vanishes identically.

We now write the linearization of our section on the disk $D \subset (0, \infty) \times (0, 1)$, where we required J to have the domain dependence. The main point here is that since we allow our J to be domain dependent, we do not need a somewhere injective curve but simply a point $z_0 \in D$ such that $dv(z_0) \neq 0$. Following the argument in [15, page 48], the linearized operator has the following form on D :

$$D\bar{\partial}(v, J) = D\bar{\partial}(v) + \frac{1}{2}S(v, z) \circ dv \circ j_D$$

Here, $D\bar{\partial}(v)$ denotes the partial derivative holding J fixed and $S(z, v) \circ v \circ j_D$ corresponds to linearization with respect to J , where $S(z, v)$ is a section of the tangent space to \mathcal{J} at J which can be identified with the subspace of $S \in \text{End}(TM)$ such that

$$SJ = -JS, \quad \omega(S \cdot, \cdot) = -\omega(\cdot, S \cdot)$$

Now, suppose that some section η is orthogonal to the image. Following [15], we can choose $S(z_0, v(z_0))$ such that

$$\langle \eta(z_0), S(z_0, v(z_0)) \circ dv(z_0) \circ j_D \rangle > 0$$

whenever $\eta(z_0) \neq 0$. We can extend $S(z_0, v(z_0))$ to a small neighborhood of z_0 by using a bump function. Note that the resulting $S(z, v)$ is domain dependent. This shows that $\eta(z_0) = 0$ for all z_0 where $dv(z_0) \neq 0$. However, such z_0 are dense in D . Since η is in the kernel of a $\bar{\partial}$ -type operator, namely $(D\bar{\partial}(v))^* \eta = 0$, it must vanish everywhere by the unique continuation principle.

Finally, assume that v is constant, thus by unique continuation \underline{u} is constant. In the next subsection, we show that the index of the linearization is 0 (Lemma 8) for the case of a constant almost complex structure of split-type, that is, it comes from an almost complex structure J_i on M_i . Notice however that in Section 2.2 (Definition 5), we have required that almost complex structure J is of split-type except on a small disk D in order to ensure transversality at the constant solutions. On the other hand, for the purpose of calculation of index we can always homotope our J to a split-type almost complex structure so the calculation of the next subsection ensures that the index of the linearization is 0 for a regular J (as in Definition 5).

We claim that for any regular J , $D\bar{\partial}(\underline{u}, J)$ is surjective. In view of the index calculation, it is enough to show that the kernel of the linearization at a constant map is zero. We may identify the image of u_i

with $0 \in \mathbb{R}^{2n_i}$. An element \underline{u}' of the kernel of the linearization is then a triple of maps u'_i from the quilt to $(\mathbb{R}^{2n_i}, \omega_0)$. (Outside of the region where J is non-split, we have that $\bar{\partial}_{J_i} u'_i = 0$ but along the disk D where J is non-split we can only speak of J -holomorphicity for the folded map.) u'_i have linear quilted Lagrangian boundary conditions. Thus, along each seam $(u'_i, u'_{i+1}) \in L'_{i,i+1} \subset \mathbb{R}^{2(n_i+n_{i+1})}$ where $L'_{i,i+1}$ is a linear Lagrangian submanifold. By construction, u'_i have exponential decay near the incoming and outgoing ends. Near the Y-end, we can identify u'_i with the folded map

$$v' : (0, \infty) \times [0, 1] \rightarrow \mathbb{C}^{n_0+2n_1+n_2}$$

which satisfies $\bar{\partial}_J v' = 0$. Let us decompose v' as

$$v' = v'_a + v'_b$$

where v'_a has exponential decay at infinity and v'_b is a constant vector in the intersection of the linear Lagrangians. In particular, the total derivative of v' has exponential decay near infinity. Note that in the part where J does not split (i.e. near the Y-end) by assumption J is compatible with the symplectic structure on

$$(\mathbb{R}^{2n_0}, -\omega_0) \times (\mathbb{R}^{2n_1}, \omega_0) \times (\mathbb{R}^{2n_1}, -\omega_0) \times (\mathbb{R}^{2n_2}, \omega_0)$$

We claim that in fact each u'_i is constant. Now, the standard symplectic form $\omega_0 = \frac{1}{2}d(xdy - ydx)$ is an exact form and the linear Lagrangians $L'_{i,i+1}$ are exact Lagrangians, therefore by Stokes' theorem we have that the symplectic area vanishes:

$$\sum_{i=1}^3 \int_{\Sigma_i} (u'_i)^* \omega_0 = 0$$

Note that the use of Stokes' theorem is justified since near the incoming and outgoing ends u'_i have exponential decay and near the Y-end we have that the total derivative of v' decays exponentially. Now, the fact that \underline{u}' is holomorphic enables us to relate the symplectic area to the energy (see [15, page 20]). Since we have that J is non-split near the Y-end, in order express the energy, as before let us divide the domain into two pieces. Near the Y-end, we have the folded map v' with domain $(0, \infty) \times [0, 1]$. Denote by $\Sigma'_i = \Sigma_i \setminus ((0, \infty) \times [0, 1])$, the domain of u'_i outside of the neighborhood of the Y-end. We can now write out the relation between symplectic area and the energy of a holomorphic \underline{u}' as follows:

$$\sum_{i=1}^3 \frac{1}{2} \int_{\Sigma'_i} |du'_i|_{J_i}^2 dvol_{\Sigma'_i} + \frac{1}{2} \int_{(0, \infty) \times [0, 1]} |dv'|_J^2 dvol = \sum_{i=1}^3 \int_{\Sigma_i} (u'_i)^* \omega_0 = 0$$

where J_i are almost complex structures on M_i and $J = (J_i)_{i=1}^3$ over $(\Sigma'_i)_{i=1}^3$.

Therefore, we conclude that the u'_i are constant. Since u'_i converges to zero at the incoming and outgoing ends, this implies that $u'_i = v' = 0$ as desired. \square

2.4 Index computation for a constant map

In this section we calculate the index of the linearization of $\bar{\partial}$ operator at a constant map. Although, everywhere else, we used Sobolev spaces $L^p_{1,\epsilon}$ and $L^p_{0,\epsilon}$ for $p > 2$, in calculating the index we take $p = 2$ as this makes the calculation of the index easier. Since the kernel and cokernel of the linearized $\bar{\partial}$ operator is smooth by elliptic regularity, this calculation immediately implies the index calculation for $p > 2$.

Lemma 8 *Let $u \in \mathcal{M}_J(\underline{x}, \underline{x})$ be a constant map, then the index of u vanishes.*

The linearized problem we will study is that of holomorphic quilts mapping into \mathbb{C}^n with linear Lagrangian boundary conditions. As preparation for the main result, we first review a standard Morse-Bott index calculation. Let $L, L' \subset \mathbb{C}^n$ be a pair of Lagrangian subspaces. Let $S = \mathbb{R} \times [0, 1]$. We consider the Fredholm map

$$\bar{\partial} : L^2_{1,\epsilon}(S; L, L') \rightarrow L^2_{0,\epsilon}(S)$$

for sufficiently small $\epsilon > 0$. Here $L^2_{1,\epsilon}(S; L, L')$ denotes the weighted Sobolev space of maps with $u(\cdot, 0) \in L$ and $u(\cdot, 1) \in L'$. Note that in view of the restriction map

$$L^2_1(S) \rightarrow L^2(\partial S)$$

we do not need u to be continuous to make sense of the linear Lagrangian boundary condition. For a general discussion of regularity for the $\bar{\partial}$ -operator with totally real boundary conditions that include the rather special case we are considering, see Theorem C.1.10 in [15].

Lemma 9 $ind(\bar{\partial}) = -dim(L \cap L')$.

Proof A function $u : \mathbb{R} \times [0, 1] \rightarrow (\mathbb{C}^n; L, L')$ may be written as

$$u(s, t) = \sum_{\lambda} f(t) \phi_{\lambda}(s)$$

where ϕ_{λ} is an eigenfunction with eigenvalue λ of the operator $-i\partial_s$ on $[0, 1]$ with $\phi_{\lambda}(0) \in L$ and $\phi_{\lambda}(1) \in L'$.

The kernel of $\bar{\partial} = \partial_t + i\partial_s$ consists of maps

$$u(s, t) = \sum_{\lambda} c_{\lambda} e^{t\lambda} \phi_{\lambda}(s)$$

for some constants c_{λ} . However, since λ is real and such solutions are required to have exponential decay for $t \rightarrow \pm\infty$, it must be that $c_{\lambda} = 0$ for all λ . Hence, we conclude that $dim \ker \bar{\partial} = 0$.

By elliptic regularity, the cokernel can be identified with the kernel of $-\partial_t + i\partial_s$ on the space of L^2_1 functions with exponential growth of at most ϵ . In addition, these functions have boundary values on iL and iL' . Therefore, such maps consist of

$$u(s, t) = \sum_{\lambda} c_{\lambda} e^{-t\lambda} \phi_{\lambda}(s)$$

However, if ϵ is smaller than the first nonzero eigenvalue λ_0 , the only maps are those with $\lambda = 0$. These are precisely the constant maps with values in $(iL) \cap (iL')$. Therefore, index $\bar{\partial} = -\dim((iL) \cap (iL')) = -\dim(L \cap L')$. \square

The calculation of the index for

$$\bar{\partial} : L^2_{1,\epsilon}(S; L, L') \rightarrow L^2_{0,\epsilon}(S)$$

computes the index for any

$$\bar{\partial} : L^p_{k,\epsilon}(S; L, L') \rightarrow L^p_{k-1,\epsilon}(S)$$

with $p \geq 2$ and $k \geq 1$. Indeed, elliptic regularity for the $\bar{\partial}$ -operator with linear Lagrangian boundary conditions implies that any element of the kernel/cokernel is smooth. Strictly speaking, one must first prove that the corresponding problem for $L^p_{k,\epsilon}$ -spaces is Fredholm. As explained in [3] (pg.58-60, 70-75), one can convert the index problem over weighted spaces to an equivalent problem for unweighted spaces where the Fredholm property is standard. The previous lemma is useful when considering Morse-Bott moduli spaces. In particular, consider the tangent space at the constant map of the moduli space of holomorphic curves with Morse-Bott boundary conditions along (L, L') . By definition, it is the kernel of the map

$$\bar{\partial} \oplus K : L^2_{1,\epsilon}(S; L, L') \oplus ((L \cap L') \times (L \cap L')) \rightarrow L^2_{0,\epsilon}(S)$$

For the calculation of the index the explicit form of the map K is not relevant since it is a compact operator. Thus we have:

Corollary 10 $ind(\bar{\partial} \oplus K) = \dim(L \cap L')$.

This is consistent with the intuition that the Morse-Bott case corresponds to constant holomorphic disks lying on $L \cap L'$.

We will make use of excision for our index calculations. This is a standard tool for computing the index of elliptic operators that goes back to the work of Atiyah and Singer on the index theorem ([1]). We review a simple version of it that is tailored to our application. For recent proofs, one may consult [2].

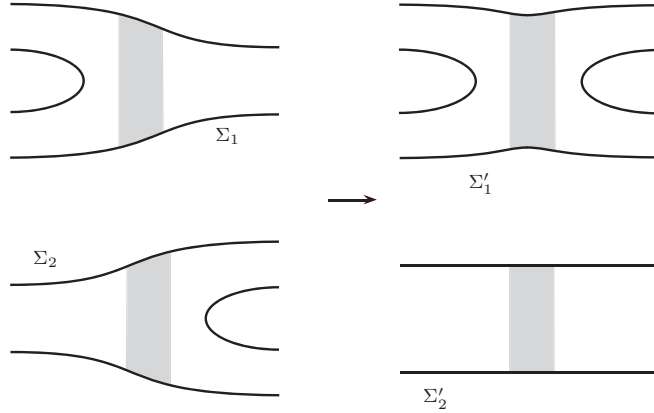


Figure 2: Excision

Suppose we are given quilts Σ_1, Σ_2 each with a pair of complex vector bundles E_i and F_i . In addition, suppose we have $\bar{\partial}$ -operators

$$\bar{\partial}_i : \Gamma(E_i) \rightarrow \Gamma(F_i)$$

over each Σ_i . At the boundaries, we assume that there are totally real boundary conditions. This amounts to a choice of a totally real subbundle T_i of each E_i over the boundary of Σ_i .

Now, assume that each Σ_i contains a separating strip $(a, b) \times [0, 1]$. We assume there are isomorphisms $F : E_{1|(a,b) \times [0,1]} \rightarrow E_{2|(a,b) \times [0,1]}$ and $G : F_{1|(a,b) \times [0,1]} \rightarrow F_{2|(a,b) \times [0,1]}$ which map $\bar{\partial}_1$ to $\bar{\partial}_2$ and T_1 to T_2 , respectively. We may excise Σ_i along the strips as in Figure 2 to form new quilts Σ'_1 and Σ'_2 with corresponding bundles and $\bar{\partial}$ -operators $\bar{\partial}'_1$ and $\bar{\partial}'_2$. The excision theorem asserts that

$$\text{ind}(\bar{\partial}_1) + \text{ind}(\bar{\partial}_2) = \text{ind}(\bar{\partial}'_1) + \text{ind}(\bar{\partial}'_2)$$

A similar discussion applies when instead of a separating strip we have a separating cylinder $(a, b) \times S^1$

We are ready to compute the index of the linearization at a constant map. Note that for this linearization all maps are into \mathbb{C}^n with the standard complex structure and the nonlinear Lagrangian boundary conditions are replaced by their tangent spaces in \mathbb{C}^n . Consider the nine figures drawn in Figure 3. Let m_i stand for the index of Fig i . We wish to compute m_1 . We have shown in the previous section that the kernel of the map represented by Fig 1 is zero. Similarly the kernel of Fig 2 is zero. This implies that $m_1 \leq 0$ and $m_2 \leq 0$. By additivity of index,

$$m_3 = m_1 + m_2$$

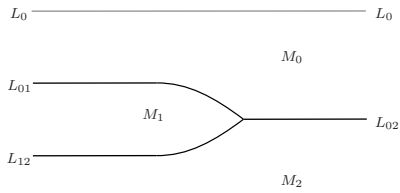


Fig 1

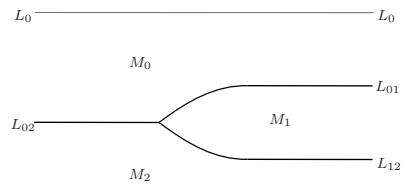


Fig 2

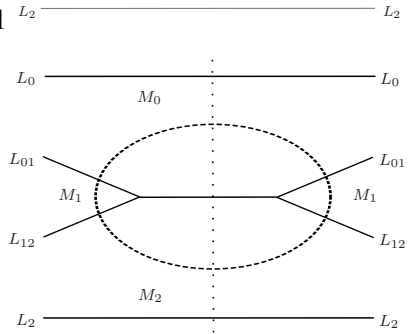


Fig 3

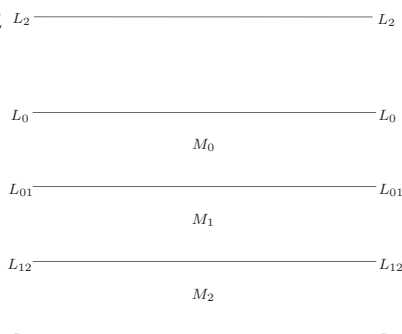


Fig 4

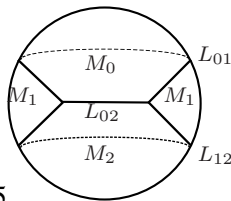


Fig 5

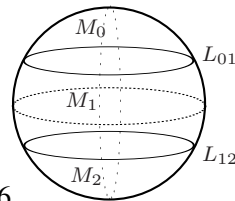


Fig 6

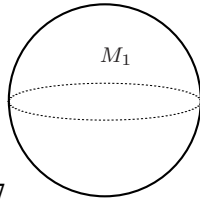


Fig 7

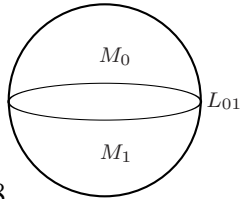


Fig 8

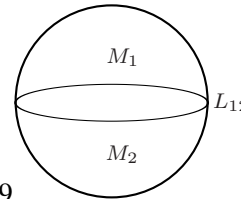


Fig 9

Figure 3: Index calculation using excision

Excising Fig 3 and 6 along a neighborhood of the dotted circles (where we use the vertical dotted circle for Fig 6) gives

$$m_3 + m_6 = m_4 + m_5$$

Now, we claim that $m_5 = \dim(L_{02})$. To see this, one simply folds to obtain a single strip with Morse-Bott Lagrangian boundary conditions on $(L_{01} \times L_{12}, L_{02} \times \Delta)$. Thus, the discussion right after Lemma 9 above gives $m_5 = \dim(L_{02})$. We have $m_4 = 0$ since it is the quilt of the identity map. To compute m_6 , note that excision (this time we use horizontal dotted circle for Fig 6) implies that

$$m_6 + m_7 = m_8 + m_9$$

By folding, we have that m_8 and m_9 represent disks so

$$m_8 + m_9 = \dim(L_{01}) + \dim(L_{12})$$

and $m_7 = \dim(M_1)$ since it is the linearization of a constant map of a sphere. Thus, $m_6 = \dim(L_{02})$ which together with $m_4 = 0$ and $m_5 = \dim(L_{02})$ gives $m_3 = 0$. This implies $m_1 = m_2 = 0$, as desired.

2.5 Completion of the proof of Theorem 1

By Proposition 7, the moduli spaces $\mathcal{M}_J(\underline{x}, \underline{y})$ are transversely cut out. To define a count we need to show that the zero dimensional moduli spaces $\mathcal{M}_J^0(\underline{x}, \underline{y})$ is compact and hence finite. Then, $\mathcal{M}_J^0(\underline{x}, \underline{y})$ allows us to define the map

$$\Phi : CF(L_0, L_{01}, L_{12}, L_2) \rightarrow CF(L_0, L_{02}, L_2)$$

We will sometimes refer to this map by Y-map. To verify that this is indeed a chain map we need to consider the 1-dimensional moduli spaces $\mathcal{M}_J^1(\underline{x}, \underline{y})$.

First note that the set of assumption (1) on second homotopy classes ensure that we cannot have any interior disk or sphere bubbles. Therefore, by Gromov compactness the boundary of $\mathcal{M}_J^0(\underline{x}, \underline{y})$ and $\mathcal{M}_J^1(\underline{x}, \underline{y})$ consists of broken configurations at the ends. In the case of $\mathcal{M}_J^0(\underline{x}, \underline{y})$, there cannot be breaking at the \underline{x} and \underline{y} ends because by our transversality assumptions such a break cannot occur in a 0-dimensional component of the moduli space. Finally, we need to argue that for both $\mathcal{M}_J^0(\underline{x}, \underline{y})$ and $\mathcal{M}_J^1(\underline{x}, \underline{y})$ there cannot be a breaking at the Y-end.

To this end, the following lemma will be useful:

Lemma 11 *Let $\delta : \mathbb{R} \times [0, 1] \rightarrow (\underline{M}; L_{01} \times L_{12}, L_{02} \times \Delta)$ be a smooth map with Lagrangian boundary conditions and at the two ends converges exponentially to points in the Morse-Bott intersection. Then*

there exists a smooth map $\tilde{\delta} \in \pi_2(\underline{\mathbf{M}}; L_{01} \times L_{12})$ such that

$$\int \delta^* \omega_M = \int \tilde{\delta}^* \omega_M$$

Furthermore, the Fredholm indices of δ and $\tilde{\delta}$ are related as follows:

$$\text{index}(\tilde{\delta}) + 2\dim(M_1) = \text{index}(\delta)$$

Proof It will be convenient to view δ as a quilted map

$$\delta_i : \mathbb{R} \times [0, 1] \rightarrow M_i, i = 1, 2, 3$$

with cyclic Lagrangian boundary conditions (L_{01}, L_{12}, L_{02}) . Thus, we have $(\delta_2(1, s), \delta_0(0, s)) \in L_{02}$, etc. Let $\delta_4 : \mathbb{R} \times [0, 1] \rightarrow M_1$ be the map with $\delta_4(s, t) = b(s)$, where $b(s)$ is the unique point on M_1 with $(\delta_2(1, s), b(s), b(s), \delta_0(0, s)) \subset L_{01} \times L_{12}$. Note that δ_4 is a smooth map which is not holomorphic but converges exponentially as $|s| \rightarrow \infty$. Furthermore, the image of δ_4 is just a path, thus δ_4 has zero area. We have now obtained a new quilt δ' with four patches δ_i and seams $(L_{01}, L_{12}, L_{01}, L_{12})$ while the area of δ and δ' is the same. We fold δ' to obtain a map

$$\mathbb{R} \times [0, 1] \rightarrow \underline{\mathbf{M}}$$

with boundary on $(L_{01} \times L_{12}, L_{01} \times L_{12})$. Alternatively, we may view this as a map

$$\tilde{\delta} : D \rightarrow \underline{\mathbf{M}}$$

where D is the unit disk and $\tilde{\delta}$ has Lagrangian boundary conditions on $L_{01} \times L_{12}$. This map satisfies the required property, since $\delta, \delta', \tilde{\delta}$ all have the same area.

To see the relation of Fredholm indices, we note that $\delta, \delta', \tilde{\delta}$ all have the same Maslov index since t -derivative of δ_4 vanishes (cf. [22] pg. 846). The Fredholm index is given by the sum of Maslov index and the dimension of the Morse-Bott intersection. To conclude, observe that the dimension of the Morse-Bott intersection for δ is $\dim((L_{01} \times L_{12}) \cap (L_{02} \times \Delta)) = \dim(M_0) + \dim(M_2)$ while it is $\dim(L_{01} \times L_{12}) = \dim(M_0) + 2\dim(M_1) + \dim(M_2)$ for $\tilde{\delta}$. \square

Back to the proof of Theorem 1, observe that a bubble at the Y-end would be a holomorphic map $\delta : \mathbb{R} \times [0, 1] \rightarrow M_0^- \times M_1 \times M_1^- \times M_2$ as in the previous lemma. However, holomorphicity ensures that it has positive area. Therefore, we would obtain an element $\tilde{\delta} \in \pi_2(\underline{\mathbf{M}}; L_{01} \times L_{12})$ which has positive area, which is impossible by the assumption (1). Thus, there cannot be a bubbling at the Y-end.

Therefore, standard gluing theory applied to $\mathcal{M}_J^1(\underline{x}, \underline{y})$ shows that Φ is a chain map.

Remark 12 Note that we do not need to consider Morse-Bott gluing as the only place where a breaking could occur is at the ends where we have transverse intersection.

To complete the proof we need to show that Φ induces an isomorphism on cohomology. Let us write $\mathcal{P}^{in} = \{(\gamma_0, \gamma_1, \gamma_2) \mid \gamma_i : [0, 1] \rightarrow M_i, \gamma_0(0) \in L_0, (\gamma_0(1), \gamma_1(0)) \in L_{01}, (\gamma_1(1), \gamma_2(0)) \in L_{12}, \gamma_2(1) \in L_2\}$. Each generator of the chain complex $CF(L_0, L_{01}, L_{12}, L_2)$ is an element of \mathcal{P}^{in} and the chain complex splits into a direct sum of chain complexes corresponding to the path components of \mathcal{P}^{in} . In what follows, we assume that \mathcal{P}^{in} is path-connected in order to avoid the notational complexity of indexing path components and carrying out the argument for each path-component separately.

As above, assumption (1) enable us to have a well-defined action functional,

$$\mathcal{A}^{in} : \mathcal{P}^{in} \rightarrow \mathbb{R}$$

$$\gamma \rightarrow \sum_{i=0}^2 \int_{[0,1]^2} (\gamma_i^t)^* \omega_i$$

where as before γ_i^t is any choice of a smooth homotopy in \mathcal{P}^{in} between γ_i and a fixed path on \mathcal{P}^{in} , and ω_i are the given symplectic forms on M_i . Therefore, the chain complex $CF(L_0, L_{01}, L_{12}, L_2)$ inherits a filtration given by \mathcal{A}^{in} . Recall that the Floer differential decreases the action functional.

Next, we have a similar filtration on $CF(L_0, L_{02}, L_2)$, where we write

$$\mathcal{P}^{out} = \{(\gamma_0, \gamma_2) \mid \gamma_i : [0, 1] \rightarrow M_i, \gamma_0(0) \in L_0, (\gamma_0(1), \gamma_2(0)) \in L_{02}, \gamma_2(1) \in L_2\}$$

and \mathcal{A}^{out} is defined as before. Note that \mathcal{P}^{out} consists of elements of \mathcal{P}^{in} such that γ_1 is constant. Therefore, the action functional $\mathcal{A}^{out} = \mathcal{A}^{in}$ whenever both are defined.

In view of the fact that constant maps are the only zero dimensional solutions which preserve the action (and they are transversely cut out as proved in Proposition 7), to conclude that Φ is an isomorphism, it suffices to show that Φ is a filtered chain map. For this, it suffices to show that if $\mathcal{M}_J^0(\underline{x}, \underline{y})$ is non-empty, then the following inequality holds :

$$\mathcal{A}^{in}(\underline{x}) \geq \mathcal{A}^{out}(\underline{y})$$

where the equality holds only if $\underline{x} = \underline{y}$ and $\mathcal{M}_J^0(\underline{x}, \underline{y})$ consists entirely of the trivial solution.

To see this, let u be a holomorphic curve in $\mathcal{M}_J^0(\underline{x}, \underline{y})$. Now, u can be considered as a path $(\gamma_i^t)_{i=0}^2$ in the path space \mathcal{P}^{in} such that γ_1^t shrinks to a constant path as we get to the Y-end and stays constant until the outgoing end. Now, since u is a holomorphic map, the action strictly decreases unless u is constant. This gives the desired inequality: $\mathcal{A}^{in}(\underline{x}) \geq \mathcal{A}^{in}(\underline{y}) = \mathcal{A}^{out}(\underline{y})$ with equality only if $\underline{x} = \underline{y}$ and u is constant. So, we have $\#\mathcal{M}_J^0(\underline{x}, \underline{x}) = 1$ with contributions coming only from constant solutions, which are cut transversely. We conclude that Φ is an isomorphims, as desired. \square

3 Extensions of the main theorem

In this section, we discuss the proof of Theorem 1 under positive and strongly negative monotonicity assumptions. This result in the positively monotone case was first proved by Wehrheim and Woodward by different techniques. However, the strongly negative monotone case is new and important for our application.

As a first step, we prove a topological lemma which will allow us to establish an a priori energy bound for pseudoholomorphic curves counted in the moduli space $\mathcal{M}_J(\underline{x}, \underline{y})$ used for defining the map $\Phi : CF(L_0, L_{01}, L_{12}, L_2) \rightarrow CF(L_0, L_{02}, L_2)$.

Lemma 13 *Let $\underline{x}, \underline{y} \in CF(L_0, L_{01}, L_{12}, L_2) \simeq CF(L_0, L_{02}, L_2)$ be two generalized intersection points. Let $\mathcal{P}(\underline{x}, \underline{y})$ be the space of maps $(-\infty, \infty) \rightarrow \mathcal{P}(L_0, L_{01}, L_{12}, L_2)$ which asymptotically converge to \underline{x} and \underline{y} . Similarly, let $\mathcal{B}(\underline{x}, \underline{y})$ be the space of smooth maps that is considered for defining the map $\Phi : CF(L_0, L_{01}, L_{12}, L_2) \rightarrow CF(L_0, L_{02}, L_2)$ (see Section 2.2) Then there is a natural inclusion map $\mathcal{B}(\underline{x}, \underline{y}) \hookrightarrow \mathcal{P}(\underline{x}, \underline{y})$ which induces an isomorphism:*

$$\pi_0(\mathcal{P}(\underline{x}, \underline{y})) \cong \pi_0(\mathcal{B}(\underline{x}, \underline{y}))$$

In particular, any homotopy classes of maps used to define $\Phi : CF(L_0, L_{01}, L_{12}, L_2) \rightarrow CF(L_0, L_{02}, L_2)$ mapping \underline{x} to \underline{y} can be represented as a concatenation of maps $\Phi = u\#c$, where $u \in \mathcal{P}(\underline{x}, \underline{y})$ and $c : CF(L_0, L_{01}, L_{12}, L_2) \rightarrow CF(L_0, L_{02}, L_2)$ is the constant map with value \underline{y} .

Proof Recall that the space of paths in the absence of Hamiltonian perturbations (which we assume, by an a priori arrangement of transversality of L_0, L_{01}, L_{12}, L_2 as before) is given by $\mathcal{P}(L_0, L_{01}, L_{12}, L_2) = \{(\gamma_1, \gamma_2, \gamma_3) | \gamma_i : [0, 1] \rightarrow M_i, \gamma_1(0) \in L_0, (\gamma_1(1), \gamma_2(0)) \in L_{01}, (\gamma_2(1), \gamma_3(0)) \in L_{12}, \gamma_3(1) \in L_2\}$. We will denote a path $\gamma : (-\infty, \infty) \rightarrow \mathcal{P}(L_0, L_{01}, L_{12}, L_2)$ in this path space by $\gamma^s = (\gamma_1^s, \gamma_2^s, \gamma_3^s) \in \mathcal{P}(\underline{x}, \underline{y})$, where $s \in (-\infty, \infty)$. Now the space $\mathcal{B}(\underline{x}, \underline{y})$ can be identified as a subspace of $\mathcal{P}(\underline{x}, \underline{y})$ where $\gamma^s = (\gamma_1^s, \gamma_2^s, \gamma_3^s) \in \mathcal{B}(\underline{x}, \underline{y})$ if and only if γ_2^s is a constant with respect to t for $s \geq 1$. More precisely, first note that any map in $\mathcal{B}(\underline{x}, \underline{y})$ can be homotoped to be constant around a neighborhood of the Y-end (because of the exponential convergence at the Y-end). Now, for a moment, let us forget about all the decorations and seam conditions on the domain of Φ as given in Figure 1, we then see a rectangle with a middle point (Y-end) removed. Let's identify this rectangle with $\mathbb{R} \times [0, 1]$ (\mathbb{R} corresponds to the horizontal direction and $[0, 1]$ corresponds to the vertical direction in Figure 1). We arrange so that the Y-end point corresponds to $(1, 1/2)$. Let's foliate this rectangle by vertical lines $L_s = \{s\} \times [0, 1]$. Now, if we look at $\Phi(L_s)$ for $s < 1$ then we get 3 paths $(\gamma_1^s, \gamma_2^s, \gamma_3^s)$ as restrictions of $\Phi|_{L_s}$. For $s > 1$, we obtain (γ_1^s, γ_3^s) as restrictions of $\Phi|_{L_s}$ such that $(\gamma_1^s(1), \gamma_3^s(0)) \in L_{02}$. We can view this alternatively, as a triple of paths $(\gamma_1^s, \gamma_2^s, \gamma_3^s)$ where γ_2^s is the constant path such that $(\gamma_1^s(1), \gamma_2^s(t)) \in L_{01}$ and $(\gamma_2^s(t), \gamma_3^s(0)) \in L_{12}$ (such a path is uniquely

determined by the composability of L_{01} and L_{12}). Finally, since we arranged that Φ is constant near the Y -end, the triple $\gamma^s = (\gamma_1^s, \gamma_2^s, \gamma_3^s)$ of path of paths extends continuously over $s = 1$ and we obtain $\gamma^s \in \mathcal{P}(\underline{x}, \underline{y})$. Note that γ_2^s does not vary with t for $s \geq 1$.

The desired equivalence of path components can be seen by noting that any path $\gamma^s = (\gamma_1^s, \gamma_2^s, \gamma_3^s) \in \mathcal{P}(\underline{x}, \underline{y})$ is homotopic to a path which is constant for $s > N$ for some sufficiently large N by the requirement of convergence as $s \rightarrow \infty$. One can then isotope γ^s so that it is constant for $s \geq \frac{1}{2}$ in both s and t . Thus, we have an inverse map $\pi_0(\mathcal{P}(\underline{x}, \underline{y})) \rightarrow \pi_0(\mathcal{B}(\underline{x}, \underline{y}))$ to the map induced by the inclusion map. This gives the desired isomorphism.

To see the last part of the statement more explicitly, express any the homotopy class $\rho \in \mathcal{P}(\underline{x}, \underline{y})$ as $(\gamma_1^s, \gamma_2^s, \gamma_3^s)$ as above with γ_i^s constant for $s > N$. Now, consider the homotopy ρ^r where $r \in [0, 1]$ given by $(\gamma_1^{s+rN}, \gamma_2^{s+rN}, \gamma_3^{s+rN})$. Then ρ^1 is a map in $\mathcal{B}(\underline{x}, \underline{y})$ which is constant for $s \geq \frac{1}{2}$, hence it is a concatenation of $u \in \mathcal{P}(\underline{x}, \underline{y})$ and the constant map with value \underline{y} as stated. \square

We are now ready to prove the extension of Theorem 1 to the monotone case:

3.1 Proof of Theorem 2

We briefly recall from [21] why the Floer cohomology groups in consideration are well-defined (independent of the choices, invariant under Hamiltonian deformations, etc.). Given $\underline{x}, \underline{y} \in L_{(0)} \cap L_{(1)}$, the monotonicity assumptions guarantee that the energy of index k holomorphic strips $u \in \mathcal{M}^k(\underline{x}, \underline{y})$ is constant. Therefore, by Gromov-Floer compactness it suffices to exclude disk and sphere bubbles. The monotonicity assumptions ensure that any non-trivial holomorphic disk must have non-zero Maslov index which excludes disk bubbles in 0-dimensional moduli space. If one assumes that the Lagrangians are orientable, the Maslov index at a disk bubble has to be at least 2, which excludes disk bubbles in 0- and 1-dimensional moduli spaces. However, to have a well-defined Floer cohomology group we also need to avoid disk bubbles in index 2 moduli spaces, hence we require the minimal Maslov index for disks to be at least 3 (the sphere bubbles are handled similarly).

Now, as before we will consider the map

$$\Phi : HF(L_0, L_{01}, L_{12}, L_2) \rightarrow HF(L_0, L_{02}, L_2)$$

which is defined by counting solutions in $\mathcal{M}_J^0(\underline{x}, \underline{y})$. Let us first study the compactness property of the moduli space $\mathcal{M}_J(\underline{x}, \underline{y})$ under our assumptions. We first need to establish an area-index relation to have an a priori energy bound so that we can apply Gromov-Floer compactness. This follows easily from Lemma 13. Namely, to compute the index of an element $\Phi \in \mathcal{M}_J(\underline{x}, \underline{y})$, we can topologically apply a homotopy as in Lemma 13 so that $\Phi = u\#c$ where u is a map contributing to the differential of the chain complex $CF(L_0, L_{01}, L_{12}, L_2)$ and $c \in \mathcal{M}_J(\underline{y}, \underline{y})$ is the *constant* map at \underline{y} . In Section 2.4,

we computed the index of c to be equal to zero. (Indeed, this computation is the non-trivial part of the argument that we are giving here). Therefore, by excision,

$$\text{index}(\Phi) = \text{index}(u) + \text{index}(c) = \text{index}(u)$$

Now, by the area-index relation for the moduli space that u belongs to (this follows from the monotonicity assumptions, see [21, Remark 5.2.3]), the energy of index k holomorphic strips is constant. Since the energy of Φ is equal to the energy u , it follows that the energy of index k maps in $\mathcal{M}_J(\underline{x}, \underline{y})$ is constant. This last statement is the main output of monotonicity assumptions and it is what we mean by area-index relation for holomorphic curves in $\mathcal{M}_J(\underline{x}, \underline{y})$.

In view of the area-index relation for maps in $\mathcal{M}_J(\underline{x}, \underline{y})$, we have energy bounds on all trajectories of index 0 and 1 and thus Gromov-Floer compactification holds. Therefore, the compactification includes broken configurations at the ends, possibly also including the Y-end, and disk and sphere bubbles. However, as before our monotonicity assumptions ensure that disk and sphere bubbles do not arise in the compactification of index 0 and 1 moduli spaces. Recall that \mathcal{M}_J^0 is used to define the map Φ and \mathcal{M}_J^1 is used to check that it is a chain map.

We now need to deal with bubbles at the Y-end. Given a sequence of trajectories $u_i \in \mathcal{M}_J^1(\underline{x}, \underline{y})$ breaking along the Y-end, by Gromov-Floer compactness, we get in the limit a pair (u_∞, δ) where $u_\infty \in \mathcal{M}_J(\underline{x}, \underline{y})$ (possibly broken at the incoming and outgoing ends) and δ is a holomorphic strip with Morse-Bott boundary conditions along $(L_{01} \times L_{12}, L_{02} \times \Delta)$. Note that δ can also be broken but that does not affect the argument. Now, if δ is non-constant, it will have non-zero energy, therefore we have

$$E(u_\infty) < E(u_i)$$

By an application of Lemma 11, from δ we obtain a map $\tilde{\delta}$ with $E(\delta) = E(\tilde{\delta})$ which represents a class in $\pi_2(\underline{\mathbf{M}}, L_{01} \times L_{12})$ therefore if δ is non-constant, by energy-index relation $\tilde{\delta}$ has at least index 2 since $L_{01} \times L_{12}$ are assumed to be orientable. By the index computation in Lemma 11 we conclude that δ has index at least 2 if it is non-constant. Again by the energy-index relation proved above for maps in $\mathcal{M}_J(\underline{x}, \underline{y})$, this implies

$$\text{index}(u_\infty) \leq \text{index}(u_i) - 2 = -1$$

Since the index is additive, there exists at least one unbroken holomorphic piece in u_∞ with negative index. However, since these moduli spaces are cut out transversely, this cannot occur. Therefore, \mathcal{M}_J^0 and \mathcal{M}_J^1 cannot have any broken configuration with a bubble at the Y-end. This concludes the argument that the map Φ is well-defined.

Now, to check that Φ is an isomorphism, we construct an approximate inverse to Φ . Let

$$\Psi : HF(L_0, L_{02}, L_2) \rightarrow HF(L_0, L_{01}, L_{12}, L_2)$$

be the map given by counting index 0 holomorphic maps from the quilt that is obtained by reversing the orientation of the quilt that is used to define Φ . Arguments identical to those for Φ , show that Ψ is a chain map. We claim that

$$\Psi \circ \Phi = I + K$$

where I is the identity and K is a nilpotent map. This will prove that Φ is an isomorphism. The diagonal entries of $\Psi \circ \Phi$ are obtained by counting pairs of broken trajectories (u_1, u_2) with u_1 starting at a critical point x and u_2 ending at the same critical point. In addition, u_1 has the same endpoint as the starting point of u_2 . By the area-index relation, the only such trajectories of index 0 are the constants. More generally, given a sequence of such broken pairs $(u_1, u_2), (u_3, u_4) \cdots (u_{N-1}, u_N)$ such that the endpoint of u_i is the starting point of u_{i+1} and the starting point of u_1 is the same as the endpoint of u_N we have that the only index 0 trajectory is the constant one. Indeed, since index and area are additive, any such trajectory with nonzero area has positive index.

Let N_0 denote the number of intersection points, i.e. cardinality of $\mathcal{I}(\underline{L})$. Consider a broken trajectory, that is a sequence of holomorphic curves that contribute to $\Psi \circ \Phi$, of index zero with no constant pairs. In other words, these are the holomorphic curves that contribute to $K = \Psi \circ \Phi - I$. Any such trajectory with more than $N_0 - 1$ pairs must have a repeated intersection point. Thus such trajectories do not arise, since they would include a segment which has positive index, as we just explained. Now, if $K^k(x)$ is nonzero there must be a broken trajectory consisting of k pairs connecting x to some critical point y . This trajectory consists of non-constant pairs. This is easily seen by induction. First, $K(x)$ lies in the span of critical points connected to x by a non-constant pair. Suppose that $K^i(x)$ lies in the span of critical points y_i connected to x by a non-constant broken path of pairs of length $i - 1$. Any nonzero matrix element $\langle y, K(y_i) \rangle$ gives rise to a critical point y connected to y_i by a non-constant pair. It is thus connected to x by a non-constant path of pairs of length i as desired. Therefore, we may conclude that $K^{N_0} = 0$ since it is contained in the span of elements coming from broken pairs of length N_0 . This completes the proof that Φ is an isomorphism. \square

3.2 Proof of Theorem 3

In this case, we follow the same steps as in the positively monotone case. The only difference is the way we handle various exclusions of bubbles. Namely, we exclude bubbling by first arranging the transversality for the moduli spaces of *simple* sphere bubbles and *simple* disk bubbles. (Recall that simple means not multiply-covered). The strongly negative monotonicity assumption is the assumption that the expected dimension of these moduli spaces is negative therefore when transversality holds (which can be arranged by choosing the almost complex structure J in the target generically), we guarantee that these moduli spaces are empty. A lemma of McDuff ([15],

Proposition 2.51) and the decomposition lemma of Kwon-Oh [8] and Lazzarini ([9]) allows us to lift this to non-simple sphere and disk bubbles. More specifically, the lemma of McDuff states that any pseudoholomorphic sphere factors through a simple pseudoholomorphic sphere, so the existence of the former one implies the existence of the latter. Similarly, Kwon-Oh and Lazzarini's lemma implies that the existence of any pseudoholomorphic disk ensures the existence of a simple pseudoholomorphic disk. Now, recall that the expected dimension of unparameterized moduli space of spheres in M_i in the homology class $[u]$ is $2(\langle [u], c_1(TM_i) \rangle + m_i - 3)$. As part of the hypothesis, we assumed that this number is negative, in fact we assumed that this number is strictly less than -2 to exclude bubbling in $\mathcal{M}_J^k(\underline{x}, \underline{y})$, for $k = 0, 1, 2$. This is required to ensure that the Floer cohomology groups that we are considering are well-defined and independent of the auxiliary choices. Similarly, to avoid disk bubbles, recall that by the real-analyticity of the seams, any disk bubble in a quilted map can be seen as a disk bubble in $M_i \times M_{i+1}$ with boundary on L_{i+1} for some i . The expected dimension for unparameterized simple disks in the homology class u is given by $\mu_{L_{i+1}}([u]) + (m_i + m_{i+1}) - 3$. We assumed that this number is strictly less than -2 to avoid disk bubbles in $\mathcal{M}_J^k(\underline{x}, \underline{y})$, for $k = 0, 1, 2$ for the same reason as before.

Therefore, these considerations imply that the Floer cohomology groups are well-defined. Furthermore, the negative monotonicity assumption gives an area-index relation as before, which guarantees an a priori energy bound on the moduli space $\mathcal{M}_J^k(\underline{x}, \underline{y})$, hence Gromov-Floer compactness applies. Since we excluded the possibility of the sphere and disk bubbled configurations in the compactification of the moduli spaces \mathcal{M}_J^0 and \mathcal{M}_J^1 , to finish off the only remaining issue is to exclude the bubbling at the Y-end. We will follow the notation given in the proof of Theorem 2. We need to exclude non-constant δ bubbles. Recall that

$$\delta : \mathbb{R} \times [0, 1] \rightarrow (\underline{M}; L_{01} \times L_{12}, L_{02} \times \Delta)$$

is a strip with Lagrangian boundary conditions and at the two ends converges exponentially to points in the Morse-Bott intersection. Note that we can always ensure the transversality of the moduli space of such J -holomorphic curves by choosing our J to be t -dependent near the Y-end (cf. [5]).

Lemma 14 $\text{index}(\delta) \leq 0$.

Proof Via the construction given in Lemma 11 we will relate the index of δ to that of a disk in \underline{M} with boundary on $L_{01} \times L_{12}$. The desired conclusion will then follow from the monotonicity assumptions of Theorem 3. Recall from Lemma 11 that we view δ as a quilt of maps

$$\delta_i : \mathbb{R} \times [0, 1] \rightarrow M_i, i = 1, 2, 3$$

with cyclic Lagrangian boundary conditions (L_{01}, L_{12}, L_{02}) . We introduce $\delta_4 : \mathbb{R} \times [0, 1] \rightarrow M_1$ to be the map with $\delta_4(s, t) = b(s)$, where $b(s)$ is the unique point on M_1 with $(\delta_2(1, s), b(s), b(s), \delta_0(0, s)) \in L_{01} \times L_{12}$. This defines a quilted map δ' with four patches δ_i and seams $(L_{01}, L_{12}, L_{01}, L_{12})$. Note

that we have equality of energies $E(\delta) = E(\delta')$ since the image of δ_4 has zero area. We fold δ' to obtain a map

$$\delta'' : \mathbb{R} \times [0, 1] \rightarrow \underline{\mathbf{M}}$$

with boundary on $(L_{01} \times L_{12}, L_{01} \times L_{12})$. Alternatively, we may view this as a map

$$\delta''' : D \rightarrow \underline{\mathbf{M}}$$

where D is the unit disk and δ''' has Lagrangian boundary conditions on $L_{01} \times L_{12}$. Now observe that $\text{index}(\delta''') = \text{index}(\delta'')$ and by the monotonicity assumptions,

$$\text{index}(\delta''') = \mu_{L_{01} \times L_{12}}([\delta''']) + m_0 + 2m_1 + m_2 \leq 0$$

To conclude, we know by Lemma 11 that $\text{index}(\delta) + 2\dim(M_1) = \text{index}(\delta''')$. Putting all this together, we get

$$\text{index}(\delta) + 2\dim(M_1) \leq 0$$

We conclude that $\text{index}(\delta) \leq 0$ as desired. \square

Since δ is assumed to be non-constant, it cannot have expected dimension zero since translations contribute one dimension to the moduli space. Therefore, $\text{index}(\delta) < 0$. Such δ cannot occur in view of the transversality of the moduli space. Having excluded bubbling at the Y-end, we argue as in the positively monotone case to conclude that the map Φ gives the desired isomorphism. The crucial point is again to exclude broken non-constant trajectories with the same endpoints. While in the positive monotone case these gave rise to moduli spaces of index greater than zero, under the negative monotone assumptions, the sum of the expected dimension of such trajectories is negative. This means that at least one unbroken trajectory in the sequence has negative index. This violates the transversality. \square

4 Application

In Theorem 25 of [10], the first author proves an isomorphism of Floer groups associated with (Y, f) where Y is a closed 3-manifold with $b_1 > 0$ and f is a circle valued Morse function with connected fibres and without maxima or minima. Furthermore, the fibres are arranged so that their genera are in decreasing order as one travels clockwise and counter-clockwise from -1 to 1 . Such an f is called a *broken fibration* in [10], a term which we adopt here. We will denote by Σ_{\max} and Σ_{\min} two fixed regular fibres which have maximal genus g and minimal genus k among fibres of f . We also write $\mathcal{S}(Y|\Sigma_{\min})$ for the set of Spin^c structures \mathfrak{s} on Y , which satisfy $\langle c_1(\mathfrak{s}), \Sigma_{\min} \rangle = \chi(\Sigma_{\min})$.

In the proof of this isomorphism, an important step is an application of Theorem 3. This does not follow from the work of Wehrheim and Woodward in [21], and was the main motivation for finding

an alternative argument which extends the previous results in order to prove Theorem 3. Therefore, we finally have the desired form of Theorem 25 from [10] as follows:

Theorem 15 (cf. Theorem 25 of [10]) *Suppose that Y admits a broken fibration with $g < 2k$. Then for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$,*

$$QFH'(Y, f; \mathfrak{s}, \Lambda) \simeq QFH(Y, f; \mathfrak{s}, \Lambda)$$

Once we have Theorem 3 and the results obtained in [10], the proof of this result is a matter of verifying that the Lagrangian correspondences and symplectic manifolds involved in the definition of above groups satisfy the hypotheses of Theorem 3. For this purpose, we briefly recall the definitions of the groups $QFH'(Y, f; \mathfrak{s}, \Lambda)$ and $QFH(Y, f; \mathfrak{s}, \Lambda)$.

We remark that the hypothesis $g < 2k$ is required for the group $QFH(Y, f; \mathfrak{s}, \Lambda)$ to be well-defined. Without this assumption, we do not know how to avoid the possibility of disk bubbles. On the other hand, the group $QFH'(Y, f; \mathfrak{s}, \Lambda)$ has an alternative cylindrical formulation a la Lipshitz ([13]) which makes it well defined in general. (See [10] for a detailed exposition of this issue).

4.1 Quilted Floer homology of a broken fibration

Given a Riemann surface (Σ, j) and a non-separating embedded curve $L \subset \Sigma$, denote by Σ_L the surface obtained after surgery along L , which is given by removing a tubular neighborhood of L and gluing in a pair of discs. We also choose an almost complex structure \bar{j} on Σ_L which agrees with j outside a neighborhood of L . This is a canonical construction in the sense that the parameter space is contractible.

Perutz equips $\text{Sym}^n(\Sigma)$ with a Kähler form ω_Σ in the cohomology class $\eta_\Sigma + \lambda\theta_\Sigma$ ([16]). Here, η_Σ is the Poincaré dual to $\{pt\} \times \text{Sym}^{k-1}(\Sigma)$, $\theta_\Sigma - g\eta_\Sigma$ is Poincaré dual to $\sum_{i=1}^g \alpha_i \times \beta_i \times \text{Sym}^{n-2}(\Sigma)$ where $\{\alpha_i, \beta_i\}$ is a symplectic basis for $H_1(\Sigma)$, and $\lambda > 0$ is a small positive number.

To such data, Perutz in [16] associates a Lagrangian correspondence V_L in $\text{Sym}^n(\Sigma) \times \text{Sym}^{n-1}(\Sigma_L)$, for any $n \geq 1$. The symplectic forms ω and ω_L on the two spaces are Kähler forms in the respective cohomology class $\eta + \lambda\theta$, where $\lambda > 0$ is a common parameter.

The correspondence is obtained as a vanishing cycle for a symplectic degeneration of $\text{Sym}^n(\Sigma)$. One first considers a holomorphic Lefschetz fibration E over D^2 with regular fibre Σ and just one vanishing cycle, isotopic to L ; thus Σ collapses, in the fibre over the origin, to a nodal curve Σ_0 . The correspondence V_L arises from the vanishing cycle of the relative Hilbert scheme of n points associated with this Lefschetz fibration. Furthermore, Perutz proves that the correspondences V_L are canonically determined up to Hamiltonian isotopy [16].

Using Perutz's constructions, in [10] the first author defined the group $QFH(Y, f, \Lambda)$. Given a 3-manifold Y and a broken fibration $f : Y \rightarrow S^1$, the *quilted Floer homology* of Y , $QFH(Y, f)$ is a quilted Floer homology of the cyclic set of Lagrangian correspondences associated with f and a Spin^c structure on Y obtained by the Perutz's construction above.

More specifically, we will restrict our attention to broken fibrations f which have connected fibres and the genera of the fibres are in decreasing order as one travels clockwise and counter-clockwise from -1 to 1 . As before, we denote by Σ_{\max} a genus g surface identified with the fibre above -1 and by Σ_{\min} a genus k surface identified with the fibre above 1 . On Σ_{\max} , we denote by $\alpha_1, \dots, \alpha_{g-k}$, the disjoint attaching circles of the two handles corresponding to the critical points of f above the northern semi-circle, and similarly we have disjoint attaching circles $\beta_1, \dots, \beta_{g-k}$ for the critical points of f above the southern semi-circle.

Let p_1, \dots, p_k and q_1, \dots, q_k be critical values of f in the northern and southern semicircle respectively. Now, pick points p_i^\pm, q_i^\pm in a small neighborhood of each critical point p so that the fibre genus increases from p_i^- (resp. q_i^-) to p_i^+ (resp. q_i^+). For $\mathfrak{s} \in \text{Spin}^c(Y)$, let $\nu : S^1 \setminus \text{crit}(f) \rightarrow \mathbb{Z}_{\geq 0}$ be the locally constant function defined by $\langle c_1(\mathfrak{s}), [F_s] \rangle = 2\nu(s) + \chi(F_s)$, where $F_s = f^{-1}(s)$. We denote the Lagrangian correspondences obtained by Perutz's construction by $V_{\alpha_i} \subset \text{Sym}^{\nu(p_i^+)}(F_{p_i^+}) \times \text{Sym}^{\nu(p_i^-)}(F_{p_i^-})$ and $V_{\beta_i} \subset \text{Sym}^{\nu(q_i^+)}(F_{q_i^+}) \times \text{Sym}^{\nu(q_i^-)}(F_{q_i^-})$

With this notation, $QFH(Y, f)$ is given as the quilted Floer homology of the Lagrangian correspondences $V_{\alpha_1}, \dots, V_{\alpha_{g-k}}$ and $V_{\beta_1}, \dots, V_{\beta_{g-k}}$. Note that we use the complex structures of the form $\text{Sym}^r(j_s)$ on the symplectic manifolds $\text{Sym}^r(\Sigma)$ where j_s is a generic path of almost complex structures on Σ . The fact that these complex structures achieve transversality is standard, see Proposition A.5 of [13] for an argument in a related set-up. As remarked before, to avoid disk bubbles in this setting, we need to assume $g < 2k$. (Note that in the upcoming work [12], $QFH(Y, f)$ is shown to be independent of the choice of f).

Next, we restrict our attention to Spin^c structures which lie in $\mathcal{S}(Y|\Sigma_{\min})$. The following lemma can be found in Appendix B of [11] (see also [12] for an improved exposition):

Lemma 16 *For $g > k$, $V_{\alpha_1} \circ \dots \circ V_{\alpha_{g-k}}$ and $V_{\beta_1} \circ \dots \circ V_{\beta_{g-k}}$ are respectively Hamiltonian isotopic to $\alpha_1 \times \dots \times \alpha_{g-k}$ and $\beta_1 \times \dots \times \beta_{g-k}$ in $\text{Sym}^{g-k}(\Sigma)$ equipped with a Kähler form ω which lies in the cohomology class $\eta + \lambda\theta$ with $\lambda > 0$. \square*

Because of this result, one defines the group $QFH^l(Y, f)$ as the Floer homology of the Lagrangians $\alpha_1 \times \dots \times \alpha_{g-k}$ and $\beta_1 \times \dots \times \beta_{g-k}$ in $\text{Sym}^{g-k}(\Sigma)$.

4.2 Proof of Theorem 15

To complete the proof of Theorem 15, we need to show that $HF(V_{\alpha_1}, \dots, V_{\alpha_{g-k}}, V_{\beta_{g-k}}, \dots, V_{\beta_1})$ and $HF(V_{\alpha_1} \circ \dots \circ V_{\alpha_{g-k}}, V_{\beta_{g-k}} \circ \dots \circ V_{\beta_1})$ are isomorphic.

For this, we need to recall the calculations from [17], [10] to verify the assumptions (3) and (4) in Theorem 3. The area-index relation follows from Section 4.2 in [10]. Indeed, in the construction of the Lagrangian correspondences V_{α_i} and V_{β_i} one can arrange by isotopy of the curves α_i and β_i to obtain a strongly admissible configuration. This ensures that two trajectories with the same index and the same asymptotic limits have the same area with respect to a symplectic form ω' that is a compactly supported perturbation of the given symplectic form ω . Furthermore, by inspection of the construction of ω' one shows that both ω and ω' tames the same regular complex structure. This is sufficient for our purposes and the proof of Theorem 15 goes through. It implies that there is a well-defined action functional (defined using ω') that is S^1 -valued. A complete discussion of monotonicity can be found in Section 4.2 of [10].

Next, in order to ensure *strongly* negative monotonicity condition, that is assumptions (4) in Theorem 3, we need to verify that the index of holomorphic spheres and disks are sufficiently negative as stated in the assumptions of Theorem 3. We show next that this follows from the assumption that $g < 2k$. Note that the symplectic manifolds that we are dealing with are $M = \text{Sym}^n(\Sigma)$ where $n = g(\Sigma) - k$ and $g(\Sigma)$ takes values between $g(\Sigma_{\max}) = g$ and $g(\Sigma_{\min}) = k$. We equip M with a symplectic form in the class $\eta + \lambda\theta$ where $\lambda > 0$ is a fixed parameter that is determined by the monotonicity condition as follows:

The monotonicity constant τ is determined by the equation:

$$[\eta + \lambda\theta] = \tau[c_1(\text{Sym}^n(\Sigma))] = \tau[(n + 1 - g)\eta - \theta]$$

Therefore, $\tau = \frac{1}{n+1-g} < 0$ is the fixed monotonicity constant which is the same for any of the symplectic manifolds we consider since $n - g = -g(\Sigma_{\min})$.

Now, Perutz calculates in Section 4 of [17] that for $n > 1$ the Hurewicz map $\pi_2(\text{Sym}^n(\Sigma)) \rightarrow H_2(\text{Sym}^n(\Sigma))$ has rank 1 and generated by a class h which satisfies $\eta(h) = 1$ and $\theta(h) = 0$. On the other hand, $c_1(\text{Sym}^n(\Sigma)) = (n + 1 - g(\Sigma))\eta - \theta$. Therefore, any simple holomorphic sphere would have $[u] = h$ and its index would be :

$$2(\langle c_1(\text{Sym}^n(\Sigma)), h \rangle + n - 3) = 4n - 2g(\Sigma) - 4 = 2g(\Sigma) - 4k - 4$$

The assumption $g < 2k$ now implies that this quantity is strictly less than -4 , which suffices for our purpose. (For $n = 1$, we can't have any holomorphic spheres since $\pi_2(\Sigma) = 0$)

Similarly, for a disk bubble we need to verify the assumptions for our Lagrangian $V_L \subset \text{Sym}^n(\Sigma) \times \text{Sym}^{n-1}(\Sigma_L)$, where as before $n = g(\Sigma) - k = g(\Sigma_L) + 1 - k$. In light of the fact that Perutz proves in Lemma 3.18 of [16] that any disk in $\pi_2(\text{Sym}^n(\Sigma) \times \text{Sym}^{n-1}(\Sigma_L), V_L)$ lifts to a sphere it follows that

$$\mu_{V_L}([u]) = 2(\langle c_1(\text{Sym}^n(\Sigma) \times \text{Sym}^{n-1}(\Sigma_L)), [u] \rangle)$$

Now, the positive area disks u for which the value $\mu_{V_L}([u])$ is maximal, have index given by

$$2(n + 1 - g(\Sigma)) + (2n - 1) - 3 = 2g(\Sigma) - 4k - 2$$

Again, the assumption $g < 2k$ ensures that this value is strictly less than -2 , which guarantees that the non-existence of disk bubbles in the $\mathcal{M}_j^i(\underline{x}, \underline{y})$ for $j = 0, 1, 2$.

This completes the verification of the hypothesis of Theorem 3. Therefore, it applies to give the desired isomorphism. \square

It is worth pointing out that the results of [10] now can be put together without the additional hypothesis on figure-eight bubbles to give the isomorphism of $QFH(Y, f; \mathfrak{s})$ with the Heegaard Floer homology groups $HF^+(Y, \mathfrak{s})$ ([10] Theorem 33).

References

- [1] **M F Atiyah I M Singer** *The index of elliptic operators. I.* Ann. of Math. **87** (1968) 484–530.
- [2] **B Charbonneau** *Analytic aspects of periodic instantons*, PhD thesis MIT (2004).
- [3] **S Donaldson** *Floer homology groups in Yang-Mills theory*, Cambridge University Press.
- [4] **A Floer** *An unregularized gradient flow for the symplectic action* Comm. Pure Appl. Math **41** (1987) 775–813.
- [5] **A Floer H Hofer D Salamon** *Transversality in elliptic Morse theory for the symplectic action* Duke Math. J. **80** (1995) 251–292.
- [6] **H Hofer V Lizan J-C. Sikorav** *On genericity for holomorphic curves in four-dimensional almost-complex manifolds* J. of Geom. Anal. **7** (1998) 149–159.
- [7] **S Ivashkovich V Shevchishin** *Complex curves in almost-complex manifolds and meromorphic hulls* Schriftenreihe des Graduiertenkollegs Geometrie und mathematische Physik der Universität Bochum **Heft 36** (1999).
- [8] **D Kwon Y. G. Oh**, *Structure of the image of (pseudo)-holomorphic discs with totally real boundary conditions* Comm. Anal. Geom. **8** (2000), 31–82.
- [9] **L Lazzarini** *Existence of a somewhere injective pseudo-holomorphic disc* Geom. Funct. Anal. **10** (2000) 829–862.
- [10] **Y Lekili** *Heegaard Floer homology of broken fibrations over the circle* (2009) Preprint. arXiv: 0903.1773.
- [11] **Y Lekili** *Broken Lefschetz fibrations, Lagrangian matching invariant and Ozsváth Szabó invariants* PhD Thesis, MIT (2009).

- [12] **Y Lekili, T Perutz** *Lagrangian correspondences and invariants of three-manifolds with boundary*, In preparation.
- [13] **R Lipshitz** *A cylindrical reformulation of Heegaard Floer homology* *Geometry & Topology* **10** (2006) 955–1096.
- [14] **C Manolescu, C Woodward** *Floer homology on the extended moduli space*, to appear in *Perspectives in Analysis, Geometry and Topology*, *Progress in Mathematics*, Birkhauser (2009).
- [15] **D McDuff, D Salamon** *J-holomorphic curves and symplectic topology* Amer. Mathematical Society, (1994).
- [16] **T Perutz** *Lagrangian matching invariants for fibred four-manifolds: I*, *Geometry & Topology* **11** (2007), 759–828.
- [17] **T Perutz** *Lagrangian matching invariants for fibred four-manifolds: II*, *Geometry & Topology* **12** (2008), 1461–1542.
- [18] **T Perutz** *A symplectic Gysin sequence*, (2008) Preprint. arXiv:0807.1863v1.
- [19] **P Seidel** *Fukaya categories and Picard-Lefschetz theory* European Mathematical Society.
- [20] **K Wehrheim, C Woodward** *Pseudoholomorphic quilts*, *J. of Symp. Geom.* (to appear).
- [21] **K Wehrheim, C Woodward** *Quilted Floer cohomology*, *Geometry & Topology* **14** (2010), 833–902.
- [22] **K Wehrheim, C Woodward** *Floer cohomology and geometric composition of Lagrangian Correspondences*, (2009) Preprint. arXiv:0905.1368.
- [23] **K Wehrheim, C Woodward** *Orientations for pseudoholomorphic quilts* Preprint, available from www.math.rutgers.edu/~ctw/papers.html.
- [24] **K Wehrheim, C Woodward** *Quilted Floer trajectories with constant components: corrigendum to “Quilted Floer cohomology”* *Geom. Topol.* **16** (2012) 127–154.
- [25] **C Woodward** *Gauged Floer theory of toric moment fibers*. *Geom. Funct. Anal.* **21** (2011) 680–749.

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