Computing Symplectic Cohomology via Mirror Symmetry

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based on joint work with Kazushi Ueda

Symplectix
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An example computation

Consider a Kleinian singularity

$$\mathbf{w}(x,y,z) = \begin{cases} x^{n+1} + y^2 + z^2 & A_n \\ x^{n-1} + xy^2 + z^2 & D_n \\ x^4 + y^3 + z^2 & E_6 \\ x^3 + xy^3 + z^2 & E_7 \\ x^5 + y^3 + z^2 & E_8 \end{cases}$$

$V = \mathbf{w}^{-1}(1)$ Milnor fiber with its Liouville structure

Theorem. (L. - Ueda)

$$\mathrm{SH}^*(V) = \begin{cases} \mathbb{C}^\mu & \text{if } * \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

where $\mu = n$ for A_n and D_n and $6, 7, 8$ for E_6, E_7, E_8.

Free loop spaces

-Good reference: Latschev-Oancea

Fix Q a smooth manifold of dimension n.

Let $\mathcal{L}Q := \mathrm{Map}(S^1, Q)$ the *free loop space* of Q.

Clasically, we have the *energy functional* $E : \mathcal{L}Q \to \mathbb{R}$ given by

$$E(\gamma) = \int_{S^1} \|\dot{\gamma}(t)\|^2$$

whose critical points are the closed geodesics in Q.

This satisfies Palais-Smale condition, and it can be used to compute the homology of the free loop space (see Milnor, Bott).

Free loop spaces

-Free loop space of S^2

$$\mathcal{L}S^2 \simeq S^2 \cup \mathbb{R}P^3_1 \cup \mathbb{R}P^3_3 \cup \mathbb{R}P^3_5 \cdots \ldots$$

Cohen-Jones-Yan used Leray-Serre spectral sequence to obtain

$$H_{2-*}(\mathcal{L}S^2) = \Lambda(x) \otimes \mathbb{Z}[y,z]/(y^2, xy, 2yz)$$

with $|x| = 1, |y| = 2, |z| = -2$.

$$H_0 \otimes \mathbb{C} = \mathbb{C} \cdot y$$
$$H_1 \otimes \mathbb{C} = \mathbb{C} \cdot x$$
$$H_2 \otimes \mathbb{C} = \mathbb{C} \cdot 1$$
$$H_3 \otimes \mathbb{C} = \mathbb{C} \cdot xz$$
$$H_4 \otimes \mathbb{C} = \mathbb{C} \cdot z$$

$$H_{1+2i} \otimes \mathbb{C} = \mathbb{C} \cdot xz^{i+1} \text{ and } H_{2+2i} \otimes \mathbb{C} = \mathbb{C} \cdot z^i$$

Free loop spaces

-Burghelea–Fiedorowicz, Goodwillie, ...

$\Omega Q := Map_*(S^1, Q)$ based loop space of Q.

$C_{-*}(\Omega Q)$ is a DG-algebra (Pontryagin product).

Elaborating on the fibration $\Omega Q \to \mathcal{L}Q \to Q$, one obtains

$$H_{n-*}(\mathcal{L}Q) = HH^*(C_{-*}(\Omega Q), C_{-*}(\Omega Q))$$

For $Q = S^2$, we have $C_{-*}(\Omega S^2) \simeq \mathbb{C}\langle x^\vee \rangle$ with $|x^\vee| = -1$.

Free loop spaces

-Koszul duality, Jones, ...

For Q simply connected, $C_{-*}(\Omega Q)$ is (derived) Koszul dual to $C^*(Q)$.

$$\mathrm{H}_{n-*}(\mathcal{L}Q) = \mathrm{HH}^*(C^*(Q), C^*(Q))$$

For $Q = S^2$, $C^*(Q)$ is quasi-isomorphic to $\mathbb{C}[x]/(x^2)$ with $|x| = 2$.

Koszul bimodule complex has generators $(x^\vee)^i \otimes 1$ and $(x^\vee)^i \otimes x$ for $i \geq 0$. The differential can be computed as:

$$d((x^\vee)^i \otimes 1) = (1 + (-1)^{i+1})(x^\vee)^{i+1} \otimes x$$
$$d((x^\vee)^i \otimes x) = 0$$

(cf. Etgü-L. - Koszul duality patterns in Floer theory G&T 2017)

Symplectic Cohomology

Given a Liouville manifold V, one defines *symplectic cohomology*

$$\mathrm{SH}^*(V)$$

as a Hamiltonian Floer cohomology group associated with a time-dependent Hamiltonian with quadratic growth

A generalization of Quantum Cohomology to non-compact symplectic manifolds

Very roughly, in addition to Morse critical points capturing $H^*(V)$, there are generators corresponding to Reeb orbits along ∂V.

Symplectic Cohomology

-Viterbo isomorphism

Generally speaking, it is hard to compute $\mathrm{SH}^*(V)$ explicitly. However, for $V = T^*Q$, we have *Viterbo isomorphism*

$$\mathrm{SH}^*(T^*Q) = \mathrm{H}_{n-*}(\mathcal{L}Q) \tag{1}$$

Thus, for example, we can use this to compute $\mathrm{SH}^*(T^*S^2)$.

* Refinements of (1) exist. It should hold in all glory as a quasi-isomorphism of BV_∞-algebras.

Symplectic Cohomology

-Morse-Bott spectral sequence

The Milnor fiber of $x^4 + y^4 + z^4$ can be compactified to a quartic K3 surface in $\mathbb{P}^3$ by adding a smooth divisor of genus 3.

					q
	$\mathbb{C}^6$	0	0	0	$\vdots$
	$\mathbb{C}$	$\mathbb{C}$	0	0	9
	0	$\mathbb{C}^6$	0	0	$\vdots$
	0	$\mathbb{C}^6$	0	0	7
	0	$\mathbb{C}$	$\mathbb{C}$	0	6
	0	0	$\mathbb{C}^6$	0	5
	0	0	$\mathbb{C}^6$	0	4
	0	0	$\mathbb{C}$	0	3
	0	0	0	$\mathbb{C}^{27}$	2
	0	0	0	0	1
	0	0	0	$\mathbb{C}$	0
p	...	-2	-1	0	

We immediately conclude that $\mathrm{SH}^0 = \mathbb{C}$, $\mathrm{SH}^1 = 0$, $\mathrm{SH}^2 = \mathbb{C}^{28}$, $\mathrm{SH}^3 = \mathbb{C}^6$, and with a bit more work $\mathrm{SH}^i = \mathbb{C}^7$ for $i > 3$.

Symplectic Cohomology as Hochschild Cohomology

-Wrapped Fukaya category

Let $\mathcal{W}(V)$ denote the wrapped Fukaya category. This has objects exact Lagrangians L with controlled behaviour at infinity. In analogy with $\mathrm{SH}^*(V)$

$$hom(L_1, L_2)$$

has generators not only the intersection points between L_1 and L_2 but also Reeb chords from L_1 to L_2. The following is a (vast) generalization Burghelea-Fiedorowicz-Goodwillie result.

$$\mathrm{SH}^*(V) = \mathrm{HH}^*(\mathcal{W}(V), \mathcal{W}(V))$$

This is a culmination of many people's work. Notably, Bourgeois-Ekholm-Eliashberg, Abouzaid, Ganatra, Chantraine-Dimitroglou Rizell-Ghiggini-Golovko,...

Problems

- How do we compute $\mathcal{W}(V)$?
- How do we compute $\mathrm{HH}^*(\mathcal{W}(V), \mathcal{W}(V))$?

BEE surgery formula

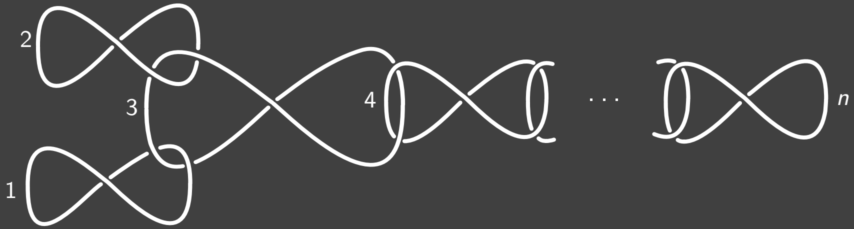
Let Λ be a Legendrian on the boundary of a subcritical Liouville manifold. If $V = V_\Lambda$ the result of Legendrian surgery on Λ, then

$$\mathcal{W}(V) = \mathrm{Perf}(CE^*(\Lambda))$$

where $CE^*(\Lambda)$ is the Chekanov-Eliashberg dg-algebra of Λ.

* This fundamental result is due to Bourgeois-Ekholm-Eliashberg (G&T 2012) but this particular formulation of their result is easier to extract from Ekholm-L, where an extension of this result to partially wrapped Fukaya categories was given.

Chekanov-Eliashberg algebra for plumbings



Lagrangian projection of a Legendrian for D_n Milnor fiber.

Ginzburg algebra

$\mathscr{G}^n_Q$ of a quiver Q is a model of the n-Calabi–Yau completion of the path algebra A_Q. Consider the path algebra of the graded quiver $\overline{Q}$ with same vertices as Q and arrows consisting of

- the original arrows $g \in Q_1$ in degree 1,
- the opposite arrows g^* for each arrow $g \in Q_1$ in degree $1-n$,
- loops h_v at each vertex $v \in Q_0$ in degree $1-n$,

equipped with the differential d given by

$$dg = dg^* = 0 \text{ and } dh = \sum_{g \in Q_1} g^* g - g g^* \tag{2}$$

where $h = \sum_{v \in Q_0} h_v$.

Theorem. (Etgü-L., Ekholm-L.) For a simple singularity of Dynkin type Q,

$$CE^*(\Lambda) \simeq \mathscr{G}^n_Q$$

Invertible polynomials

A weighted homogeneous polynomial $\mathbf{w} \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ with an isolated critical point at the origin is *invertible* if there is an integer matrix $A = (a_{ij})_{i,j=1}^{n+1}$ with non-zero determinant such that

$$\mathbf{w} = \sum_{i=1}^{n+1} \prod_{j=1}^{n+1} x_j^{a_{ij}}. \tag{3}$$

The *transpose* of $\mathbf{w}$ is defined as

$$\check{\mathbf{w}} = \sum_{i=1}^{n+1} \prod_{j=1}^{n+1} x_j^{a_{ji}}, \tag{4}$$

For example, the transpose of

$$x^{n-1} + xy^2 + z^2 \quad \text{is} \quad x^{n-1}y + y^2 + z^2$$

(The latter is equivalent to $x^{2n-2} + y^2 + z^2$).

Invertible polynomials

-HMS conjecture

The group

$$\Gamma_{\mathbf{w}} := \{(t_0, t_1, \ldots, t_{n+1}) \in (\mathbb{G}_m)^{n+2} | \\ t_1^{a_{1,1}} \cdots t_{n+1}^{a_{1,n+1}} = \cdots = t_1^{a_{n+1,1}} \cdots t_{n+1}^{a_{n+1,n+1}} = t_0 t_1 \cdots t_{n+1}\}$$

acts naturally on $\mathbb{A}^{n+2} := \operatorname{Spec}\mathbb{C}[x_0, \ldots, x_{n+1}]$.

$\operatorname{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 \cdots x_{n+1})$ denote the idempotent completion of the dg category of $\Gamma_{\mathbf{w}}$-equivariant coherent matrix factorizations of $\mathbf{w} + x_0 \cdots x_{n+1}$ on $\mathbb{A}^{n+2}$

Conjecture (L.-Ueda '2019) For any invertible polynomial $\mathbf{w}$, one has a quasi-equivalence

$$\operatorname{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 \cdots x_{n+1}) \simeq \mathcal{W}(\check{\mathbf{w}}^{-1}(1)). \tag{5}$$

Matrix factorizations

-example

By a matrix factorization, we mean a pair (f, g) of matrices with entries in $\mathbb{C}[x_0, x_1, \ldots, x_n]$ such that

$$f \cdot g = (\mathbf{w} + x_0 x_1 \ldots x_{n+1}) Id \text{ and } g \cdot f = (\mathbf{w} + x_0 x_1 \ldots x_{n+1}) Id$$

The matrix factorization associated with the structure sheaf of the critical locus of $f(x, y, z, w) = x^{n+1} + y^2 + z^2 + xyzw$ can be computed as

$$f = \begin{pmatrix} x^n & -y & xyw + z & 0 \\ -y & -x & 0 & xyw + z \\ z & 0 & -x & y \\ 0 & z & y & x^n \end{pmatrix},$$

$$g = \begin{pmatrix} x & -y & xyw + z & 0 \\ -y & -x^n & 0 & xyw + z \\ z & 0 & -x^n & y \\ 0 & z & y & x \end{pmatrix}.$$

Matrix factorizations

-HMS for simple singularity

If $\mathbf{w}$ is a polynomial defining a Kleinian singularity or a stabilization of such a polynomial (simple singularity), then

$$\mathbf{w} + x_0 x_1 \dots x_n \text{ and } \mathbf{w}$$

are right-equivalent. By a theorem of Orlov, this implies

$$\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 \cdots x_{n+1}) = \mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})$$

Theorem. (L.-Ueda) If w is a polynomial for a simple singularity,

$$\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}) = \mathrm{Perf}\mathscr{G}_Q^n$$

Matrix factorizations

-Hochschild Cohomology

$$V := \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_{n+1}.$$

Then (by Dyckerhoff, Ballard-Favero-Katzarkov,...) we have $\mathrm{HH}^t(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma, \mathbf{w}))$ is isomorphic to

$$\left(\bigoplus_{\substack{\gamma \in \ker \chi,\ l \geq 0 \\ t - \dim N_\gamma = 2u}} H^{-2l}(d\mathbf{w}_\gamma) \otimes \chi^{\otimes(u+l)} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right.$$

$$\left. \oplus \bigoplus_{\substack{\gamma \in \ker \chi,\ l \geq 0 \\ t - \dim N_\gamma = 2u+1}} H^{-2l-1}(d\mathbf{w}_\gamma) \otimes \chi^{\otimes(u+l+1)} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)^\Gamma . \quad (6)$$

Matrix factorizations
-Hochschild Cohomology

Here $H^i(d\mathbf{w}_\gamma)$ is the i-th cohomology of the Koszul complex

$$C^*(d\mathbf{w}_\gamma) := \{\cdots \to \Lambda^2 V_\gamma^\vee \otimes \chi^{\otimes(-2)} \otimes S_\gamma \to V_\gamma^\vee \otimes \chi^\vee \otimes S_\gamma \to S_\gamma\},$$

where the rightmost term S_γ sits in cohomological degree 0, and the differential is the contraction with

$$d\mathbf{w}_\gamma \in (V_\gamma \otimes \chi \otimes S_\gamma)^\Gamma .$$

The vector space V_γ is the subspace of γ-invariant elements in V, S_γ is the symmetric algebra of V_γ, $\mathbf{w}_\gamma$ is the restriction of $\mathbf{w}$ to $\mathrm{Spec} S_\gamma$, and N_γ is the complement of V_γ in V so that $V \cong V_\gamma \oplus N_\gamma$ as a Γ-module.

Hochschild Cohomology

-Let's do an example

Let $\mathbf{w} = x_1^3 + x_2^2 + x_3^2$. $\mathrm{Jac}_{\mathbf{w}} = \mathbb{C}[x_1]/(x_1^2)$.
We have $\Gamma = \{(t_0, t_1, t_2, t_3) : t_1^3 = t_2^2 = t_3^2 = t_0 t_1 t_2 t_3\}$.
$\chi = t_1^3 = t_2^2 = t_3^2 = t_0 t_1 t_2 t_3$.

We compute the summands of the formula (6) for each $\gamma \in \mathrm{Ker}\chi$ and check directly that the only contributions are (for $m \in \mathbb{N}$)

$$(1,1,1,1): \quad x_0^{6m},\ x_0^{4+6m} x_1 \in \mathrm{HH}^{-4m}, \mathrm{HH}^{-4m-2}$$

$$x_0^\vee x_0^{6m+1},\ x_0^\vee x_0^{5+6m} x_1 \in \mathrm{HH}^{-4m+1}, \mathrm{HH}^{-4m-1}$$

$$(1,1,-1,-1): \quad x_0^{3+6m} x_2^\vee x_3^\vee,\ x_0^{1+6m} x_1 x_2^\vee x_3^\vee \in \mathrm{HH}^{-4m-2}, \mathrm{HH}^{-4m},$$

$$x_0^\vee x_0^{4+6m} x_2^\vee x_3^\vee,\ x_0^\vee x_0^{2+6m} x_1 x_2^\vee x_3^\vee \in \mathrm{HH}^{-4m-1}, \mathrm{HH}^{-4m+1}$$

$$(e^{2\pi i/3}, e^{-2\pi i/3}, -1, -1): \quad x_0^\vee x_1^\vee x_2^\vee x_3^\vee \in \mathrm{HH}^2$$

$$(e^{-2\pi i/3}, e^{2\pi i/3} - 1, -1): \quad x_0^\vee x_1^\vee x_2^\vee x_3^\vee \in \mathrm{HH}^2$$

End