Homework 6:

Below, if unspecified, take the coefficient ring R to be $\mathbb{Z}$.

1) Let $f, g : C_* \to D_*$ be chain maps and $s : C_* \to D_{*+1}$ is a chain homotopy between them. Let $f', g' : D_* \to E_*$ be chain maps and $t : D_* \to E_{*+1}$ is a chain homotopy between them. Using s and t construct a chain homotopy between $f' \circ f$ and $g' \circ g$.

2) If $p : E \to B$ is a covering map, then we know that $\Pi(p) : \pi_1(E, e) \to \pi_1(E, p(e))$ is injective. Is it true that $p_* : H_1(E) \to H_1(B)$ is injective? Prove or disprove.

3) Let Σ_g be a closed genus g surface. Compute $\pi_1(\Sigma_g)$ and $H_1(\Sigma_g)$ for all $g \geq 0$.

4) Show that for every knot $K \subset S^3$, $H_1(S^3 \backslash K) = \mathbb{Z}$.

5) Suppose $A \subset \mathbb{R}^n$ is a retract of $\mathbb{R}^n$, i.e. there exists a map $r : \mathbb{R}^n \to A$ such that $r_{|A} = id_A$. Compute $H_*(A)$.

6) Compute the first homology group of the $n-$torus $T^n = (S^1)^n$. Use this to show that there exists a surjective homomorphism from the group of homotopy equivalences of T^n to the group $GL_n(\mathbb{Z})$. Show that for $n = 1$ its kernel consists of maps homotopic to the identity map.

7) If $\sigma : \Delta_n \to X$ is a simplex, define $\overline{\sigma} : \Delta_n \to X$ by

$$\overline{\sigma}(t_0, \ldots, t_n) := \sigma(t_n, \ldots, t_0).$$

Define a map $T : C_n(X) \to C_n(X)$ by $T(\sigma) := (-1)^{n(n+1)/2}\overline{\sigma}$. Show that $T : C_*(X) \to C_*(X)$ is a chain map, i.e. $T \circ d = d \circ T$. Show that there exists a chain homotopy from T to the identity. (Hint: You don't need to construct the chain homotopy explicitly.)