Homework 2:

1) Show that the isomorphisms $\beta_h : \pi_1(X, x) \to \pi_1(X, y)$ coincide for all h connecting x to y if and only if $\pi_1(X, x)$ is abelian.

2) Let X and Y be topological spaces. Let $X \times Y$ be equipped with product topology. Prove that there is an isomorphism of groupoids $\Pi(X \times Y) \cong \Pi(X) \times \Pi(Y)$. (Recall that the product of two categories $\mathscr{C}$ and $\mathscr{D}$ has objects as pairs (x, y) and $Mor((x_1, y_1), (x_2, y_2))$ is just the cartesian product $Mor(x_1, y_1) \times Mor(x_2, y_2)$ of sets.)

3) Suppose $i_1 : K \to G$ and $i_2 : K \to H$ are group homomorphisms, prove that the amalgamated product $G *_K H$ fits into a push-out square for i_1 and i_2. (In particular, you need to check the universal property for push-outs.)

The following series of exercises gives an alternative computation of $\pi_1(S^1, *)$ *(due to Ronald Brown) to the more standard argument based on coverings that we will see next week.*

4) Prove that the composite of two pushouts is a pushout. In other words, suppose that we are give a diagram as follows:

$$\begin{array}{ccccc} A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 \\ \downarrow & & \downarrow & & \downarrow \\ A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 \end{array}$$

such that the square consisting of A_i and B_i is a push-out, and the square consisting of B_i and C_i is a push-out. Show that the square consisting of A_i and C_i is a push-out.

5) Let $U \subset X$ be a path-connected open subset and A be a set of base points in X. Let $A \cap U = B \sqcup C$ is a disjoint union. Then, prove that there is a push-out diagram in the category of groupoids, where the vertical maps are induced by inclusions and the horizontal maps are induced by retractions (constructed similarly to those we used in the proof of van Kampen theorem).

$$\begin{array}{ccc} \Pi(U, A \cap U) & \longrightarrow & \Pi(U, B) \\ \downarrow & & \downarrow \\ \Pi(X, A) & \longrightarrow & \Pi(X, B) \end{array}$$

6) Consider S^1 as embedded in $\mathbb{C}$. Let $U = S^1 \backslash \{i\}$ and $V = S^1 \backslash \{-i\}$. $A = \{-1, 1\}$ and $A \cap U = B \sqcup C$ where $B = \{1\}$ and $C = \{-1\}$. By using the

groupoid version of the van-Kampen and the results of exercises 4 and 5, show that there is a pushout diagram:

$$\begin{array}{ccc} \{0,1\} & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow \\ \mathscr{I} & \longrightarrow & \pi_1(S^1,1) \end{array}$$

where $0, 1$ is the discrete groupoid, the objects are 0 and 1 and the only morphism are identity elements; 0 is the trivial group, it has a unique object and a unique morphism, that is, the identity morphism; and $\mathscr{I}$ is the groupoid from lecture notes - it has two objects 0 and 1 and, in addition to the identity elements, there is a morphism $t \in Mor(0,1)$ and $t^{-1} \in Mor(1,0)$

7) Conclude that $\pi_1(S^1,1) \cong \mathbb{Z}$ by using the universal property of pushout diagram constructed in exercise 6.

(Suggested approach for exercises 4-7: First try the exercises without consulting any resources. If you find yourself wasting too much time, try reading Ronald Brown's book. After getting some hints from there, try to reconstruct the proof by yourself and write it up in your own words.)

8) (Optional) Consider the non-Hausdorff space L called "line with two origins". This is the space obtained by gluing $\mathbb{R} \times \{0\}$ with $\mathbb{R} \times \{1\}$ according to the equivalence relation $(x,0) \sim (x,1)$ if and only if $x \neq 0$. This space has a single point for each non-zero real number and two origins corresponding to $(0,0)$ and $(0,1)$. Show that $\pi_1(L,*) \simeq \mathbb{Z}$, where $*$ is any base-point.