Homework 5:

1) For all $g \geq 1$ and $n \geq 2$, compute all the higher homotopy groups $\pi_n(\Sigma_g)$ where Σ_g is the closed surface of genus g.
(Bonus[1]: Do the same for $g = 0$.)

2) Show that chain homotopy of chain maps is an equivalence relation.

3) Let X be a topological spaace. Let $I = [0,1]$ be the unit interval and let $I^n = [0,1]^n$ the n-cube. For $n = 0$, this is interpreted to be a single point.

Define a *cubical chain* to be a continuous map $\sigma : I^n \to X$. Work over the ground ring $R = \mathbb{Z}$. Define the cubical chain group $C_n(X)$ to be the abelian group generated by cubical chains.

Immitating the construction of the boundary map in singular homology construct a chain complex :

$$\to C_n(X) \xrightarrow{d_n} C_{n-1}(X) \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} C_0(X) \to 0$$

In particular, verify that $d_n \circ d_{n+1} = 0$. Call the resulting homology $H_n^{\square}(X)$.

4) Show that the homology groups $H_n^{\square}(X)$ do not agree with singular homology $H_n(X)$ by computing $H_*^{\square}(pt.)$

5) Compute $H_1^{\square}(S^1)$.

6) Let $D_n(X)$ be the subgroup of $C_n(X)$ generated by degenerate cubical chains. (A cubical chain $\sigma : I^n \to X$ is called degenerate if there exists a coordinate x_i such that $\sigma(x_1, \dots, x_n)$ is independent of x_i.)

Show that $D_*(X)$ is a sub-complex of $C_*(X)$, that is, it is preserved by d. Construct a new chain complex by letting $Q_n(X) = C_n(X)/D_n(X)$ with the induced differential. Show that for this new chain complex $H_*(pt.)$ agrees with singular homology.

[1]A really big one! :)