Homework 3:

1) Let $f : S^1 \to S^1$ be a continuous map. Define the degree of f to be index of the subgroup $\Pi(f)(\pi_1(S^1, *)) \subset \pi_1(S^1, f(*))$. In other words, as a map from $\mathbb{Z} \to \mathbb{Z}$, $\Pi(f)$ sends 1 to $deg(f) \cdot 1$.

Show that if $deg(f) \neq 1$, the $f : S^1 \to S^1$ has a fixed point.

2) Suppose G is a topological group, that is a group together with a topology such that the group's binary operation and the group's inverse function are continuous functions with respect to the topology.

a) Show that $\pi_1(G, id)$ can be equipped with two different group structures.

b) (Eilenberg's miracle) Let $(E, *, \cdot)$ be a set equipped with two different group structures, denoted by $*$ and $\cdot$, such that:

$$\begin{cases} (a \cdot b) * (a' \cdot b') = (a * a') \cdot (b * b') \quad \forall a, a', b, b' \in E \\ \exists e \in E, \forall a \in E, a \cdot e = e \cdot a = a \\ \exists e \in E, \forall a \in E, a * e = e * a = a \end{cases}$$

Then show that $* = \cdot$ as operations, and $\forall a, b \in E, a \cdot b = b \cdot a$.

c) Show that fundamental group of a topological group is always abelian.

3) Hatcher page 54, problem 14.

4) Hatcher page 55, problem 22.

Covering spaces:

Assume that the total space E and the base space B of coverings are path-connected. In case, I don't get to say this in class. We say that $p : E \to B$ is the universal covering of B if E is simply connected.

5) Construct a universal covering of $S^1 \vee S^1$ and $S^1 \vee S^2$.

6) Show that the composition of two finite sheeted coverings is a covering. Give an example of a covering $p : X \to Y$ and a covering $q : Y \to Z$ such that $q \circ p : X \to Z$ is not a covering.

7) Let G be (path-connected) topological group (as in Problem 2) and $p : \tilde{G} \to G$ is a covering of G by a path-connected topological space $\tilde{G}$. Let $e \in G$ be its unit element and $\tilde{e}$ be a point in $\tilde{G}$ such that $p(\tilde{e}) = e$. Show that there exists a unique continuous multiplication $\tilde{m} : \tilde{G} \times \tilde{G} \to \tilde{G}$ that makes $\tilde{G}$ into a topological group with the unit element $\tilde{e}$, such that $p : \tilde{G} \to \tilde{G}$ is a group homomorphism. (Hint: To construct the group multiplication and the inverse maps, consider the maps: $\tilde{G} \times \tilde{G} \to G$ and $\tilde{G} \to G$ given by $(a, b) \to p(a) \cdot p(b)$ and $a \to p(a)^{-1}$ respectively, and show that these maps can be lifted in a way so that the group axioms are satisfied.)