Homework 1:

1) Let X be a topological space, and $A \subset X$ be a subspace. A retraction of X onto A is a map $r : X \to X$ such that $r(X) = A$ and $r_{|A} = id$. Show that $r^2 = r$. Show that if X is contractible and $r : X \to X$ is a retraction of X onto A, then A is also contractible.

2) A *deformation retraction* of a space X onto a subspace A is a family of maps $f_t : X \to X$, $t \in I$, such that $f_0 = id$, $f_1(X) = A$ and $f_{t|A} = id$ for all t. As usual, the family f_t should be continuous, that is, the associated map $X \times I \to X$ is continuous. Construct a deformation retraction of $\mathbb{R}^{n+1} \setminus \{0\} \to S^n$.

3) The "dunce cap" is the quotient of a triangle (and interior) obtained by identifying all the three edges in an inconsistent manner. That is, if the vertices of the triangle are p, q, r then we identify the line segment (p, q) with (q, r) and with (p, r) in the orientation indicated by the order given. Show that the dunce cap is contractible.

4) Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of spaces indexed by a set I. Let $p_\alpha \in X_\alpha$ be basepoints. Define the wedge sum $\bigvee_\alpha X_\alpha$ as the quotient space of the disjoint union of X_α by the equivalence relation $p_\alpha \sim p_\beta$ for $\alpha, \beta \in I$. Show that, for $n \geq 1$, the complement of m distinct points in $\mathbb{R}^n$ is homotopy equivalent to the wedge sum of m copies of the sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

5) A space X is said to be *locally contractible* if for all $x \in X$, there exists an open neighborhood V of x that is contractible.

Let $B = \bigvee_\infty S^1$ be a wedge of infinitely many circles S^1. That is to say, an attachment of infinitely many circles at a point p. This is naturally a CW-complex with a unique 0-cell and infinitely many 1-cells attached at p.

A shrinking wedge of circles is the union $X = \bigcup_{n \geq 1} C_n$ of circles C_n in $\mathbb{R}^2$ of radius $1/n$ centered at $(1/n, 0)$ equipped with a subspace topology inherited from $\mathbb{R}^2$.

Show that X is not locally contractible but B is locally contractible.

Is there a continuous bijection from X to B, what about B to X?

6) Show that $\mathbb{R}P^2 \setminus \{*\}$ is homeomorphic to a Möbius band. Deduce that $\mathbb{R}P^2 \setminus \{*\}$ is homotopy equivalent to a circle. (Think of $\mathbb{R}P^2$ as a quotient of the sphere S^2).

7) Show that a CW complex X is connected if and only if its 1-skeleton X^1 is connected. Show that a CW complex is connected if and only if it is path-connected.

8) Let X be a finite CW complex. Prove that there exists an embedding $f : X \to \mathbb{R}^N$ for some N large enough.

9) (Optional) Show that the algebraic curve defined by the homogeneous equa-

tion

$$y^2 z = x^3 + x^2 z$$

in $\mathbb{C}P^2$ is homotopy equivalent to $S^2 \vee S^1$. (Hint: Use the rational parametrization: $x = t^2 - 1, y = t^3 - t$.)

10) (Optional) Let X be any topological space, $cat_1 X$ is defined as the minimal cardinality of sets I with $X = \cup_{\alpha \in I} X_\alpha$ such that X_α are closed, and the inclusion maps $i_\alpha : X_\alpha \to X$ and the constant maps $c_\alpha : X_\alpha \to X$ that maps all of X_α to a point $x_\alpha \in X$ are homotopic for every $\alpha \in I$. (In other words, i_α is null-homotopic.)

$cat_2 X$ is defined similarly, by requiring X_α to be closed and X_α to be contractible to a point (instead of the nullhomotopy condition above). Are $cat_1 X$ and $cat_2 X$ invariants up to homotopy equivalence? Let K be the two-dimensional sphere with 3 of its points identified. Is then $cat_1 K = cat_2 K$? (Read about Lusternik-Schnirelmann category if you need help.)