Course on Algebraic Topology

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1 Introduction

Recollections from point-set topology:

A topology on a set is a way of measuring nearness of points. Recall that a topological space is a set with a preferred collection of subsets, the open sets, such that arbitrary unions of opens sets are open, finite intersections of open sets are open, and $\emptyset, X$ are open. This gives a notion of closeness without requiring a notion of distance - points are close if they tend to be in the same open sets.

It suffices to think of metric spaces, where there is a distance function

$$d : X \times X \to \mathbb{R}$$

and open sets are *generated by* the basis $U_\epsilon(x) = \{y : d(x,y) < \epsilon\}$ This means that any open set is a union of these basic open sets.

Recall that a continuous map $f : X \to Y$ between topological spaces is a map such that $f^{-1}(U)$ is open for all open sets U of Y. Intuitively, this means that nearby points are mapped to nearby points under f with the precision of the word "nearby" is dictated by the topologies on X and Y.

When sets X and Y are equipped with topologies, any map between them will be assumed to be continuous by convention. We denote the set of continuous maps between X and Y by $C^0(X,Y)$. This set carries a natural topology, called *compact-open topology*, which is generated by the basis of open sets:

$$\mathcal{U}(K,U) =: \{f \in C^0(X,Y) | f(K) \subset U\}$$

where K is a compact set in X and U open in Y.

A *homeomorphism* is a continuous map $f : X \to Y$ with a continuous inverse $g : Y \to X$, that is, $g \circ f = \mathrm{Id}_X$, $f \circ g = \mathrm{Id}_Y$. In topology, one considers homeomorphic topological spaces equivalent. Classical topology asks:

Given two topological spaces X and Y, can you decide whether X and Y are homeomorphic?

Key definition of algebraic topology: Homotopy equivalence

In algebraic topology, a weaker notion of equivalence is used. Namely, one asks given two topological spaces X and Y, can you decide whether they are *homotopy equivalent*?

Definition 1.1. *Let $f, g : X \to Y$ are continuous maps. We say that f and g are* homotopic *and write $f \sim g$ whenever there exists a family of maps:*

$$H_t : X \to Y \text{ for } t \in [0,1]$$

such that:

$$H_0 = f \quad \text{and} \quad H_1 = g,$$

$$H : X \times [0,1] \to Y \text{ given by } H(x,t) = H_t(x) \text{ is continuous.}$$

This is a formal way of saying that f can be continuously deformed into g. The relation of homotopy is a relation of equivalence (Check!).

We use the notation $[X, Y]$ for the homotopy classes of maps from $X \to Y$.

Definition 1.2. *Two spaces X and Y are homotopy equivalent, written $X \sim Y$, if there exists a map $f : X \to Y$ with an inverse up to homotopy, $g : Y \to X$, that is, $g \circ f \sim Id_X$, $f \circ g \sim Id_Y$.*

It is easy to show that homotopy equivalence is an equivalence relation in the usual. An equivalence class of homotopy equivalent spaces is a *homotopy type.* One of the main themes of this course is to construct and compute algebraic invariants of topological spaces that distinguish their homotopy types.

In this course, we will deal with "reasonable" topological spaces. An important class of "reasonable" topological spaces is furnished by *manifolds.*

Definition 1.3. *A (topological) manifold is a Hausdorff topological space locally homeomorphic to $\mathbb{R}^n$ for some fixed n, the dimension of the manifold.*

For example, the torus or two-holed torus are 2-manifolds, but the double cone or pinched torus are not manifolds (since they have special points no neighbourhood of which looks like $\mathbb{R}^n$). Manifolds have been something of a 20^{th} century obsession but they are also popular because algebraic topology works well on them. Even though, the class of spaces that one cares about in geometry is really manifolds or some singular generalizations of them, several natural constructions in algebraic topology does not respect this class of spaces. For homotopy theory, the most useful class of spaces, generalizing that of manifolds, is the class of *CW complexes* defined as follows:

Definition 1.4. *(Whitehead '49) A CW-complex is a topological space X which is represented as a disjoint union:*

$$X = \bigcup_{p=0}^{\infty} \bigcup_{i \in I_p} e_i^p$$

of cells e_i^p, if there exists a family of continuous mappings $f_i^p : D^p \to X$ (where D^p is the p-dimensional closed ball) called the characteristic mapping *of e_i^q, such that the restriction of f_i^p to Int D^p is a homeomorphism Int $D^p \simeq e_i^p$ and $f_i^p(\partial D^p)$ is contained in the union of the cells* *of smaller dimensions :* $f_i^p(\partial D^p) \subset \bigcup_{q=0}^{p-1}\bigcup_{i\in I_q} e_i^q$*. Furthermore, the following axioms have to be satisfied:*

(C) (closure finite) The closure of each cell meets only a finite number of cells;

(W) (weak topology) a subset $U \subset X$ *is open if and only if for each* e_i^p *the preimage* $(f_i^p)^{-1}(U) \subset D^p$ *is open in* D^p*.*

Note that the topology given by the axiom (W) is the weakest one among the topologies for which the characteristic mappings are continuous.

A CW-complex X has a *filtration* by subcomplexes:

$$\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \ldots \subset$$

where $X^i = \bigcup_{p=0}^{i}\bigcup_{i\in I_p} e_i^p$ is called the i-th skeleton.

Furthermore, X^n is obtained by attaching a bunch of n-cells to X^{n-1}, that is,

$$X^n = X^{n-1} \cup_{(\sqcup_i\ j_i^n)} (\sqcup_i\ D_i^n) \tag{1}$$

Here $j_i^n : \partial D_i^n \to X^{n-1}$, *attaching maps*, are the restrictions of f_i^n to $\partial D_i^n = S_i^{n-1}$.

This viewpoint leads to another way to define a CW-complex. Namely, a CW-complex X is a space which is a union of an expanding sequence of subspaces X^n such that, inductively, X^0 is a set of points equipped with discrete topology and X^n is the pushout as given in (1) by attaching disks D_i^n along the attaching maps j_i^n. Furthermore, the pushouts are given weak topology as in the axiom (W).

The axiom (C) is automatic from this point of view by the following proposition:

Proposition 1.5. *Suppose* X *is constructed inductively as a (possibly infinite) sequence of pushouts, and given the weak topology as in (W). Then, a compact subspace of* X *intersects only finite number of cells.*

Proof. Let K be a compact subset of X. Suppose on the contrary that there exists an infinite sequence of points x_i in K which lie in different cells of X. The set $S = \{x_1, x_2, \ldots\}$ be the union of these points. Assume by induction on n that $S \cap X^{n-1}$ is closed. Then for each n-cell e_i^n, we have that $(f_i^p)^{-1}(S)$ is closed in ∂D^p by induction hypothesis and $(f_i^p)^{-1}(S)$ has at most 1 more point. Hence, it is closed. Thus, $S \cap X^n$ is closed, and hence S is closed in X. Now, the same argument shows that any subset of S is closed, hence the subspace topology on S is the discrete topology. On the other hand, S is a closed subset of a compact set K, hence S is compact. It follows that S must be finite - a contradiction.

Remark 1.6. *When one builds a topological space as a CW complex, it is useful to use the inductive definition via pushouts. On the other hand, if one has been provided with an ambient topological space and is asked to put a CW complex structure compatible with the given topology, the first definition is often more practical.*

Examples:

- A graph is just a CW complex of dimension ≤ 1.

- The n-dimensional sphere S^n may be represented as a union $e^0 \cup e^n$, a point in S^n and its complement $e^n = S^n \backslash e^0$.

- The Hawaiian earring, i.e. the union $X = \bigcup_{n\geq 1} C_n$ of circles C_n in $\mathbb{R}^2$ of radius $1/n$ centered at $(1/n, 0)$, is naturally a CW-complex, but the topology inherited from $\mathbb{R}^2$ does not make it a CW complex. The set $U = \bigcup C_{2n}\backslash\{0\}$ is open in the weak topology but not the subspace topology.

- The real projective space $\mathbb{R}P^n$ of dimension n can be given as a CW-complex. The filtration is

$$\mathbb{R}P^0 \subset \mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \ldots \subset \mathbb{R}P^n$$

$\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching the upper hemisphere and glueing the boundary to $\mathbb{R}P^{n-1}$.

- Similarly, complex projective space $\mathbb{C}P^n = (\mathbb{C}^{n+1}\backslash\{0\})/\mathbb{C}^\times$ is a cell complex with cells:

$$e^0 \cup e^2 \cup \cdots \cup e^{2n}$$

Let us work out this example more carefully. Define

$$X^{2k} = \{[z_0, \ldots, z_n] \in \mathbb{C}P^n : z_j = 0 \text{ for all } j > k\}$$

Thus $X^0 \subset X^2 \subset \ldots \subset X = \mathbb{C}P^n$, and $X^{2k} = \mathbb{C}P^k$. We will exhibit X^{2k} as the $2k$-skeleton of a cell decomposition. If $[z] \in X^{2k}\backslash X^{2(k-1)} \subset \mathbb{C}P^n$ then $z_{k+1} = \ldots = z_n = 0$ but $z_k \neq 0$, so $[z] = [w_1, \ldots, w_{k-1}, 1, 0 \ldots, 0]$ for a unique $(w_0, \ldots, w_{k-1}) \in \mathbb{C}^k$. Thus $X^{2k}\backslash X^{2(k-1)} \simeq \mathbb{C}^k$. Thus $\mathbb{C}P^n$ is a disjoint union of open cells, one of each even dimension up to $2n$. To see that they are attached in the proper way, think of D^{2k} as $\{w = (w_0, \ldots, w_{k-1}) \in \mathbb{C}^k : |w| \leq 1\}$ and define $f_{2k} : D^{2k} \to X^{2k}$ as follows:

$$f_{2k}(w) = (w_0, \ldots, w_{k-1}, (1 - |w|^2)^{1/2}, 0, \ldots, 0)$$

This map extends to a homeomorphism $D^{2k} \cup_{f_{2k}} X^{2k-2} \to X^{2k}$ which restricts to the inclusion on X^{2k-2} , where $f_{2k} : S^{2k-1} \to X^{2k-2}$ is given by $f_{2k}(\xi_0, \ldots, \xi_k) = [\xi_0, \ldots, \xi_k, 0, \ldots, 0]$.

Remark 1.7. *One can always equip a smooth manifold X with a CW structure. (cf. Morse theory). In dimension > 4, every topological manifold can also be given a CW structure. As far as I know, the question of whether every topological 4-manifold admits a CW-structure is open. (Note that for $n < 4$ topological and smooth manifolds are all the same.)*

Remark 1.8. *In a certain precise sense, CW complexes capture all there is to capture about homotopy types of topological spaces. A poor man's statement of this fact is that any topological space is weakly equivalent to a CW complex, and any weak equivalence between CW complexes is a homotopy equivalence. (A weak equivalence is a map which induces isomorphisms on all homotopy groups, see below.) More sophisticated statements of this sort can be found in Chapter 10 of Weibel's Introduction to homological algebra.*

2 Fundamental groupoid

Definition 2.1. *A topological space X is* contractible *if it is homotopy equivalent to a point.*

Convex subsets of $\mathbb{R}^n$ are contractible but S^1 is not. An easy way to see the latter fact is via studying the *fundamental group* of the circle. In this section, we will construct an invariant of a (pointed) topological space, namely its fundamental group, that is sensitive to its homotopy type.

Since many of you have seen the construction of the fundamental group already, I will take a somewhat more advanced point of view and define the fundamental groupoid.

Recall that a category $\mathcal{C}$ consists of a collection objects: $x, y, z \ldots$ and a set of morphisms $Mor(x, y)$ between any objects x and y. Furthermore, there is a composition law:

$$Mor(y, z) \times Mor(x, y) \to Mor(x, z)$$

that is associative. In addition, each set $Mor(x, x)$ must contain an identity element e_x.

For example, $\mathscr{T}$ of topological spaces is a category, where objects $X, Y, \ldots$ are topological spaces and morphisms $Mor(X, Y)$ are continuous mappings from X to Y. Another category of interest is $h\mathscr{T}$, where the objects are topological spaces as before, but the morphisms are homotopy classes of maps.

A *groupoid* is a category where every morphism $f \in Mor(x, y)$ has a two-sided inverse $g \in Mor(y, x)$, i.e. $f \cdot g = c_y$ and $g \cdot f = c_x$. Note that it follows in particular that for each object x of a groupoid, the endomorphisms of the object x, that is, $Mor(x, x)$ is a group.

Given a topological space X, we would like to construct the fundamental groupoid $\Pi(X)$. Here is a first try:

Let $\mathcal{C}$ be a 'category' whose objects are the points $x \in X$, and the morphisms between the objects x and y be the space of continuous paths $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. We can define a composition via concatenation of paths:

$$(g \cdot f)(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t-1) & 1/2 \leq t \leq 1 \end{cases} \tag{2}$$

The constants paths c_x give identity elements. However, in general, we have

$$(f \cdot g) \cdot h \neq f \cdot (g \cdot h)$$

That is, associativity of the composition fails. Luckily, this is easy to remedy, if we relax the notion of equivalence in our morphism spaces. Let us, finally, define the category $\Pi(X)$ where the morphisms $Mor(x, y)$ are the equivalence classes of continuous maps $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = y$ under the relation of homotopy rel end points. (Two maps f_0 and f_1 with end points x and y are said to be homotopic rel end points if there is a homotopy f_t with $f_t(0) = x$ and $f_t(1) = y$ for all t). We will denote the equivalence class of a map $f : [0, 1] \to X$ by $[f]$.

Proposition 2.2. *The composition of morphisms given by* $[g] \cdot [f] = [g \cdot f]$ *for* $[f] \in Mor(x, y)$, $[g] \in Mor(y, z)$ *is well-defined and is associative. The constant paths* $[c_x] \in Mor(x, x)$ *are identity elements. Furthermore, every morphism* $[f] \in Mor(x, y)$ *has an inverse* $[f^{-1}] \in Mor(y, x)$, *defined by* $f^{-1}(t) = f(1 - t)$, *and these satisfy* $[f^{-1}][f] = [c_x]$ *and* $[f][f^{-1}] = [c_y]$

Proof. These can all be proven by picturing the domain of the required homotopies.

Definition 2.3. *The groupoid just defined, denoted by* $\Pi(X)$ *is called the fundamental groupoid. If* $x \in X$ *is an object of* $\Pi(X)$, *then* $Mor(x, x) =: \pi_1(X, x)$ *is called the fundamental group of* X *based at* x.

Dependence on the basepoint: Let h be a path connecting x to y. Define $\beta_h : \pi_1(X, x) \to \pi_1(X, y)$ via

$$[f] \to [h] \cdot [f] \cdot [h^{-1}]$$

It is easy to check that β_h is a homomorphism with inverse $\beta_{h^{-1}}$. Hence, it is an isomorphism. This isomorphism depends on the choice of the path h. By changing h to another path that is not homotopic to h, we usually get different isomorphisms. To understand this, suppose k is a path connecting y to x, we see that

$$\beta_{k \cdot h}([f]) = \beta_k \cdot \beta_h([f]) = [k \cdot h][f][(k \cdot h)^{-1}]$$

Thus, in general, the isomorphisms if h and h' are two paths connecting x to y β_h and $\beta_{h'}$ differ by conjugation by $(h')^{-1}h$. So, the isomorphisms of $\pi_1(X, x)$ and $\pi_1(X, y)$ would be canonical if the fundamental group is abelian.

More generally, we can take $A \subset X$ to be any subset in X, and consider the full sub-groupoid $\Pi(X, A)$ of $\Pi(X)$. The objects of $\Pi(X, A)$ are the points $x \in A$ and the morphisms $Mor(a, b)$ for $a, b \in A$ are homotopy classes of paths in X connecting a and b (rel end points).

Homotopy invariance: A functor $F : \mathcal{C} \to \mathcal{D}$ between two categories assigns an object $F(X)$ of $\mathcal{D}$ for each object X of $\mathcal{C}$ and a morphism $F(f) : F(X) \to F(Y)$ for each morphism $f : X \to Y$. As usual, this should respect the composition $F(g \circ f) = F(g) \circ F(f)$ and it should send identity elements to identity elements : $F(id_X) = id_{F(X)}$.

The construction $X \to \Pi(X)$ is a functor from the category of topological spaces to the category of groupoids. We write this as:

$$\Pi : \mathscr{T} \to \mathscr{G}\mathscr{P}$$

To see this note that we have already constructed this functor on each object $\mathscr{T}$ by sending X to its fundamental groupoid $\Pi(X)$. Now, if we have a morphism $p : X \to Y$, that is, a continuous function, we need to construct a morphism of groupoids $\Pi(p) : \Pi(X) \to \Pi(Y)$. This is straightforward: For each object x of $\Pi(X)$, a point in X, we let $\Pi(p)(x) = p(x) \in Y$ to be the point under the image of p and for each morphism $f \in Mor(x_1, x_2)$, that is, a path $f : [0, 1] \to X$ connecting x_1 to x_2, we send it to the path $p(f) : [0, 1] \to Y$ connecting $p(y_1)$ to $p(y_2)$. It is easily checked that if $f \sim g$ then $p(f) \sim p(g)$.

Corollary 2.4. *If X is homeomorphic to Y then $\Pi(X)$ is isomorphic to $\Pi(Y)$.*

Thus, the fundamental groupoid $\Pi(X)$ is an invariant of a topological space. Next, we would like to understand the behaviour of this invariant under homotopy equivalences.To understand this, we need to define a notion of "homotopy equivalence" of groupoids. (Believe me, this is not as abstract as it sounds).

To define a homotopy of functors between two categories $\mathscr{C}$ and $\mathscr{D}$, we need a category corresponding to the unit interval. Let $\mathscr{I}$ be the groupoid consisting of two objects: 0 and 1 , and a single morphism $t \in Mor(0,1)$ and its inverse $t^{-1} \in Mor(1,0)$. A *homotopy* of functors from $\mathscr{C}$ to $\mathscr{D}$ is a functor:

$$F : \mathscr{C} \times \mathscr{I} \to \mathscr{D}$$

Let us unwind this a little bit: We have two functors

$$f = F(\ ,0) : \mathscr{C} \to \mathscr{D},$$

$$g = F(\ ,1) : \mathscr{C} \to \mathscr{D}$$

Furthermore, for each object x of $\mathscr{C}$, we have the invertible elements $\theta_x = F(x,t) \in Mor(f(x), g(x))$ such that the following diagram has to commute for all x, y and $a \in Mor(x,y)$:

$$\begin{tikzcd} f(x) \arrow[r, "f(a)"] \arrow[d, "\theta_x"] & f(y) \arrow[d, "\theta_y"] \\ g(x) \arrow[r, "g(a)"] & g(y) \end{tikzcd} \tag{3}$$

The map $\theta : f \to g$ is also called a *natural isomorphism* in category theory. Having defined what it means to be a homotopy between functors. We can define, as before, two categories $\mathscr{C}$ and $\mathscr{D}$ are homotopy equivalent, if there exists functors $f : \mathscr{C} \to \mathscr{D}$ and $g : \mathscr{D} \to \mathscr{C}$ such that :

$$g \circ f \sim id_{\mathscr{C}} \text{ and } f \circ g \sim id_{\mathscr{D}}$$

Now, we are ready to state the behaviour of the fundamental groupoid under homotopy equivalences of spaces:

Proposition 2.5. *Suppose $p : X \to Y$ is a homotopy equivalence of spaces, then $\Pi(p) : \Pi(X) \to \Pi(Y)$ is a homotopy equivalence of groupoids.*

The proof of this proposition is immediate from the following:

Lemma 2.6. *Let $f, g : X \to Y$ be homotopic maps of topological spaces, then $\Pi(f), \Pi(g) : \Pi(X) \to \Pi(Y)$ are homotopic functors.*

Proof. Let $F : X \times I \to Y$ be a homotopy from f to g. We need to construct a functor $\Pi X \times \mathscr{I} \to \Pi Y$. As explained above, this in turn corresponds to giving invertible elements:

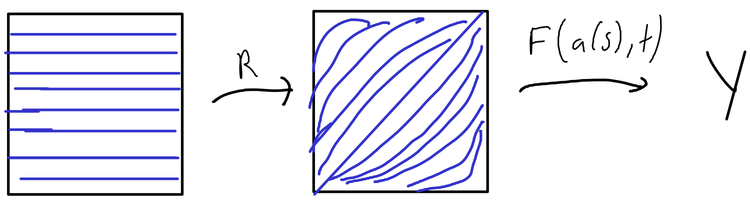
$$\theta_x \in Mor(f(x), g(x))$$

which satisfy the commutativity enforced by the diagrams (3). The complication is only in the notation. Namely, we just let θ_x to be defined by the homotopy class of the obvious path: $\theta_x : [0,1] \to Y$ given by

$$\theta_x(t) = [F(x,t)]$$

This is a path that connects the point $f(x) = F(x,0)$ to $g(x) = F(x,1)$.

To check the commutativity, it suffices to show that, for a path $a : [0,1] \to X$ connecting x to y, $g(a) \circ \theta_x$ is homotopic to $\theta_y \circ f(a)$ as paths from $f(x)$ to $f(y)$. This homotopy is provided by the map $H : I \times I \to Y$ given by pre-composing the map $(s,t) \to F(a(s),t)$ by a reparametrization $R : I \times I \to I \times I$ as in the following figure:



A more succinct way of constructing the homotopy between the functors $\Pi(f)$ and $\Pi(g)$ is obtained by the following composition of functors:

$$\Pi X \times \mathscr{I} \to \Pi X \times \Pi I \to \Pi(X \times I) \to \Pi Y,$$

where the first arrow is induced by the inclusion functor from $\mathscr{I} \to \Pi I$, the second arrow is given via the inverse of the isomorphism $\Pi(X \times I) \cong \Pi X \times \Pi I$ induced by projections to factors and the last arrow is given by $\Pi F : \Pi(X \times I) \to \Pi Y$.

In particular, if $p : X \to Y$ and $q : Y \to X$ are homotopy inverses, i.e. $q \circ p \sim id_X$ and $p \circ q \sim id_Y$ then the functors $f = \Pi(p)$ and $g = \Pi(q)$ are homotopy inverses. An immediate corollary of this is the following:

Corollary 2.7. *Suppose $p : X \to Y$ is a homotopy equivalence, then $\Pi(p) : \Pi(X) \to \Pi(Y)$ induces bijections $Mor(x,y) \to Mor(p(x), p(y))$ for all $x, y \in X$.*

Proof. Let $f = \Pi(p)$ be the homotopy equivalence constructed above and $g : \Pi(Y) \to \Pi(X)$ be a homotopy inverse of f so that $g \circ f \sim id$ and $f \circ g \sim id$. Consider the functions:

$$Mor(x,y) \to Mor(fx, fy) \to Mor(gfx, gfy) \to Mor(fgfx, fgfy)$$

The composite of the first two and the last two are bijections. It follows that each function is a bijection.

In particular, if $x \in X$, then $\Pi(p) : \pi_1(X,x) \to \pi_1(Y, p(x))$ is an isomorphism of fundamental groups.

3 van Kampen theorems

3.1 Pushouts

Let $\mathscr{C}$ be a category. A diagram

$$\begin{tikzcd} C_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & C_1 \arrow[d, "u_1"] \\ C_2 \arrow[r, "u_2"'] & C \end{tikzcd}$$

of morphisms of $\mathscr{C}$ is called a pushout of (i_1, i_2) if the diagram is commutative and u_1 and u_2 are universal with respect to this commutativity. This means that if there is another object $\tilde{C}$ and maps v_1, v_2 that completes the diagram to a commutative square, then there exists a *unique* map $v : C \to \tilde{C}$ such that $v_1 = v \circ u_1, v_2 = v \circ u_2$. One can express this as a commutative diagram:

$$\begin{tikzcd} C_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & C_1 \arrow[d, "u_1"'] \arrow[ddr, bend left, "v_1"] & \\ C_2 \arrow[r, "u_2"] \arrow[drr, bend right, "v_2"] & C \arrow[dr, "v"] & \\ & & \tilde{C} \end{tikzcd}$$

Note that in an arbitrary category, given $i_1 : C_0 \to C_1$ and $i_2 : C_0 \to C_2$, an object C and morphisms $u_1 : C_1 \to C$ and $u_2 : C_2 \to C$ fitting into a push-out diagram need not exist. On the other hand, it is easy to show that arbitrary pushouts exist in the categories, $\mathscr{T}$ of topological spaces, $\mathscr{G}$ of groups, $\mathscr{GP}$ of groupoids. As May puts it, the proof is a worthwhile exercise.

Examples:

- Let U and V be open subspaces of a topological space X such that $X = U \cup V$. Then the following is a push-out diagram in the category of topological spaces:

$$\begin{tikzcd} U \cap V \arrow[r, hook] \arrow[d, hook] & U \arrow[d, hook] \\ V \arrow[r, hook] & X \end{tikzcd}$$

where each of the maps are natural inclusion maps.

- The construction of n-skeleton of CW-complex from its $n-1$-skeleton and the attaching maps is an example of a push-out in the category of topological spaces. The pushout diagram for this is as follows:

$$\begin{tikzcd}
\sqcup_i S_i^{n-1} \arrow[r, hook] \arrow[d, "j_i^n"'] & \sqcup_i D_i^n \arrow[d, "f_i^n"] \\
X^{n-1} \arrow[r, hook] & X^n
\end{tikzcd}$$

- In the category of groups, push-outs are also called *amalgamated products*. Suppose $i_1 : K \to G$ and $i_2 : K \to H$ are group homomorphisms. Let N be the normal subgroup of the free product $G * H$ generated by the elements of the form $i_1(k)i_2(k)^{-1}$ for $k \in K$; then the amalgamated product is:

$$G *_K H := (G * H)/N$$

One can understand this via group presentations. Suppose that we are given presentations $G = \langle \mathcal{G} | \mathcal{R} \rangle$ and $H = \langle \mathcal{H} | \mathcal{S} \rangle$. Then, the amalgamated product has a presentation:

$$G_1 *_H G_2 = \langle \mathcal{G} \cup \mathcal{H} | \mathcal{R} \cup \mathcal{S} \cup \{i_1(k)i_2(k)^{-1} | k \in K\} \rangle$$

Note that the free product $G * H$ of groups is the push-out of the diagram $G \leftarrow \{1\} \rightarrow H$.

Here is a version of the van Kampen theorem (for groupoids):

Theorem 3.1. *Let $X = U_1 \cup U_2$ for some open sets U_1 and U_2 and let $A \subset X$ of base points such that A contains at least one point on each path-component of U_1, U_2 and $U_1 \cap U_2$. Then the following is a push-out in the category of groupoids:*

$$\begin{tikzcd}
\Pi(U_1 \cap U_2, A) \arrow[r, "i_1"] \arrow[d, "i_2"'] & \Pi(U_1, A) \arrow[d, "u_1"] \\
\Pi(U_2, A) \arrow[r, "u_2"'] & \Pi(X, A)
\end{tikzcd}$$

Note that here by abuse of notation we write $\Pi(U_1, A)$ for $\Pi(U_1, A \cap U_1)$ and similarly for others.

Proof. First, we give the proof for $A = X$. We need to verify the universal property of push-outs in the category of groupoids. So, let $\mathscr{G}$ be a groupoid such that one has a commutative diagram:

$$\begin{tikzcd}
\Pi(U_1 \cap U_2) \arrow[r, "i_1"] \arrow[d, "i_2"'] & \Pi(U_1) \arrow[d, "v_1"] \\
\Pi(U_2) \arrow[r, "v_2"'] & \mathscr{G}
\end{tikzcd}$$

We need to show that there is a unique map $v : \Pi(X) \to \mathscr{G}$ such that $v_1 = v \circ u_1$ and $v_2 = v \circ u_2$.

Let $x \in X$ be a point (equivalently an object of $\Pi(X)$). If $x \in U_i$, then define $v(x) = u_i(x)$. Note that if $x \in U_1 \cap U_2$ then we have $u_1(x) = u_2(x)$ hence there is no ambiguity. Similarly, if a path $a \in Mor(x, y)$ lies entirely in U_i, then we must define $v(a) = u_i(a)$. Next, any path $f : [0, 1] \to X$ can be factorized as a composite :

$$f = f_1 \cdot f_2 \cdot \ldots \cdot f_k,$$

for some k, such that the image of f_i lies entirely in either U_1 or U_2. This is obtained by subdividing the domain $[0, 1]$ and using compactness. We can then define:

$$v([f]) = v([f_1]) \cdot v([f_2]) \cdot \ldots \cdot v([f_k])$$

where $v([f_i])$ had already been defined. Indeed, this definition is forced on us by the required restriction properties of the map. To conclude, we need to show that this specification is well-defined. Let $H : I \times I \to X$ be a homotopy from f to g. We may subdivide the square $I \times I$ to subsquares so that each of the subsquares are mapped by H to either U_1 or U_2 (again by using the compactness of the domain). Furthermore, by subdividing more if necessary we can ensure that the subdivision in $I \times \{0\}$ and $I \times \{1\}$ refines the subdivisions used to define f and g. Now the relation $[f] = [g]$ can be seen as a composite of finite number of relations each of which holds in either $\Pi(U)$ or $\Pi(V)$ hence we conclude that $v(f) = v(g)$ as required (I will explain this last part in more detail in class).

To prove the more general case for $A \neq X$, we need to again check the universal property of the push-outs. Suppose, as before, that $\mathscr{G}$ is a groupoid which fits into a commutative diagram with $\Pi(U_1 \cap U_2, A)$, $\Pi(U_1, A)$ and $\Pi(U_2, A)$.

We construct a retraction of $r : \Pi(X) \to \Pi(X, A)$. Namely, since A meets every path-component of $U_1 \cap U_2$, for every $y \in U_1 \cap U_2$ we can choose a path $y \to ry$ that connects y to a point in $ry \in A$. Similarly, since A intersects every path component of U_i, for every $x \in U_i \backslash (U_1 \cap U_2)$ we can find an $rx \in A$ such that there is a path in U_i connecting $x \to rx$ for $i = 1, 2$. We also choose $rx = x$ for $x \in A$. The key property that is needed for choosing these paths is that if a path connects a point y in some U, where U is either U_1 or U_2 or $U_1 \cap U_2$ to some $ry \in A$, then the entire path is contained in that U.

This key property ensures that we get compatible retraction functors (the solid diagonal arrows are retractions) :

$$\begin{tikzcd}
\Pi(U_1 \cap U_2) \arrow[rrr] \arrow[dddd] \arrow[dr] & & & \Pi(U_1) \arrow[dddd] \arrow[dl] \\
& \Pi(U_1 \cap U_2, A) \arrow[r] \arrow[d] & \Pi(U_1, A) \arrow[d] & \\
& \Pi(U_2, A) \arrow[r] & \mathscr{G} & \\
& & & \\
\Pi(U_2) \arrow[rrr] \arrow[uur] & & & \Pi(X) \arrow[uul, dashed]
\end{tikzcd}$$

The dashed arrow is the morphism $\Pi(X) \to \mathscr{G}$ that is induced by the universal property of the pushout for $\Pi(X)$, that was proven in the first part. Now, composing with the inclusion morphism:

$$\Pi(X, A) \to \Pi(X) \to \mathscr{G}$$

we get the required morphism $v : \Pi(X, A) \to \mathscr{G}$. Finally, uniqueness of v follows from the uniqueness of $\Pi(X) \to \mathscr{G}$, which we proved before. $\square$

Remark 3.2. *It is possible to generalize the argument to an arbitrary union: $X = \bigcup_{\alpha \in I} U_\alpha$. It suffices the generalize the condition on A as follows: A should contain at least one point on each path-component of two-fold $U_\alpha \cap U_\beta$ or three-fold intersections $U_\alpha \cap U_\beta \cap U_\gamma$. For this generalization, the push-outs should also be replaced by coequalizers of the maps induced from the inclusions $U_\alpha \to X$. Note that we do not need to assume that any of these intersections are path-connected for groupoid version of van-Kampen's theorem. (May assumes this on page 17 of his book and it is indeed unnecessary!). We won't use this generalization in this course.*

When U, V and $U \cap V$ are path-connected, we can take A to be a single base-point in $U \cap V$, the above theorem then gives the more familiar version of Seifert-van Kampen theorem for fundamental groups:

Corollary 3.3. *Let $X = U \cup V$, and suppose U, V and $U \cap V$ are path-connected. Let $*$ be a base-point in $U \cap V$, then the following is a push-out in the category of groups:*

$$\begin{tikzcd}
\pi_1(U \cap V, *) \arrow[r] \arrow[d] & \pi_1(U, *) \arrow[d] \\
\pi_1(V, *) \arrow[r] & \pi_1(X, *)
\end{tikzcd}$$

In particular, it follows that $\pi_1(X, *)$ is isomorphic to an amalgamated product of $\pi_1(U, *)$ and $\pi_1(V, *)$.

We give a couple of example applications of van Kampen theorem:

Corollary 3.4. *Let $X = X_1 \vee X_2$ be a wedge of path-connected based spaces (X_i, p_i) such that for each i, p_i has a neighborhood in X_i that is contractible. Then $\pi_1(X)$ is isomorphic to a free product $\pi_1(X_1, p_1) * \pi_1(X_2, p_2)$.*

Proof. Let V_i be contractible neighborhoods of p_i in X_i. Take $U_1 = X_1 \cup V_2$ and $U_2 = X_2 \cup V_2$ and apply the van-Kampen theorem.

Proposition 3.5. *Let $X = U \cup V$ where U, V and $U \cap V$ are path connected and $\pi_1(V) = 0$. Then, $\pi_1(X, *)$ is isomorphic to $\pi_1(U, *)/N$ where N is the smallest normal subgroup of $\pi_1(U, *)$ that contains the image of $\pi_1(U \cap V, *) \to \pi_1(U, *)$.*

This is just a restatement of van Kampen's theorem with an extra assumption on V. As a corollary, this allows us to compute the fundamental group of any finite CW complex from its 2-skeleton.

Corollary 3.6. *Let X be a (finite) connected CW-complex, then $\pi_1(X, *) \simeq \pi_1(X^2, *) \simeq \pi_1(X^1, *)/N$ where N is the normal subgroup of $\pi_1(X^1, *)$ determined by the attaching maps $j_i^2 : \partial D_i^2 \to X^1$.*

If we had proven van Kampen theorem in full generality, we would be able to remove the finiteness assumption above.

We will next compute the fundamental group of a circle. In view of the last corollaries, we obtain an algorithmic way of computing the fundamental group of all (finite) CW complexes.

4 Covering spaces

4.1 $\pi_1(S^1, *) = \mathbb{Z}$

A computation of $\pi_1(S^1, *)$ is given in the second set of homework problems based on groupoid version of van Kampen's theorem (the key ingredient there is to be able to use several base points). Here, we will give the more standard version based on a covering argument.

Warning: In this section, we will assume that all the spaces are connected and locally path-connected. I will try to specify these but the reader has been warned that these hypothesis may have been omitted.

Definition 4.1. *Let E and B be connected and locally path-connected spaces. A map $p : E \to B$ is a covering if it is surjective and if each point $b \in B$ has an open neighborhood U such that if $p^{-1}(U) = \sqcup_{i \in I} U_i$ is a decomposition to the connected components of $p^{-1}(U)$, the restriction $p_{|U_i} : U_i \to U$ is a homeomorphisms for each $i \in I$. E is called the total space and B is called the base space of the covering. We also write $F_b = p^{-1}(b)$ for the fiber of the covering at b.*

Note that by connectedness the cardinality of points in F_b remains constant as b varies. This cardinality is called the number of *sheets* of the covering.

Examples:

The projection $p : \mathbb{R} \to S^1$ given by $p(s) = e^{2\pi is}$ is a covering. Similarly, each $p_n : S^1 \to S^1$ sending $e^{2\pi is} \to e^{2\pi ins}$ is a covering. The projection $p : S^n \to \mathbb{R}P^n$ obtained by identifying antipodal points is a covering.

Theorem 4.2. $\pi_1(S^1, *) = \mathbb{Z}$

The proof will be obtained by studying the covering map $p : \mathbb{R} \to S^1$ given by $p(s) = e^{2\pi is}$. The next proposition shows that the *homotopy lifting property* holds for covering maps $p : E \to B$.

Proposition 4.3. *Let S be any locally path-connected topological space. Let $p : E \to B$ be a covering. Given a homotopy $f_t : S \to B$ and a lift $\tilde{f}_0 : S \to E$, that is a map that satisfies $\tilde{f}_0 = p \circ f_0$, then there exists a unique homotopy $\tilde{f}_t : S \to E$ of $\tilde{f}_0$ that lifts f_t.*

Proof. By definition of a covering space, we can cover B with a collection of open sets U_α such that $p^{-1}(U_\alpha) = \sqcup_{i \in I_\alpha} U_{\alpha,i}$ such that $p_{|U_{\alpha,i}} : U_{\alpha,i} \to U_\alpha$ is a homeomorphism.

Let us write $F : S \times I \to B$ for the homotopy f_t and we are seeking to lift this to $\tilde{F} : S \times I \to E$ where $\tilde{F}_{|S \times \{0\}}$ is already specified to be $\tilde{f}_0$.

Given a point $s \in S$, let us first construct a lift $\tilde{F} : N \times I \to E$ in a neighborhood N of $\{s\}$. Since I is compact, we can find a subdivision of I to $I_i = [t_k, t_{k+1}]$ for $0 = t_0 < t_1 < \ldots < t_n = 1$ and a neighborhood N of $\{s\}$ in S such that for each i there exists a U_α such that $F(N \times I_k)$ lies entirely in some U_α. Assume inductively that we have defined a lift $\tilde{F}$ at $N \times \{t_k\}$. Now, in the preimage $p^{-1}(U_\alpha)$ we also get a distinguished open set U_{α,i_k} which contains $F(s, t_k)$. By shrinking N if necessary, we can assume that $\tilde{F}(N \times \{t_k\}) \subset U_{\alpha,i_k}$. We can then extend, $\tilde{F}$ to all of $N \times [t_k, t_{k+1}]$ by composing $F : N \times [t_k, t_{k+1}] \to B$ with the homeomorphism $p^{-1} : U_\alpha \to U_{\alpha,i_k}$. Thus by, induction we get a lift $\tilde{F} : N_s \times I \to E$ in a neighborhood of every point $s \in S$.

We note that we did not have any choice in constructing the lift. Therefore, for different N_s and $N_{s'}$ the lifts $\tilde{F} : N_s \times I \to E$ and $\tilde{F} : N_{s'} \times I \to E$ have to agree as they agree at $(N_s \cap N_{s'}) \times \{0\}$ by construction. Hence, all these lifts assemble together to give the desired lift $\tilde{F} : S \times I \to E$. □

In homotopy theory, surjective maps $p : E \to B$ between two topological spaces E and B for which the homotopy lifting property holds without the uniqueness condition on the lift $\tilde{F}$ is called a *fibration*. The previous proposition shows that covering maps are examples of fibrations. The fibration property can be expressed by the following diagram:

$$\begin{tikzcd} S \arrow[r, "\tilde{f}_0"] \arrow[d, hook, "i_0"'] & E \arrow[d, "p"] \\ S \times I \arrow[r, "F"'] \arrow[ur, dashed, "\tilde{F}"] & B \end{tikzcd}$$

By taking S to a point or $[0,1]$, we get the following important special cases:

Corollary 4.4. *Let $p : E \to B$ be a covering and $b \in B$ and $\tilde{b} \in F_b$.*

1. *A path $f : [0,1] \to B$ with $f(0) = b$ lifts uniquely to a path $\tilde{f} : [0,1] \to E$ such that $\tilde{f}(0) = \tilde{b}$ and $p \circ \tilde{f} = f$.*
2. *For each homotopy $f_t : [0,1] \to B$ of paths starting at b there is a unique lifted homotopy $\tilde{f}_t : [0,1] \to E$ of paths starting at $\tilde{b}$.*

Let $p : E \to B$ a covering with $p(e) = b$ as before. The homotopy lifting property implies the following properties of the functor $\Pi(p) : \Pi(E) \to \Pi(B)$.

Proposition 4.5. *i) $\Pi(p) : Mor(x,y) \to Mor(p(x), p(y))$ for $x, y \in B$ is injective.*
ii) Let $e' \in E$ be another point such that $p(e') = b$, then $\Pi(p)(\pi_1(E, e'))$ is conjugate to $\Pi(p)(\pi_1(E, e))$ and all conjugates are obtained this way.

The proof of this proposition is immediate from homotopy lifting property. We now turn to the computation of fundamental group of the circle:

Proof of Theorem 4.2 We denote by $p : \mathbb{R} \to S^1$ the covering given by $p(s) = e^{2\pi i s}$. We will use the basepoint $1 \in S^1$.

Let $f_n : [0,1] \to S^1$ be loop defined by $f_n(s) = e^{2\pi i s}$. It is easy to check that $[f_m] \cdot [f_n] = [f_{m+n}]$ hence we get a group homomorphism: $i : \mathbb{Z} \to \pi_1(S^1, 1)$ by sending $i(n) = [f_n]$. We will show that this is an isomorphism.

Let $\tilde{f}_n : [0,1] \to \mathbb{R}$ be the lift of f_n given by $\tilde{f}_n = ns$ so that $f_n = p(\tilde{f}_n)$.

Now, suppose $f : [0,1] \to S^1$ be an arbitrary loop with $f(0) = f(1) = 1$. By the previous corollary, we can lift this uniquely to $\tilde{f} : [0,1] \to \mathbb{R}$ such that $\tilde{f}(0) = 0$ and it is necessarily true that $\tilde{f}(1) = m$ for some integer $m \in Z$. But now $\tilde{f}t + (1-t)\tilde{f}_m$ is a homotopy between these paths rel endpoints hence by composing with $p : \mathbb{R} \to S^1$, we get a homotopy between f and f_m hence, $[f] = [f_m]$. This proves surjectivity.

To prove injectivity, suppose $f : [0,1] \to S^1$ and $g : [0,1] \to S^1$ are homotopic rel end point points. Then, as before, we can lift them uniquely to $\mathbb{R}$, and we need to show that $\tilde{f}(1) = \tilde{g}(1)$. But, if H is homotopy between f and g, by the homotopy lifting property it lifts to $\tilde{H} : [0,1] \times [0,1] \to \mathbb{R}$. Since the preimage of $H(1,t) = f(1) = g(1)$ is the discrete set $\mathbb{Z}$ in $\mathbb{R}$, the lift $\tilde{H}(1,t)$ has to be constant. It follows that $\tilde{f}(1) = \tilde{g}(1)$ as required. □

Digression : Amusing applications.

Theorem 4.6. *(Fundamental theorem of algebra) A polynomial*

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in \mathbb{C}[x]$$

has a complex root.

Proof. Look at the homotopy $f(tx)/|f(tx)|$ from $f(x)/|f(x)|$ to constant if f doesn't have a zero in $|x| \leq 1$. On the other hand,

$$t^n f(x/t)/|t^n f(x/t)|$$

is a homotopy from $f(x)/|f(x)|$ to $x^n/|x^n|$ if f doesn't have zero in $|x| \geq 1$.

Theorem 4.7. *(Borsuk-Ulam) For very continuous map $f : S^2 \to \mathbb{R}^2$, there exists a point $x \in S^2$ such that $f(x) = f(-x)$.*

Proof. Suppose, by contradiction, that $f(x) \neq f(-x)$ for any $x \in S^2$. Consider the map $F : S^2 \to S^1$ defined by

$$F(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

This is well-defined and continuous by our assumption. It satisfies the property of being an odd function. That is, $F(-x) = -F(x)$. Restricting F to an equator $S^1 \subset S^2$. We get a map $g = F_{|S^1} : S^1 \to S^1$. This is null-homotopic because it extends to a map from $F_{|D^2} : D^2 \to S^1$. Viewing $g : [0,1] \to S^1$ as a map from $[0,1]$ with $g(0) = g(1)$, we can lift it uniquely to $\tilde{g} : I \to \mathbb{R}$ with $\tilde{g}(0) = 0$, such that $g = p \circ \tilde{g}$ where $p : \mathbb{R} \to S^1$ is the covering map as above. Since g is null-homotopic, we conclude that $\tilde{g}(1) = \tilde{g}(0) = 0$.

On the other hand, since g is an odd map, we have $g(s + 1/2) = -g(s)$ for $s \in [0, 1/2]$. Hence, for all $s \in [0, 1/2]$, there exists an integer m such that $\tilde{g}(s+1/2) = \tilde{g}(s)+m+1/2$. Note that m does not depend on s, since $\tilde{g}(s+1/2)-\tilde{g}(s)-1/2$ is a continuous function on $[0, 1/2]$ with values in $\mathbb{Z}$, hence it has to be the constant function. By evaluating at $s = 0$ and $s = 1/2$, we conclude that $\tilde{g}(1)-\tilde{g}(0) = 2m+1$ is a non-zero integer. This gives us the desired contradiction. $\square$

4.2 Fundamental lifting theorem

We have seen that homotopies f_t lift from the base of a covering space to its total space as soon as we have a lift of f_0. Therefore, it is important to understand when can lift just a map $f = f_0$. May, understandably, calls the following the fundamental theorem of covering theory:

Proposition 4.8. *Let $p : E \to B$ be a covering. Let $f : S \to B$ be a continuous map from a path-connected and locally path-connected space. Let $s \in S$ and $b \in B$ and $e \in E$ be basepoints so that $f(s) = b = p(e)$. Then there exists a lift $\tilde{f} : S \to E$ with $f(s) = e$ if and only if*

$$\Pi(f)(\pi_1(S, s)) \subset \Pi(p)(\pi_1(E, e))$$

in $\pi_1(B, b)$. Furthermore, if a lift exists, then it is unique.

Proof. If a lift exists, then, by definition, $p \circ \tilde{f} = f$. Applying the fundamental groupoid functor, we get $\Pi(p) \circ \Pi(\tilde{f}) = \Pi(f)$ hence, $\Pi(f)(\pi_1(S, s)) \subset \Pi(p)(\pi_1(E, e))$.

Conversely, let x be any point of S. Choose a path $g : [0,1] \to S$ such that $g(0) = s$ and $g(1) = x$. Consider the path $f \circ g : [0,1] \to B$. By the homotopy lifting property, this path lifts to $\widetilde{f \circ g} : [0,1] \to E$ at e. Note that if a lift $\tilde{f}$ exists, by the uniqueness of homotopy lifting property, it must be that $\tilde{f} \circ g = \widetilde{f \circ g}$. Hence, we are led to define

$$\tilde{f}(x) := \widetilde{f \circ g}(1)$$

From this, we also conclude that if a lift exists, it is unique. We need to check that this definition is independent of the choice of g and continuous. Any other path can be written as $g \cdot \gamma$ for some $\gamma \in \pi_1(S, s)$. Therefor, $f \circ (g \cdot \gamma) = (f \circ g) \cdot (f \circ \gamma)$. But $[f \circ \gamma] \in \Pi(p)(\pi_1(E, e))$, hence by the homotopy lifting property, there exists a loop $\beta : [0,1] \to E$ with $\beta(0) = \beta(1) = e$, such that $p(\beta) = f \circ \gamma$. But then, $\widetilde{f \circ g} \cdot \beta$ is the unique lift of $f \circ (g \cdot \gamma)$ and we have $\widetilde{f \circ g} \cdot \beta(1) = \widetilde{f \circ g}(1)$, as desired. Thus $\tilde{f} : S \to E$ is well-defined. Continuity of $\tilde{f}$ is an easy consequence of local path-connectedness of S, which we skip. □

The argument given in the proof also shows that:

Proposition 4.9. *The index of the subgroup $\Pi(p)(\pi_1(E, e)) \subset \pi_1(B, b)$ equals the cardinality of $p^{-1}(b)$.*

Proof. To see this, for any loop $\gamma : [0,1] \to B$ with $\gamma(0) = \gamma(1) = b$, consider its lift $\tilde{\gamma} : [0,1] \to E$ such that $\gamma(0) = e$. Now, consider the element in $F_b = p^{-1}(b)$ associated to $\tilde{\gamma}(1)$. First of all, $\tilde{\gamma}(1) = e$ if and only if $[\gamma] \in \Pi(p)(\pi_1(E, e))$. As, then $\tilde{\gamma}$ is a loop, hence $[\gamma] = \Pi(p)[\tilde{\gamma}]$. On the other hand, since E is connected, we can choose an arc $\alpha : [0,1] \to E$ with $\alpha(0) = e$ and $\alpha(1)$ is any element in F_b. Then $[p(\alpha)] \in \pi_1(B, b)$. Two such arcs $\alpha, \beta : [0,1] \to E$ with $\alpha(0) = \beta(0) = e$ give the same coset $[p(\alpha)] = [p(\beta)] \in \pi_1(B, b)/\Pi(p)(\pi_1(E, e))$ if and only if they differ by a loop hence this holds if and only if $\alpha(1) = \beta(1)$. □

In particular, note that any two fibers F_b and $F_{b'}$ have the same cardinality.

We also obtain a *transitive* action of $G = \pi_1(B, b)$ on the fiber F_b, as the fiber F_b is identified with a coset G/G_s where $G_s = \Pi(\pi_1(E, e))$ is the stabilizer.

One can upgrade this to a functor:

$$\Pi(B) \to \mathscr{S}$$

from the fundamental groupoid of B to the category of sets by sending an object $x \in B$ to the set F_x, and a morphism $f \in Mor(x, y)$ to a morphism of sets $T_f : F_x \to F_y$ by setting $T_f(e) = e'$ where $e' = \tilde{f}(1)$ for the unique lift $\tilde{f}$ of f at e.

4.3 Classification of Covering spaces

Definition 4.10. *Let B be a path-connected and locally path-connected topological space. We write $Cov(B)$ for the category with objects coverings $p : E \to B$ with path-connected, locally path-connected spaces E. A morphism between two coverings $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ is $f : E_1 \to E_2$ such that $f \circ p_2 = p_1$.*

The morphisms in $Cov(B)$ can be described by the following diagram:

$$\begin{tikzcd} & E_2 \arrow[d, "p_2"] \\ E_1 \arrow[ur, dashed, "f"] \arrow[r, "p_1"'] & B \end{tikzcd}$$

Thus, constructing a morphism from $E_1 \to E_2$ is the same thing is as constructing a lift of $p_1 : E \to B$. The following is a direct consequence of the fundamental theorem of covering space theory:

Corollary 4.11. *Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ are covering maps with basepoints so that $p(e_1) = b = p(e_2)$. Then, there exists a homeomorphism $h : E_1 \to E_2$ with $p_2 \circ h = p_1$ and $h(e_1) = e_2$ if and only if $\Pi(p_1)(\pi_1(E_1, e_1)) = \Pi(p_2)(\pi_1(E_2, e_2))$ in $\pi_1(B, b)$. Hence two coverings of B are isomorphic if and only if they define conjugate subgroups of $\pi_1(B, b)$.*

Note that from this and Proposition 4.5 ii) it follows easily that $f : E_1 \to E_2$ is itself a covering map.

Thus, any two simply connected coverings of B are isomorphic. A simply connected covering is called *universal covering*. We shall see that under an additional mild hypothesis, semi-locally simply connectedness, on B in addition to locally path-connectedness, there always exists a universal covering. These conditions are satisfied, for example, when B is a CW-complex.

If $p : E \to B$ is an object of the category $Cov(B)$, the automorphisms of this object are, written as $Aut(E/B)$, called *deck transformations*. Choose basepoint $b \in B$ and $e \in E$, as before, such that $p(e) = b$.

The following proposition, in particular, allows us to understand $Aut(E/B)$ via restricting to the fibre:

Proposition 4.12. *Let $p : E \to B$ be a covering. Then $Aut(E/B)$ is naturally isomorphic to the group of automorphisms of F_b considered as a G-set, where $G = \pi_1(B, b)$.*

Proof. Given a deck transformation $f : E \to E$, we get an induced automorphism $f_{|F_b} : F_b \to F_b$ by restriction. It is easy to see that this is an automorphism as a G-set by unique lifting of paths. Furthermore, viewing $f : E \to E$ as a lift of $p : E \to B$, by uniqueness of lifts f is completely determined by its restriction to the fiber. Let $e \in E$ be a basepoint in F_b and H be the subgroup $\Pi(p)(\pi_1(E, e))$ of $G = \pi_1(B, b)$. To see that, any automorphism of F_b as a G-set comes from a deck transformation, write $F_b = G/H$ as coset of $H \subset G$. Now, any map from $G/H \to G/H$ that is a G-map has to be of the form: $gH \to gmH$ for some $m \in G$, where $m \in G$ satisfies $mH = Hm$ (check this!). In other words, G-maps from $G/H \to G/H$, can be identified with the Weyl group NH/H where NH is the normalizer of H in G. Let $\tilde{m}$ be a lift at a point e' of $m \in \pi_1(B, b)$ such that $e = \tilde{m}(1)$. Then, we have

$$\Pi(p)(\pi_1(E, e')) = \Pi(p)(\tilde{m}^{-1}\pi_1(E, e)\tilde{m}) = m^{-1}Hm = H = \Pi(p)(E, e)$$

Hence, by the previous result there exists a homeomorphism of $E \to E$ sending e to e'. $\square$

Definition 4.13. *We say that $p : E \to B$ is a normal (or regular or Galois) covering if $\Pi(p)(\pi_1(E, e)) \subset \pi_1(B, b)$ is a normal subgroup.*

Exercise: Show that $p : E \to B$ is normal if and only if $Aut(E/B)$ acts transitively on F_b, i.e. for any $e, e' \in F_b$, there exists an $f \in Aut(E/B)$ such that $f(e) = e'$.

Now, to organize our thoughts a little bit, let's review what we have. If $p : E \to B$ is a covering space of locally path-connected, path-connected spaces with basepoints e, b such that $p(e) = b$, then we get :

1) A subgroup $H = \Pi(p)(\pi_1(E, e)) \in \pi_1(B, b) = G$

2) A transitive G-set F_b which is identified with the cosets of G/H of H.

Given a group G, one can construct an *orbit category* $\mathcal{O}(G)$ whose objects are subgroups $H \subset G$. An object gives the set G/H of left cosets which is a transitive G-set. Given two objects $H, K \subset G$, one defines the morphisms $Mor(H, K)$ to be the maps of G-set $G/H \to G/K$.

The main theorem that we'll prove in this section is:

Theorem 4.14. *Suppose (B, b) is a based space with a universal cover $p : U \to B$ and a basepoint $u \in U$ with $p(u) = b$. Then there exists an equivalence of categories:*

$$\mathcal{E} : \mathcal{O}(\pi_1(B, b)) \to Cov(B)$$

Proof. Let us write $G = \pi_1(B, b)$. By the previous proposition, we have the isomorphism $Aut(U/B) = G$ since F_b is isomorphic to G as a G-set. Therefore, subgroups of G can be identified with subgroups of $Aut(U/B)$. Define $\mathcal{E}(H)$ to be the orbit space U/H. This comes with a projection map $\mathcal{E}(H) = U/H \to U/G = B$. Since $U \to U/G$ is a covering, it follows easily that $\mathcal{E}(H) \to B$ is a covering. The fiber of this covering over b is canonically identified with G/H, and $\pi_1(\mathcal{E}(H), [u])$ maps to H under this covering map.

Next, if $H, K \subset G$ are subgroups of G, we can generalize the argument given in the previous proposition to see that $Mor(\mathcal{E}(H), \mathcal{E}(K))$ can be identified with maps of G-sets from $G/H \to G/K$. Namely, any such G-set map is of the from $gH \to gmK$ for some m with $m^{-1}Hm \subset K$. If $p_H : \mathcal{E}(H) \to B$ and $p_K : \mathcal{E}(K) \to K$ are the associated coverings with base points e_H and e_K and let $\tilde{m}$ be the lift of m at a point e'_H such that $e_H = \tilde{m}(1)$. Then we have:

$$\Pi(p_H)(\pi_1(\mathcal{E}(H), e'_H)) = m^{-1}Hm \subset K = \Pi(p_K)(\pi_1(\mathcal{E}(K), e_K)$$

Hence, by the fundamental lifting theorem, there exists a unique lift of p_H to a morphism in $Mor(\mathcal{E}(H), \mathcal{E}(K))$ which sends the base point e'_H to e_K, hence sends $e_H \to m \cdot e_K$.

Thus, we have constructed the functor $\mathcal{E}$. Both injectivity and essential surjectivity follows from Corollary 4.11. $\square$

4.4 Existence of universal covers

The last topic in our discussion of covering spaces will be about the existence of universal covers. As before, we assume that our base spaces B are path-connected and locally path-connected. It turns out that a sufficient and necessary condition for the existence of a universal cover is *semi-locally simply-connectedness* of B. This means that B has a basis of topology such that for each open set U of the basis the natural map $\pi_1(U) \to \pi_1(X)$ is zero; this means that loops in U can be contracted within X (but not necessarily in U). This condition is obviously implied by locally contractible spaces such as CW-spaces.

Theorem 4.15. *Let B be a path-connected, locally path-connected and semi-locally simply-connected topological space, then B has a universal cover, that is, there exists a covering map $p : \mathcal{U} \to B$ from a simply connected topological space $\mathcal{U}$.*

Proof. Let $b \in B$ be a basepoint. To motivate the construction of $\mathcal{U}$, let us assume for a moment that $p : \mathcal{U} \to B$ is a universal cover. Since $\mathcal{U}$ is simply connected, for any point $x \in \mathcal{U}$, there exists a unique homotopy class of paths that start at u and end at x. This path, in turn, is a unique lift of a path in B that starts in b. Motivated by this, we define $\mathcal{U}$ as a set by the set of equivalence classes of paths $f : [0,1] \to B$ with $f(0) = b$, where the equivalence is as usual given by a homotopy rel end points. We then have a projection $p : \mathcal{U} \to B$ given by sending $[f]$ to $f(1)$. Next, we need to define a topology on $\mathcal{U}$ which makes it simply connected and such that $p : \mathcal{U} \to B$ is a covering map. Recall that by the assumption of semi-locally simply-connectedness B has a basis of open sets such that for each member U of this basis $\pi_1(U) \to \pi_1(B)$ is the zero map. For each such U and a path $f : [0,1] \to B$ such that $f(0) = b$ and $f(1) \in U$, we put

$$U([f]) = \{[g] : [g] = [c \cdot f] \text{ for some } \ c : [0,1] \to U\}$$

It is easy to see that these form a basis of topology on $\mathcal{U}$.

Note that since $\pi_1(U) \to \pi_1(B)$ is the zero map, for each point $u \in U$, there exists a unique class $[g]$ in each $U([f])$ that maps to u. Furthermore, if we choose a basepoint $u \in U$, the preimage

$$p^{-1}(U) = \bigsqcup_{[f] \in Mor(b,u)} U([f])$$

Therefore, $p : \mathcal{U} \to B$ is a covering map. Finally, we need to see that $\mathcal{U}$ is simply connected. Let $e = [c_b]$ be the constant path at b, taken as a basepoint of $\mathcal{U}$. Given any path $f : [0,1] \to B$ starting at b, we can construct its lift to $\mathcal{U}$ at e as $\tilde{f} : [0,1] \to \mathcal{U}$ by defining $\tilde{f}(s) = [f_s]$ where $f_s : [0,1] \to B$ is the path $f_s(t) = f(st)$. $\tilde{f}$ is clearly continuous and is the unique lift of f. It sends the basepoint e to $[f]$. Hence, fixes the base point only if f is null-homotopic. This means that the action of $\pi_1(B, b)$ is free on the fibre $F_b = p^{-1}(b)$, which is equivalent to the simply-connectedness of $\mathcal{U}$.

□

Note that as we have seen in the previous section, when it exists, universal cover is unique up to isomorphism.

4.5 A brief interlude: higher homotopy groups

Let (X, x) be a based space. We can express

$$\pi_1(X, x) = [S^1, X]$$

where $[A, B]$, for A and B be (based) spaces, stands for (based) homotopy classes of (based) maps.

There are natural generalization of this construction to all $n \geq 0$ defined by

$$\pi_n(X, x) = [S^n, X]$$

where S^n is the based n-sphere. These are called homotopy groups of X. Note that for $n = 0$, $\pi_0(X)$ is simply the set of path components of X. For $n \geq 1$, the homotopy groups are indeed groups and for $n \geq 2$ they are abelian groups. (The latter is the first easy fact that one learns about higher homotopy groups, see Hatcher Section 4.1).

These groups are easy to define but notoriously difficult to compute for $n \geq 2$. Even $\pi_n(S^2)$ is not known for all n. Here is a short list of the first few that we know:

$$\pi_1(S^2) = 0, \pi_2(S^2) = \mathbb{Z}, \pi_3(S^2) = \mathbb{Z}, \pi_4(S^2) = \mathbb{Z}_2, \pi_5(S^2) = \mathbb{Z}_2, \pi_6(S^2) = \mathbb{Z}_{12}, \ldots$$

It is absolutely despicable that we don't know these groups for all n.

We probably won't have time to cover higher homotopy groups in more depth in this course but it is indeed a fascinating topic. Just like the fundamental group, these higher groups are functorially associated to topological spaces. One of the key definitions that goes with is the following:

Definition 4.16. *A map $f : X \to Y$ is said to be a* weak-equivalence *if for all x, the induced maps:*

$$\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$$

are isomorphisms for all $n \geq 0$.

This definition is important in view of Whitehead's theorem that states that a weak-equivalence between CW-complexes is a homotopy equivalence. Proof of Whitehead's theorem is not too hard but it uses some more familiarity with the basics of homotopy groups and notably the cellular approximation theorem.

In this interlude, we will content ourselves with the following relation to covering spaces:

Proposition 4.17. *Let $p : E \to B$ be a covering, then p induces isomorphisms*

$$\pi_n(E) \to \pi_n(B)$$

for all $n \geq 2$.

Proof. Since S^n is simply connected for $n \geq 2$. Any map from $S^n \to B$ lifts by the fundamental lifting theorem. This implies surjectivity. Injectivity is a direct consequence of homotopy lifting theorem. □

5 Homology

Homology groups are basic computable invariants of spaces. Unlike homotopy groups these are stable invariants which is what makes them computable.

We associate invariants to topological spaces in two steps: first we map the space X to a chain complex, then we take the homology of this complex. In professional life, one should never do the second step but amateurs do, and we tolerate them.

5.1 Algebraic preliminaries:

Let R be commutative ring (for example $R = \mathbb{Z}$). Let *mod-R* be the category of R-modules. An object of this category is an R-module M and a morphism between two modules M and N, is an R-module map.

Definition 5.1. *A chain complex over R is a sequence of maps of R-modules*

$$\ldots \to C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \to \ldots$$

such that $d_i \circ d_{i+1} = 0$.

We will usually drop the indices on d_i and refer to all of them as d. One often writes C_* to refer to the direct sum $\oplus_i C_i$ of R-modules with i^{th} graded piece given by C_i.

An element of the kernel of d_i is called a *cycle* and an element of the image of d_{i+1} is called a *boundary*. The submodule of cycles in C_i are denoted by $Z_i(C) = \ker(d_i)$ and the submodule of boundaries in C_i are denoted by $B_i(C) = \mathrm{im}(d_{i+1})$. The condition $d_i \circ d_{i+1} = 0$ guarantees that $B_i \subset Z_i$. The i^{th} homology group is defined to be the quotient module:

$$H_i(C) = Z_i(C)/B_i(C)$$

As before, we write $H_*(C)$ for the direct sum $\oplus_i H_i(C)$. We say that element of $Z_i(C)$ are *homologous* if they differ by an element in $B_i(C)$.

One can form a category $\mathscr{C}h$ of chain complexes over R. A morphism between chain complexes C_* and C'_* is given by a *chain map* : $f : C_* \to C'_*$. This is a sequence of maps $f_i : C_i \to C'_i$ such that the diagram below is commutative for all i:

$$\begin{tikzcd}
C_i \arrow[r, "d_i"] \arrow[d, "f_i"'] & C_{i-1} \arrow[d, "f_{i-1}"] \\
C'_i \arrow[r, "d'_i"'] & C'_{i-1}
\end{tikzcd}$$

That is, $d'_i \circ f_i = f_{i-1} \circ d_i$ for all i.

The latter relation implies that $f_i(Z_iC) \subset (Z_iC')$ and $f_i(B_iC) \subset (B_iC')$ hence we get induced maps

$$H_i(f_i) : H_i(C) \to H_i(C')$$

Hence, for each $i \in \mathbb{N}$ we have a homology functor:

$$H_i : \mathscr{C}h \to mod\text{-}R$$

So, one could aim to construct a functor

$$\mathscr{T} \to \mathscr{C}h$$

from category of topological spaces $\mathscr{T}$ to the category of chain complexes $\mathscr{C}h$ and apply the homology functors H_i to get invariants of topological spaces. Just like in the case of fundamental groupoid, the correct source and target of our functor will be homotopy categories. That is,

$$h\mathscr{T} \to h\mathscr{C}h$$

Recall that $h\mathscr{T}$ is the homotopy category of topological spaces, where the objects are topological spaces and morphisms are homotopy equivalence classes of maps between topological spaces. To define the notion of homotopy equivalence between chain complexes, we need to say what it means to be two chain maps $f, g : C_* \to C'_*$ to be homotopic. This is an algebraic version of homotopy and is of key importance in homological algebra:

Definition 5.2. *A chain homotopy s between chain maps $f, g : C_* \to C'_*$ is a sequence of R-module maps $s_i : C_i \to C'_{i+1}$ for each i such that*

$$d'_{i+1} \circ s_i + s_{i-1} \circ d_i = f_i - g_i$$

It is easy to see that chain homotopy is an equivalence relation. The category $h\mathscr{C}h$ has objects chain complexes and morphisms given by the equivalence classes of chain maps up to chain homotopy.

Lemma 5.3. *Chain homotopic maps $f, g : C_* \to C'_*$ induce the same homomorphism on homology.*

Pf. Let s be tha chain homotopy, then for $x \in Z_i(C)$

$$f_i(x) - g_i(x) = d'_{i+1} s_i(x)$$

so that $f_i(x)$ and $g_i(x)$ are homologous.

5.2 Singular homology

In this section we construct an explicit functor from $\mathscr{T} \to \mathscr{C}h$ and we will also see that this descends to a functor from $h\mathscr{T} \to h\mathscr{C}h$.

Remark 5.4. *: Note that Hatcher's Chapter 1 begins with simplicial homology. We omit this, because it is largely subsumed by cellular homology which we'll see later on. On the other hand, you may find it useful to read through Hatcher to develop your intuition.*

Let X be a topological space (think a CW-complex). Intuitively, the resulting homology $H_i(X)$ will be i-dimensional pieces of X without boundary, and two generators will be equivalent if they are the boundary components of an $(i+1)$-dimensional piece. To make this precise and fairly general, we would need to build our spaces out of lego. There is some freedom in choosing the building blocks. We will use simplices (but you will see that cubes can also be used in a homework problem).

Definition 5.5. *An $n-$simplex Δ_n is the subspace of $\mathbb{R}^{n+1}$ given by*

$$\Delta_n = \{(t_0, t_1, \ldots, t_n) : 0 \leq t_i \leq 1, \sum t_i = 1\}$$

These come with face maps:

$$\delta_i : \Delta_{n-1} \to \Delta_n, 0 \leq i \leq n$$

given by

$$\delta_i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

which are homeomorphisms onto their images.

For a space X define the singular chain complex as follows: Let $C_n(X;R)$ be the free R-module generated by the set of continuous maps $\sigma : \Delta_n \to X$, that is,

$$C_n(X;R) = \{\sum_{i=1}^{k} a_i\sigma_i : a_i \in R, \sigma_i : \Delta_n \to X\}$$

Define the boundary map $d_n : C_n(X;R) \to C_{n-1}(X;R)$ via the formula:

$$d_n\sigma = \sum_{i=0}^{n}(-1)^i(\sigma \circ \delta_i)$$

(Have a look at the figure on page 105 of Hatcher to make sure that you understand the sign and orientation conventions used in this formula.)

The signs in the summation are mysterious and will not be justified intuitively at this stage but they play a vital role in the proof of the following:

Proposition 5.6. $d^2 = 0$.

Proof. The face maps satisfy the equality $\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1}$ for $i < j$. Using this, we have:

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j}(\sigma \circ \delta_j \circ \delta_i) = \sum_{j \leq i}(-1)^{i+j}(\sigma \circ \delta_j \circ \delta_i) + \sum_{i<j}(-1)^{i+j}(\sigma \circ \delta_i \circ \delta_{j-1}) = 0$$

(Set $i' = j-1, j' = i$ in the second sum to see the cancellation.) $\square$

Therefore, by passing to homology, we get an R-module $H_n(X;R)$ for each topological space X and $n \geq 0$. One often writes $H_n(X)$ when the reference to R is understood.

Note that if $f : X \to Y$ is continuous map, we get an induced map of chain complexes:

$$S_i(f) : C_i(X;R) \to C_i(Y;R)$$

by sending $\sigma \to f \circ \sigma$. The notation $\mathcal{S}(f)$ is often shortened to f_*. A moment's reflection leads to the obvious fact that $d_i S_i(f) = S_{i-1}(f) d_i$. Note also that we have $\mathcal{S}(id) = id$ and $\mathcal{S}(g \circ f) = \mathcal{S}(g) \circ \mathcal{S}(f)$.

Hence, we indeed get a functor, called singular chain complex, :

$$\mathcal{S} : \mathscr{T} \to \mathscr{C}h$$

In particular, if X and Y are homeomorphic, we have that their homology groups are isomorphic. We will soon see that singular chain complex functor induces a more desirable functor $h\mathscr{T} \to h\mathscr{C}h$, which will give us that homotopy equivalent spaces have isomorphic homology groups.

However, first let us familiarize ourselves a little bit more with the singular chain complex.

5.3 First computations:

Proposition 5.7. *$H_*(pt) = R$ for $* = 0$ and 0 otherwise.*

Proof. In this case, the target space is so simple that we can write its chain complex explicitly as:

$$\cdots R \xrightarrow{\simeq} R \xrightarrow{0} R \xrightarrow{\simeq} R \xrightarrow{0} R \to 0$$

since every chain is of the from $r\sigma_n$ where $\sigma_n : \Delta_n \to \{pt\}$ is the constant map to the point. Furthermore, the differential $d : C_n \to C_{n-1}$ is given by $d\sigma_n = \sum_i (-1)^i \sigma_{n-1}$. Thus it is zero for odd n and an isomorphism for even n. $\square$

Proposition 5.8. *$H_0(X)$ is a free R-module on as many generators as there are path components of X.*

Proof It suffices to show that $H_0(X) = R$ for X path-connected since we have $H_*(X) = \oplus_{i \in I} H_*(X_i)$ if X_i are path-components of X. So, assume X is path-connected and define a map:

$$\epsilon : C_0(X) \to R$$

given by

$$\sum_{i=1}^{k} a_i \sigma_i \to \sum_{i=1}^{k} a_i$$

Firstly, observe that

$$\epsilon(d\sigma) = \epsilon(\sigma_0 - \sigma_1) = 1 - 1 = 0$$

for any $\sigma : \Delta_1 \to X$, where we used the notation $\sigma_i = \sigma \circ \delta_i$. Hence, ϵ descends to a map from $H_0(X) \to R$. It is certainly surjective. Pick a point $x_0 \in X$ and observe that $\sigma(rx_0) = r$. To see injectivity, let $\sum_{i=1}^n a_i x_i$ be a 0-cycle in the kernel of ϵ. We want to show that it is also a boundary. Choose paths from $\sigma_i : [0,1] \to X$ such that $\sigma(0) = x_i$ and $\sigma(1) = x_0$. Then, we compute:

$$d\left(\sum_{i=1}^n a_i \sigma_i\right) = \sum_{i=1}^n a_i(x_i - x_0) = \sum_{i=1}^n a_i x_i$$

since $\sum_{i=1}^n a_i = 0$. □

The chain map $\epsilon : C_0(X) \to R$ is sometimes *augmented* to the singular chain complex to form the *reduced* chain complex:

$$\cdots \to C_2(X) \xrightarrow{d_2} C_1(X) \xrightarrow{d_1} C_0(X) \xrightarrow{\epsilon} R \to 0$$

This chain complex is denoted by $\tilde{C}_*(X)$ and its homology $\tilde{H}_*(X)$ is isomorphic to $H_*(X)$ in positive degrees. In contrast, $H_0(X) = \tilde{H}_0(X) \oplus R$.

Next we study the relationship of $H_1(X)$ and the fundamental group.
Theorem 5.9. *Let $x_0 \in X$ be a base-point. There is a homomorphism*

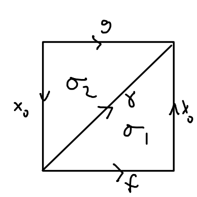
$$h : \pi_1(X, x_0) \to H_1(X; \mathbb{Z})$$

If X is path-connected, the kernel is the commutator subgroup, i.e., $H_1(X)$ is isomorphic to the abelianization of $\pi_1(X)$.

Proof. Identifying $\Delta_1 \simeq [0,1]$. We see that a loop $f : [0,1] \to X$ with $f(0) = f(1) = x_0$ defines a 1-chain, but, in fact, it is a cycle since

$$df = f(1) - f(0) = x_0 - x_0 = 0$$

So, we want to define $h(f) = [f] \in H_1(X)$. We need to see that if f is homotopic to g we get homologous cycles. Let $F : [0,1] \times [0,1] \to X$ be a homotopy from f to g. We can subdivide the domain of F into two triangles to get 2-simplices σ_1, σ_2 as in the figure: Let us also take a



2-simplex that is just the constant map to x_0. Then we have :

$$d\sigma_1 = x_0 - \gamma + f$$

$$d\sigma_2 = \gamma - g + x_0$$
$$d\sigma_3 = x_0$$

Hence,

$$d(\sigma_1 + \sigma_2 - 2\sigma_3) = f - g$$

Thus homotopic paths lead to homologous cycles and we get a well-defined map from $h : \pi_1(X, x_0) \to H_1(X)$.

This is a homomorphism. To see this, observe that for composable 1-simplices $f, g : [0,1] \to X$, one can construct a two simplex $\sigma : \Delta_2 \to X$ such that

$$d\sigma = g - (f \cdot g) + f$$

(Proof easily follows by describing the picture of the domain or formally one can define $\sigma = (f \cdot g) \circ p_1$ where $p_1 : \Delta_2 \to \Delta_1$ is the projection to a face given by $(t_0, t_1, t_2) \to (t_0 + \frac{t_2}{2}, t_1 + \frac{t_2}{2})$).

It remains to prove that this is surjective and the kernel is the commutator subgroup of $\pi_1(X, x_0)$ if X is path-connected. So, suppose now X is path-connected. For every point $p \in X$ Choose a path γ_p from x_0 to p. Now, if $\sigma = \sum_i a_i \sigma_i$ is a 1-cycle, we have:

$$0 = \sum a_i(\sigma_i(1) - \sigma_i(0)) = d\sigma$$

This means that every point occurring in the sum occur even number of times with cancelling signs. By replacing each point p with our chosen path γ_p, we see that the associated sum of 1-chains is also zero. Therefore, the 1-cycle

$$\sum a_i(\gamma_{\sigma_i(1)} + \sigma_i + \gamma^{-1}_{\sigma_i(0)})$$

is equal to σ and this is homologous to the 1-cycle consisting of images of loops of the form $\gamma_{\sigma_i(1)} \cdot \sigma_i \cdot \gamma^{-1}_{\sigma_i(0)}$. This proves surjectivity.

Finally, to compute the kernel of h. First observe that since $H_1(X)$ is abelian, and h is a group homomorphism, it is obvious that the commutator subgroup $[\pi_1, \pi_1] \subset ker(h)$.

To see the equality, let γ be a loop that is homologous to 0. This means that we have :

$$d(\sum a_i \sigma_i) = \sum a_i(\sigma_{i0} - \sigma_{i1} + \sigma_{i2}) = \gamma$$

where $\sigma_i : \Delta_2 \to X$ are 2-cycles. Hence, on the left side of the equation after collecting terms together, γ occurs with multiplicity 1 and every other path has coefficient 0.

Now, let γ_{ij}, $j = 0, 1, 2$ be the chosen paths γ_p that from x_0 to $\sigma_{i2}(0), \sigma_{i0}(0), \sigma_{i1}(1)$ - the three corners of σ_i. We then consider the loops based at x_0 given by:

$$\beta_{i0} = \gamma_{i1} \sigma_{i0} \gamma_{i2}^{-1}$$
$$\beta_{i1} = \gamma_{i0} \sigma_{i1} \gamma_{i2}^{-1}$$

$$\beta_{i2} = \gamma_{i0}\sigma_{i2}\gamma_{i1}^{-1}$$

We have that in $\pi_1(X, x_0)$.

$$[\beta_{i0}][\beta_{i1}^{-1}][\beta_{i2}] = [\gamma_{i1}\sigma_{i0}\sigma_{i1}^{-1}\sigma_{i2}\gamma_{i_1}^{-1}] = 0$$

Hence, in $\pi_1(X, x_0)$ we have

$$\sum a_i[\beta_{i0}][\beta_{i1}^{-1}][\beta_{i2}] = 0$$

On the other hand, if we are allowed to commute terms, then we know that the coefficient of γ will be 1 and others will cancel out, so we conclude that :

$$[\gamma] = 0 \in \pi_1(X, x_0)/[\pi_1, \pi_1].$$

In other words, $[\gamma] \in [\pi_1(X, x_0), \pi_1(X, x_0)]$, as required. □

Remark 5.10. *Recall that a loop $\gamma : S^1 \to X$ is homotopically trivial (i.e. $[\gamma] = 0 \in \pi_1(X)$) if and only if it extends to a map $D^2 \to X$. By refining the above discussion, one can prove that if $[\gamma] = 0 \in H_1(X)$ if and only if γ extends to a map from $(\Sigma_g \backslash D^2) \to X$ where $\Sigma_g \backslash D^2$ is genus g surface with one boundary component.*

Remark 5.11. *In a similar vein, one can construct maps $\pi_k(X) \to H_k(X)$ for all k. Hurewicz theorem stats that if X is path connected and $\pi_i(X) = 0$ for $1 \leq i < n$ for $n \geq 2$, then the map $\pi_n(X) \to H_n(X)$ is an isomorphism.*

5.4 Homotopy invariance

It is now time to fulfill our promise and show that singular homology gives a functor:

$$h\mathscr{T} \to h\mathscr{C}h$$

To see this, we need to show that homotopic maps $f, g : X \to Y$ induce chain homotopic maps $f_*, g_* : C_*(X) \to C_*(Y)$. In other words, given a homotopy $H : X \times I \to Y$ between f and g, we need to construct a chain homotopy.

Let $\eta_0, \eta_1 : X \to X \times I$ be the maps $x \to (x, 0)$ and $x \to (x, 1)$. By functoriality, it is enough to prove that

$$(\eta_0)_*, (\eta_1)_* : C_*(X) \to C_*(X \times I)$$

are chain homotopic. Since this implies that $f = H \circ \eta_0$ and $g = H \circ \eta_1$ are chain homotopic via a chain homotopy obtained by composing the chain homotopy between $(\eta_0)_*$ and $(\eta_1)_*$ with $H_* : C_*(X \times I) \to C_*(Y)$.

Thus we need to prove:

Theorem 5.12. *There exists a chain homotopy between $(\eta_0)_*, (\eta_1)_*$.*

The most natural way of proving this lemma is via "the method of acyclic models". This is what we are going to do. (See Hatcher Theorem 2.10 for another proof which uses the prism operator.)

We will need the following special case of homotopy invariance:
Lemma 5.13. *Let X be a contractible topological space, then $H_*(X) = H_*(pt.)$.*

Proof. Since X is path-connected, we know that $H_0(X) = R$. So, it suffices to show $H_n(X) = 0$ for $n > 0$. Let $x_0 \in X$ and $H : X \times I \to X$ be a deformation retraction to x_0, i.e, $H(x, 0) = x$ and $H(x, 1) = x_0$ for all $x \in X$. Define the "cone operator"

$$K : C_n(X) \to C_{n+1}(X)$$

as follows: If $\sigma : \Delta_n \to X$ is a singular n-chain, we set:

$$K(\sigma)(t_0, t_1 \ldots, t_{n+1}) = H(\sigma(\frac{t_1}{1-t_0}, \frac{t_2}{1-t_0}, \ldots, \frac{t_{n+1}}{1-t_0}), t_0)$$

(Note that since $\sum_{i=0}^{n+1} t_i = 1$, one has $\sum_{i=1}^{n+1} \frac{t_i}{1-t_0} = 1$.)

$K(\sigma)$ can be visualized by letting σ be the restriction to the $t_0 = 0$ face of an $n+1$ simplex $K(\sigma)$ and flowing σ via the homotopy H to construct the rest of the simplex $K(\sigma)$ as on the domain one flows by a linear homotopy $t_0 = 0$ face to its opposite vertex, namely the vertex defined by $t_0 = 1$.

We claim that for $dK + Kd = id_{C_n(X)}$ for $n > 0$. This then implies the result that $H_n(X) = 0$ for $n > 0$. To see the claim, we simply compute:

$$\begin{aligned} dK(\sigma) &= \sum_{i=0}^{n+1}(-1)^i K(\sigma) \circ \delta_i = \sigma + \sum_{i=1}^{n+1}(-1)^i K(\sigma) \circ \delta_i \\ &= \sigma + \sum_{i=1}^{n+1}(-1)^i K(\sigma \circ \delta_{i-1}) = \sigma - \sum_{i=0}^{n} K(\sigma \circ \delta_i) = \sigma - Kd\sigma \end{aligned}$$

□

Proof of Theorem 5.12: We will construct by induction a chain homotopy $s_i : C_i(X) \to C_{i+1}(X \times I)$ between $(\eta_0)_*$ and $(\eta_1)_*$. In other words, we will need the following equation to hold:

$$ds_i + s_{i-1}d = (\eta_1)_i - (\eta_0)_i$$

for $i \geq 0$.

To begin, define $s_0 : C_0(X) \to C_1(X \times I)$ to be the map given by $s_0(x) = x \times id$ where $id : \Delta_1 \cong I \to I$ is the identity map. Indeed, this satisfies:

$$ds_0 = (\eta_1)_0 - (\eta_0)_0$$

The idea to give a definition of $s_i : C_i(X) \to C_{i+1}(X \times I)$ is to define $s_i(\sigma)$ to be $\sigma \times id$, however the latter is a map from $\Delta_i \times I$ hence does not immediately give us a $(i+1)$-chain in $X \times I$. We will construct the required chain as the push-forward of an element $p_{i+1} \in C_{i+1}(\Delta_i \times I)$. Namely, we will define:

$$s_i(\sigma) = (\sigma \times id)_*(p_{i+1})$$

where the chains $p_i \in C_i(\Delta_{i-1} \times I)$ are required to satisfy:

$$dp_{i+1} + \sum_{j=0}^{i} (-1)^j (\delta_j \times id)_*(p_i) = id_{\Delta_i} \times \{1\} - id_{\Delta_i} \times \{0\} \in C_i(\Delta_i \times I)$$

Applying $(\sigma \times id)_* : C_i(\Delta_i \times I) \to C_i(X \times I)$ to this equation then gives the required equation for verifying that s_i is a chain homotopy for general X.

Finally, we construct the cochains p_i by induction using the contractibility of $\Delta_i \times I$. Namely, since the homology $H_i(\Delta_i \times I) = 0$, to construct p_{i+1} for $i \geq 1$, all we need to see is that:

$$d(id_{\Delta_i} \times \{1\} - id_{\Delta_i} \times \{0\} - \sum_{j=0}^{i} (-1)^j (\delta_j \times id)_*(p_i)) = 0$$

We compute this as follows:

$$d(id_{\Delta_i} \times \{1\} - id_{\Delta_i} \times \{0\} - \sum_{j=0}^{i} (-1)^j (\delta_j \times id)_*(p_i))$$

$$= \sum_{j=0}^{i} (-1)^j (\delta_j \times \{1\} - \delta_j \times \{0\} - (\delta_j) \times id)_*(dp_i)$$

by induction, this is equal to:

$$= \sum_{j=0}^{i} (-1)^j \sum_{k=0}^{i-1} (-1)^k (\delta_j \times id)_* \circ (\delta_k \times id)_*(p_{i-1}) = 0$$

where the last equality follows from $d^2 = 0$.

$\square$

Corollary 5.14. *If $f, g : X \to Y$ are homotopic then $f_*, g_* : C_*(X) \to C_*(Y)$ are chain homotopic. Hence, $f_* = g_* : H_*(X) \to H_*(Y)$.*

Corollary 5.15. *If X, Y are homotopy equivalent, then $H_*(X)$ is isomorphic to $H_*(Y)$.*

5.5 Relative Homology

For $A \subset X$, we have that $C_*(A)$ is a submodule of $C_*(X)$ which consists of those simplices that actually map into A. Furthermore, in fact, $C_*(A)$ is a sub-complex of $C_*(X)$, that is, it is preserved by the differential on $C_*(X)$. Therefore, one can form the quotient complex:

$$C_*(X, A) := C_*(X)/C_*(A)$$

with the induced differential

$$d : C_*(X)/C_*(A) \to C_{*-1}(X)/C_{*-1}(A)$$

given by $d[x] = [dx]$. As before, since $d^2 = 0$, we can form the quotient modules

$$H_i(X, A; R) = \ker d_i/\mathrm{Im} d_{i+1}$$

These are called *relative homology groups*.

If $A = \emptyset$ is empty, then by definition $H_*(X, \emptyset) \cong H_*(X)$. Slightly more interestingly, if $x_0 \in X$ is a basepoint, the relative homology groups of (X, x_0) coincide with $\tilde{H}(X)$, that is the homology of the augmented chain complex $\tilde{C}_*(X)$. In this sense, we get a generalization of singular homology to pairs (X, A).

One can form a category $\mathscr{T}^2$ pairs of topological spaces, which has objects (X, A) such that $A \subset X$ and morphisms $f : (X, A) \to (Y, B)$ are maps $f : X \to Y$ such that $f(A) \subset B$. As usual, we can construct the homotopy category $h\mathscr{T}^2$, where the objects are the same as those of $\mathscr{T}^2$ but the morphisms are equivalence classes of maps $f : (X, A) \to (Y, B)$ where the equivalence is given by homotopy equivalence. A homotopy equivalence F_t between $F_0, F_1 : (X, A) \to (Y, B)$ is by definition a homotopy $F_t : X \to Y$ such that for all values of t, $F_t(A) \subset B$.

One can easily see from the proof of Theorem 5.12 that singular chain complex gives a well-defined functor:

$$h\mathscr{T}^2 \to h\mathscr{C}h$$

The key point in the proof of Theorem 5.12 that allows us to extend it to the relative setting is that from the way we constructed the chain homotopy $s : C_*(X) \to C_{*+1}(X \times I)$, one can infer immediately that if $A \subset X$, then $s(C_*(A)) \subset C_{*+1}(A \times I)$.

Algebraic digression:

The relative singular chain complexes are not only a useful generalization of the absolute singular chain complexes but also they provide a powerful computation tool through *exact sequences*. To understand this, we need to study the category of chain complexes a little more in depth.

Definition 5.16. *An* exact sequence *is a chain complex with vanishing homology. An exact sequence* $0 \to A \to B \to C \to 0$ *is called a* short *exact sequence.*

We can also consider short exact sequences of chain complexes. Namely, given chain maps $f : C_* \to D_*$ and $g : D_* \to E_*$, we say

$$0 \to C_* \xrightarrow{f} D_* \xrightarrow{g} E_* \to 0$$

is a short exact sequence if

$$0 \to C_n \xrightarrow{f_n} D_n \xrightarrow{g_n} E_n \to 0$$

is exact for all n.

One can depict such an exact sequence as follows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C_i \arrow[r, "d"] \arrow[d, "f_i"'] & C_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\
\cdots \arrow[r] & D_i \arrow[r, "d"'] \arrow[d, "g_i"'] & D_{i-1} \arrow[r] \arrow[d, "g_{i-1}"] & \cdots \\
\cdots \arrow[r] & E_i \arrow[r, "d"'] \arrow[d] & E_{i-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 &
\end{tikzcd}$$

where the columns are exact and the rows are chain complexes but not necessarily exact. Note that the exactness of columns mean that f_i is injective, g_i is surjective and $\mathrm{Ker}(g_i) = \mathrm{Im}(f_i)$ for all i.

The useful algebra proposition that we will need is the following:

Proposition 5.17. *A short exact sequence of chain complexes* $0 \to C_* \xrightarrow{f} D_* \xrightarrow{g} E_* \to 0$ *naturally gives rise to a* long exact sequence *of R-modules :*

$$\cdots \xrightarrow{\partial} H_i(C_*) \xrightarrow{f_*} H_i(D_*) \xrightarrow{g_*} H_i(E_*) \xrightarrow{\partial} H_{i-1}(C_*) \xrightarrow{f_*} H_{i-1}(D_*) \xrightarrow{g_*} H_{i-1}(E_*) \xrightarrow{\partial} \cdots$$

Proof. The key part of the proof is the construction of a well-defined *connecting homomorphism*

$$\partial : H_i(E_*) \to H_{i-1}(C_*)$$

and checking the resulting sequence is exact.

Let $[z] \in H_i(E_*)$ be represented by $z \in E_i$, by surjectivity of $g_i : D_i \to E_i$, there exists a chain $y \in D_i$ such that $g_i(y) = z$. Now, since z is a cycle, we have:

$$0 = dz = dg_i(y) = g_{i-1}dy$$

Therefore $dy \in D_{i-1}$ is in the kernel of g_{i-1}, by the exactness of short exact sequence

$$0 \to C_{i-1} \to D_{i-1} \to E_{i-1} \to 0$$

there exists an $x \in C_{i-1}$ such that $f_{i-1}(x) = dy$. Furthermore, $f_{i-2}(dx) = d(f_{i-1}(x)) = d(dy) = 0$. But, f_{i-2} is injective, hence $dx = 0$. We then define:

$$\partial[z] = [x]$$

It is a worthwhile exercise in diagram chasing to show that ∂ is well-defined, that is, it doesn't depend on the choices of chains x, y, and z, and that ∂ is a homomorphism such that the resulting long exact sequence is really exact. (The reader is encouraged to go through this exercise until he/she is convinced of the truth of the statement.)

Finally, naturality means that a commutative diagram of short exact sequences of chain complexes gives rise to a commutative diagram of long exact sequences of R-modules, which is easily checked by the naturality of the construction of the connecting homomorphism ∂. $\square$

After this algebraic digression, we return back to our study of singular chain complex. The importance of the generalization to relative setting becomes clear from the following immediate corollary to Proposition 5.17.

Corollary 5.18. *The natural inclusion $i : A \to X$ and the projection $p : (X, \emptyset) \to (X, A)$ (here p is just the identity map on X) maps give rise to a short exact sequence of chain complexes*

$$0 \to C_*(A) \xrightarrow{i_*} C_*(X) \xrightarrow{p_*} C_*(X, A) \to 0$$

hence a long exact sequence of R-modules:

$$\cdots \xrightarrow{\partial} H_i(A) \xrightarrow{i_*} H_i(X) \xrightarrow{p_*} H_i(X, A) \xrightarrow{\partial} H_{i-1}(A) \xrightarrow{i_*} H_{i-1}(X) \xrightarrow{p_*} H_{i-1}(X, A) \xrightarrow{\partial} \cdots$$

$\square$

The long exact sequence makes precise the idea that $H_*(X, A)$ measures the difference between $H_*(X)$ and $H_*(A)$. It is useful to understand the geometric meaning of the connecting homomorphisms $\partial : H_i(X, A) \to H_{i-1}(A)$. Namely, first note that any relative cycle $z_i \in C_i(X, A)$ can be represented by a chain $\tilde{z}_i \in C_i(X)$. The fact that z_i is a cycle in $C_i(X, A)$ implies that $d\tilde{z}_i \in C_{i-1}(A)$. One then has, by definition, $\partial([z_i]) = [d\tilde{z}_i] \in H_{i-1}(A)$.

There is an easy generalization to a triple of spaces (X, A, B) such that $A \subset B \subset X$. One has an exact sequence of R-modules:

$$\cdots \xrightarrow{\partial} H_i(A, B) \xrightarrow{i_*} H_i(X, B) \xrightarrow{p_*} H_i(X, A) \xrightarrow{\partial} H_{i-1}(A, B) \to \cdots$$

This is a long exact sequence of homology groups associated to the short exact sequence of chain complexes:

$$0 \to C_*(A,B) \to C_*(X,B) \to C_*(X,A) \to 0$$

As a consequence of the *excision theorem* that we will see below, relative homology groups $H_k(X,A)$ can be identified with the absolute homology groups $H_k(X \cup CA)$ for $k > 0$, where $X \cup CA =: Cone(i)$ is the mapping cone of the inclusion map $i : A \to X$. Recall that this is given by the pushout diagram:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & X \arrow[d, hook] \\ CA \arrow[r, hook] & Cone(i) \end{tikzcd}$$

where $CA = (A \times I)/(A \times \{0\})$ and the inclusion map from $A \to CA$ is given by $a \to (a,1)$.

Moreover, for good pairs (X,A) one has a homotopy equivalence between $Cone(i)$ and X/A. Therefore, it is often the case that relative homology groups of (X,A) can be computed as absolute homology groups of X/A.

For example, if A has an open neighborhood U in X such that U deformation retracts onto A, then (X,A) is a good pair. In particular, if X is a CW complex and A is CW subcomplex, then (X,A) is a good pair. (See Chapter 0 of Hatcher.)

Remark 5.19. *More generally, $i : A \to X$ is a cofibration, then $Cone(i)$ is homotopy equivalent to X/A. We will not discuss cofibrations in this course so we omit the definition.*

We will now present an example computation. (Note that we will use the result that $H_k(X,A) = H_k(X/A)$ for $k > 0$ for good pairs (X,A), the proof of which will appear in the next section).

A computation: Homology of spheres

Let $X = D^n$, the closed unit ball, and take $A = S^{n-1}$. Note that A can be exhibited as a CW subcomplex of a CW complex structure on D^n so this is a good pair. We observe that $X/A = S^n$. The long exact sequence associated to the pair (X,A) reads as follows:

$$\xrightarrow{\partial} H_i(S^{n-1}) \xrightarrow{i_*} H_i(D^n) \xrightarrow{p_*} H_i(D^n, S^{n-1}) \xrightarrow{\partial} H_{i-1}(S^{n-1}) \xrightarrow{i_*} H_{i-1}(D^n) \xrightarrow{p_*} H_{i-1}(D^n, S^{n-1}) \xrightarrow{\partial} \cdots$$

Now we have $X/A = S^n$ and the isomorphism $H_k(X,A) = H_k(X/A)$ for $k > 0$. On the other hand, note that $H_i(D^n) = 0$ for $i > 0$ since D^n is contractible, therefore, we conclude that:

$$H_i(S^n) \cong H_{i-1}(S^{n-1}), i \geq 2, n \geq 1$$

Furthermore, the end of the long exact sequence reads:

$$0 \to H_1(S^n) \to H_0(S^{n-1}) \to H_0(D^n) \to H_0(D^n, S^{n-1}) \to 0$$

Now, since D^n is path-connected, it is easy to see that $H_0(D^n, S^{n-1}) = 0$ for all n. We also have $H_0(D^n) = \mathbb{Z}$ and $H_0(S^{n-1}) = \mathbb{Z}$ for $n \geq 2$

$$0 \to H_1(S^n) \to \mathbb{Z} \to \mathbb{Z} \to 0, n \geq 2$$

Finally, it is easy to see that the map from $H_0(S^{n-1}) \to H_0(D^n)$ is an isomorphism. So, $H_1(S^n) = 0$ for $n \geq 2$. We also know that $H_1(S^1) = \mathbb{Z}$ because this is the abelianization of $\pi_1(S^1) = \mathbb{Z}$. Collecting all these together, one concludes:

Theorem 5.20. *$H_i(S^n) = 0$ for $i \neq 0, n$, $n \geq 1$ and $H_n(S^n) = \mathbb{Z}$ for $n \geq 1$.*

In the homework you'll give a proof of the isomorphism $H_k(X, A) = H_k(X/A)$ for $k > 0$ and for good pairs (X, A) using the material from the next section.

5.6 Subdivision, Excision and Mayer-Vietoris

The main technical result we need is the following locality result:

Theorem 5.21. *Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. Let $C_*(X, \mathcal{U})$ be the subcomplex of $C_*(X)$ generated by simplices $\sigma : \Delta_* \to X$ such that the image of σ is entirely contained in some U_σ from the cover $\mathcal{U}$. (Note that d preserves $C_*(X, \mathcal{U})$). The inclusion induced chain map*

$$\iota : C_*(X, \mathcal{U}) \to C_*(X)$$

is a quasi-isomorphism, i.e. , it is an isomorphism on homology.

Proof. We will need to subdivide singular chains in order to make them fit into the open sets in $\mathcal{U}$. There is a natural way to do this called the *barrycentric subdivision*.

For a simplex Δ_q let $b_q = (\frac{1}{q+1}, \frac{1}{q+1}, \ldots, \frac{1}{q+1})$ be its *barycenter*. Let $\sigma : \Delta_{q-1} \to \Delta_q$ be any q-1 chain (not necessarily the face map). Define the cone of σ, by $K_q(\sigma) : \Delta_q \to \Delta_q$ given by:

$$K_q(\sigma)(t_0, t_1, \ldots, t_q) = t_0 b_q + (1 - t_0)\sigma(\frac{t_1}{1 - t_0}, \frac{t_2}{1 - t_0}, \ldots, \frac{t_q}{1 - t_0})$$

This is an instance of the cone operator that we have used in proving homotopy invariance of homology. Note that, as before, we have $dK_q + K_{q-1}d = \sigma$.

For every space X, we define a natural homomorphism $\beta_* : C_*(X) \to C_*(X)$ called the *barycentric subdivision* as follows:

$$\beta_0 = id,\ \beta_q(\iota_q) = K_q(\beta_{q-1}(d\iota_q)),\ \beta_q(\sigma_q) = (\sigma_q)_*(\beta_q \iota_q)$$

where $\iota_q : \Delta_q \to \Delta_q$ is the identity map and $\sigma_q : \Delta_q \to X$ is a singular chain. (By naturality, we mean the last equality. In other words, β_* is completely determined by its effect on the identity maps ι_q.)

It helps to draw a picture of $\beta_q(\iota_q)$ at this point for small q. We encourage the reader to do so (or get it for free from page 120 of Hatcher).

Lemma 5.22. *$\beta_* : C_*(X) \to C_*(X)$ is a natural chain map, and there exists a chain homotopy $s_* : C_*(X) \to C_{*+1}(X)$ between β_* and $id : C_*(X) \to C_*(X)$*

Proof. Assume by induction that $d\beta_{q-1} = \beta_{q-2}d$. We compute:

$$d(\beta_q(\sigma_q)) = d(\sigma_q)_*(\beta_q\iota_q) = (\sigma_q)_*d(\beta_q\iota_q) = (\sigma_q)_*dK_q(\beta_{q-1}(d\iota_q))$$

$$(\sigma_q)_*(\beta_{q-1}(d\iota_q)) - (\sigma_q)_*K_{q-1}d(\beta_{q-1}d(\iota_q)) = (\sigma_q)_*(\beta_{q-1}d\iota_q) = \beta_{q-1}d\sigma_q$$

Next, we show that there exists a chain homotopy $s_* : C_*(X) \to C_{*+1}(X)$ such that

$$ds_q + s_{q-1}d = id - \beta_q$$

As β is a natural homomorphism, we should define s_* also as a natural homomorphism. That is, we set $s_*(\sigma_q) = (\sigma_q)_*(s_q\iota_q)$. Therefore, we just need to construct $s_q(\iota_q)$ such that

$$ds_q(\iota_q) + s_{q-1}d(\iota_q) = \iota_q - \beta_q(\iota_q)$$

To define this inductively, first set $s_0(\iota_0) : \Delta_1 \to \Delta_0$ to be the constant map. Note that $\beta_0 = id$ and $ds_0(\iota_0) = 0$ hence this starts us off. Then using the fact that Δ_q is contractible, it suffices to prove that:

$$d(\iota_q - \beta_q(\iota_q) - s_{q-1}d(\iota_q)) = 0$$

But this again holds, by induction on q. Namely, the left hand side gives:

$$d\iota_q - d\beta_q(\iota_q) + s_{q-2}(d(d\iota_q)) - d\iota_q + \beta_{q-1}(d\iota_q) = -d\beta_q(\iota_q) + \beta_{q-1}(d\iota_q) = 0$$

□

Recall that a simplex $\Delta_n \subset \mathbb{R}^{n+1}$ and we can use the distance function on $\mathbb{R}^{n+1}$ to measure the size or the diameter of Δ_n. (Diameter of a set X by definition maximum distance of any two points $x, y \in X$.) Note that the diameter of Δ_n is 1. After subdividing Δ_n the diameter of each n-simplex in $\beta_n(\iota_n)$ is strictly less than 1. In fact, it can easily be shown that this diameter at most $n/(n+1)$, but we won't need this precision.

Note that $s_* \circ \beta_*$ is a chain homotopy between β_* and $(\beta_*)^2$ and since chain homotopy of chain maps is an equivalence relation, we see that $(\beta_*)^2$ is also chain homotopic to identity, and so as $(\beta_*)^k$ for any k. (It is, in fact, possible to write the chain homotopy between $(\beta_*)^k$ and id explicitly, namely this is given by $s_*^k := s_*((\beta_*)^{k-1} + (\beta_*)^{k-2} + \ldots + \beta_*)$ but we do not need this explicit form.)

Back to the proof of the Theorem 5.21, we need to show that $\iota_* : H_*(X, \mathcal{U}) \to H_*(X)$ is injective and surjective.

Let us first prove surjectivity: If $[z] \in H_*(X)$. Then $z = \sum a_i\sigma_i$ for a finite set of σ_i. Now by Lebesgue covering lemma since the diameter of the simplices are strictly decreasing, there exists a k_i for each σ_i such that $(\beta_*)^{k_i}(\sigma_i) \in U$ for some U in our cover $\mathcal{U}$. Let $k = max_ik_i$. Then,

$$(\beta_*)^k(z) \in C_*(X, \mathcal{U})$$

On the other hand, since $(\beta_*)^k$ is chain homotopic to identity, we have that

$$[(\beta_*)^k(z)] = [z] \in H_*(X)$$

This proves surjectivity.

Injectivity is proven in a similar way: If $z = dw$ where $z \in C_{q-1}(X,\mathcal{U})$ and $w \in C_q(X)$. We can find a sufficiently high k such that $(\beta_*)^k(w) \in C_q(X,\mathcal{U})$. Let $s_*^k : C_*(X) \to C_{*+1}(X)$ be the chain homotopy between $(\beta_*)^k$ and id. Then, we have

$$d(w - (\beta_*)^k w) = d(ds_q^k w + s_{q-1}^k dw)$$

Hence,

$$z = d((\beta_*)^k w + s_{q-1}^k(z))$$

and $(\beta_*)^k w + s_{q-1}^k(z) \in C_*(X,\mathcal{U})$ since both β_* and s_* are natural, they preserve $C_*(X,\mathcal{U})$. This proves injectivity, and completes the proof. □

Remark 5.23. *In the statement of Theorem 5.21, we can replace the open cover $\mathcal{U}$ by any collection of subspaces $\{A_i\}_{i\in I}$ of X such that $\{Int(A_i)\}_{i\in I}$ is an open cover. In other words, the argument that we have given proves that the inclusions*

$$C_*(X, \{Int(A_i)\}_{i\in I}) \to C_*(X, \{A_i\}_{i\in I}) \to C_*(X)$$

are quasi-isomorphisms. This mild generalization is useful to keep in mind in order to avoid unnecessary point-set topological elaborations.

Remark 5.24. *Note that Hatcher proves the finer statement that $\iota : C_*(X,\mathcal{U}) \to C_*(X)$ is a chain homotopy equivalence, but quasi-isomorphism is sufficient for our purposes. In fact, it can be shown abstractly that any quasi-isomorphism between* free complexes *(note that singular chain complex is indeed a freely generated by singular chains) is a chain homotopy equivalence.*

Corollary 5.25. *(Mayer-Vietoris) If $X = U \cup V$, $\mathcal{U} = \{U, V\}$, then there exists a natural short exact sequence of chain complexes:*

$$0 \to C_*(U \cap V) \to C_*(U) \oplus C_*(V) \to C_*(X,\mathcal{U}) \to 0$$

where the two nontrivial maps are $\sigma \to (\sigma, \sigma)$ and $(\sigma, \tau) \to \sigma - \tau$ respectively. Hence, we have a LES of R-modules:

$$\cdots \to H_i(U \cap V) \to H_i(U) \oplus H_i(V) \to H_i(X) \to H_{i-1}(U \cap V) \to \cdots$$

□

The naturality means that if $X = U \cup V$ and $Y = U' \cup V'$ and $f : X \to Y$ is map with the property that $f(U) \subset U'$ and $f(V) \subset V'$ then there is an induced map of Mayer-Vietoris sequences, where all the squares commute.

Mayer-Vietoris sequence can be viewed as analogous to van-Kampen theorem, since if $U \cap V$ is connected, one can deduce that $H_1(X) = (H_1(U) \oplus H_1(V))/\delta_*(H_1(U \cap V))$ where δ_* is the map induced by the diagonal embedding $\sigma \to (\sigma, \sigma)$. This is the abelianized version of van-Kampen.

Corollary 5.26. *(Excision) If $Z \subset A \subset X$ such that $cl(Z) \subset int(A)$, then $H_*(X, A) \cong H_*(X - Z, A - Z)$.*

Proof. Intuitively, we wish to "excise" Z. If all singular simplices which are not completely within A were disjoint from Z, then we could discard simplices touching Z and still be able to compute $H_*(X, A)$. The point is that $X - A$ and Z are separated from each other. So, the problem arises from "big simplices" but we have developed the tool of barycentric subdivision, which allows us to replace big simplices with small simplices.

More formally, we argue as follows: Let $U = X - Z$, $V = A$, and consider the cover $\mathcal{U} = \{U, V\}$ (cf. Remark 5.23) . Note that $X - Z = U$ and $A - Z = U \cap V$. We have $X = Int(U) \cup Int(V)$ since $cl(Z) \subset int(A)$, hence we have a quasi-isomorphism:

$$C_*(X,\mathcal{U}) \to C_*(X)$$

Now, $C_*(A)$ is a subcomplex of both sides and the quasi-isomorphism $\beta_* : C_*(X) \to C_*(X)$ of Lemma 5.22 induced by barycentric subdivision preserves $C_*(A)$, hence tracing through the proof one can easily conclude that we get an induced quasi-isomorphism:

$$C_*(X,\mathcal{U})/C_*(A) \to C_*(X)/C_*(A)$$

We now elaborate on the details of this:

Given a homology class in $H_q(X, A)$ represent it with a cycle $z = \sum a_i\sigma_i \in C_q(X,\mathcal{U})$. This means that any σ_i that is not entirely contained in $X\backslash Z$ must map into A, hence would be a chain in $C_*(A)$ so we can drop them when we regard z as a relative cycle in $C_*(X, A)$. This proves that

$$H_*(X - Z, A - Z) \to H_*(X, A)$$

is surjective.

To see injectivity, let z be a relative cycle in $C_q(X - Z, A - Z)$ such that $[z] = 0 \in H_q(X, A)$. Then, we have

$$z = a + dw \in C_*(X)$$

where $a \in C_q(A)$ and $w \in C_{q+1}(X)$. Now, take k such that $(\beta_*)^k(w) = w_1 + w_2 \in C_*(X,\mathcal{U})$ where the image of w_1 is contained in $X - Z$ and the image of w_2 is contained in A. Therefore, we have:

$$(\beta_*)^k z - dw_1 = (\beta_*)^k a + dw_2$$

The left hand side is contained in $X - Z$ and the right hand side is contained in A, therefore, both sides are contained in $A - Z$. Hence, we conclude that, the cycle $(\beta_*)^k z$ is trivial in $C_*(X - Z, A - Z)$. This concludes the proof of injectivity. □

5.7 Degree theory and applications:

We can now pause to give some applications of the theory that we have developed so far.

One immediate application of the computation of homology groups of the spheres is that:

Theorem 5.27. *If $n \neq m$, $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$.*

Proof. If $\mathbb{R}^n$ and $\mathbb{R}^m$ were homeomorphic, their one-point compactifications S^n and S^m would also be homeomorphic but this cannot be the case since $H_n(S^n) = \mathbb{Z}$ and $H_n(S^m) = 0$. □

Let $f : S^n \to S^n$ be a continuous map. Then we have an induced map

$$f_* : H_n(S^n) = \mathbb{Z} \to \mathbb{Z} = H_n(S^n)$$

This is a homomorphism of abelian groups, hence it has to a multiplication by some integer.

Definition 5.28. *For $f : S^n \to S^n$, we define $deg(f) \in \mathbb{Z}$ to be the integer such that*

$$\begin{aligned} f_* : H_n(S^n) &\to H_n(S^n) \\ 1 &\to deg(f) \cdot 1 \end{aligned}$$

Note that $deg(f \circ g) = deg(f) \cdot deg(g)$. Furthermore, if $f : S^k \to S^k$ is not surjective, then we can write it as a composition $S^k \to \mathbb{R}^k \to S^k$, and hence conclude that the $deg(f) = 0$.

Let us study some examples. Viewing $S^n \subset \mathbb{R}^{n+1}$ as the set of vectors of length 1, we can see that the orthogonal group $G = O(n+1)$ acts on S^n. Any element of G consists of composition of some rotations and some reflections. The rotations are homotopic to identity. (One connects a rotation by an angle θ to identity by a homotopy of rotations that rotates $t\theta$.). The composition of two reflections is a rotation. The continuous $det : G \to \{1, -1\}$ takes value -1 on a reflection. Hence, G has two components distinguished by the determinant function.

Lemma 5.29. *Let $g \in O(n+1)$, then $deg(g) = det(g) = \pm 1$.*

Proof. Since rotations are homotopic to identity by homotopy invariance, they have degree 1. Therefore, all we need to show is that a reflection has degree -1.Let $r : S^n \to S^n$ be the reflection given by $(x_0, \ldots, x_n) \to (-x_0, x_1, \ldots x_n)$. Write $S^n = U \cup V$ where U, V are homeomorphic to balls D^n which are preserved by the reflection such that $U \cap V$ deformation retracts onto an equatorial sphere S^{n-1} on which r acts by reflection. For example, take $U = \{x_n > -\epsilon\}$ and $V = \{x_n < \epsilon\}$ for some small $\epsilon > 0$. Applying the Mayer-Vietoris sequence and using the naturality with respect to r, for $n > 1$, we get:

$$\begin{tikzcd} H_n(S^n) \arrow[r, "\cong"] \arrow[d, "r_*"'] & H_{n-1}(S^{n-1}) \arrow[d, "r_*"] \\ H_n(S^n) \arrow[r, "\cong"'] & H_{n-1}(S^{n-1}) \end{tikzcd}$$

Therefore, we reduced the calculation of degree to the case of $n = 1$. But, for $n = 1$, by the isomorphism $\pi_1(S^1) = H_1(S^1)$, we know that a loop that goes around once in counter-clockwise can be taken to be a generator. Applying a reflection to that loop, one gets the negative of it. Hence $deg(f) = -1$. $\square$

Corollary 5.30. *The antipodal map $a : S^n \to S^n$ given by $a(x) = -x$ has degree $(-1)^{n+1}$*

Proof. The determinant of a is $(-1)^{n+1}$.

This has a very nice application to vector fields on S^n. For our purpose, we will define a vector field on S^n to be a continuous map $v : S^n \to \mathbb{R}^{n+1}$ such that for any $p \in S^n \subset \mathbb{R}^{n+1}$, the point p and $v(p)$ are perpendicular in $\mathbb{R}^{n+1}$.

Theorem 5.31. *(Hairy ball theorem) S^n has a nowhere vanishing vector field if and only if n is odd.*

Proof. For $n = 2m + 1$ odd, we just exhibit a nowhere vanishing vector field as:

$$v(x_0, x_1, \ldots, x_{2m+1}) = (-x_1, x_0, -x_3, x_2, \ldots, -x_{2m+1}, x_{2m})$$

Now, in general, suppose $v(p) \neq 0$ for all $p \in S^n$. Then, consider the normalized vector

$$w(p) = v(p)/|v(p)|$$

as a map from $S^n \to S^n$. Using this, we can construct a homotopy $F : S^n \times I \to S^n$ given by:

$$F(p, t) = p \cos(\pi t) + w(p) \sin(\pi t)$$

This is a homotopy between $F(p, 0) = p$ and $F(p, 1) = -p$. Thus, the identity map is homotopic to the antipodal map. But the antipodal map has degree $(-1)^{n+1}$ and identity map has degree 1. So, this can only happen if n is odd. $\square$.

Remark 5.32. *In the case n is odd, one can ask how many linearly independent non-vanishing vector fields are there on S^n. This is a harder problem and was solved by Adams in 1962 using K-theory (that you might see on a second course on algebraic topology.) The answer is that the number of such vector fields is the same as the number of linear vector fields. The latter can be computed via a computation in linear algebra (cf. Clifford algebras).*

Local degree:

Intuitively, for a map $f : S^n \to S^n$, $deg(f)$ measures how many times the domain wraps around the target. At least superficially, this seems related to the number of preimages of a generic point in the target. On the other hand, we can observe from $n = 1$ case that there is a subtle point. Namely, wrapping can be positive or negative. Correspondingly, we should count the number of preimages of a generic point with signs. The idea of local degree makes this intuition precise.

Definition 5.33. *Let $f : S^n \to S^n$ be a map and $U \subset S^n$ be an open set (in the domain). Let $q \in S^n$ be a point such that $f^{-1}(q) \cap U = \{p_1, \ldots, p_k\}$ is a finite set of points in the domain. Consider the composite map:*

$$H_n S^n \to H_n(S^n, S^n - f_{|U}^{-1}(q)) \cong H_n(U, U - f_{|U}^{-1}(q)) \xrightarrow{f_*} H_n(S^n, S^n - \{q\}) \cong H_n S^n$$

where all the unlabelled maps are the obvious ones - they either come from excision or the exact sequences of pairs.

The composite map has the form $1 \to deg_q(f_{|U}) \cdot 1$, for some integer $deg_q(f_{|U})$, which is called the local degree. If U contains a unique preimage p_i, then we write, the local degree as $deg_{p_i}(f)$. This is the local degree of f at p_i.

The justify the name "local degree", we have the following proposition:

Proposition 5.34. *If $f_{|U}^{-1}(q) \subset K \subset V \subset U$, where K is a compact and V is a neighborhood of K, then the local degree $deg_q(f_{|U})$ is also given by the composite:*

$$H_n(S^n) \to H_n(S^n, S^n - K) \cong H_n(V, V - K) \xrightarrow{f_*} H_n(S^n, S^n - \{q\}) \cong H_n(S^n)$$

In other words, we can replace $f_{|U}^{-1}(q)$ by any larger compact set K inside U and we can shrink U to any neighborhood of $f_{|U}^{-1}(q)$.

Proof. The proof follows immediately from the commutativity of the following diagram:

$$\begin{tikzcd}
H_n(S^n) \arrow[r] & H_n(S^n, S^n - f_{|U}^{-1}(q)) \arrow[r, "\cong"] & H_n(U, U - f_{|U}^{-1}(q)) \arrow[r, "f_*"] & H_n(S^n, S^n - \{q\}) \cong H_n(S^n) \\
H_n(S^n) \arrow[r] \arrow[u, "="] & H_n(S^n, S^n - K) \arrow[r, "\cong"] \arrow[u] & H_n(V, V - K) \arrow[r, "f_*"] \arrow[u] & H_n(S^n, S^n - \{q\}) \cong H_n(S^n) \arrow[u, "="']
\end{tikzcd}$$

In particular, taking $K = S^n = V$, we conclude that $deg_q(f_{|S^n}) = deg(f)$.

The next proposition shows that local degrees add up to global degree. Hence, local degree computations gives a way to compute the degree of a map.

Proposition 5.35. *Let $f : S^n \to S^n$ and $q \in S^n$ be such that $f^{-1}(q) = \{p_1, \ldots, p_k\}$ is a finite set of points. Choose open sets U_i for $i = 1, \ldots k$, such that $p_j \in U_i$ if and only if $j = i$. Then,*

$$deg(f) = \sum_{i=1}^{k} deg_q(f_{|U_i})$$

Proof. By the previous proposition, we may suppose that by shrinking U_i, if necessary, we may suppose that they are disjoint. Put $f_i = f_{|U_i}$ and $U = \bigsqcup_i U_i$. The proof follows from the commutativity (which is easy to check) of the following diagram:

$$\begin{tikzcd}
H_n(S^n) \arrow[r] \arrow[d, "\{id\}"'] & H_n(S^n, S^n - f^{-1}(q)) \arrow[r, "\cong"] \arrow[d, "\{(\iota_i)_*\}"'] & H_n(U, U - f^{-1}(q)) \arrow[r, "f_*"] & H_n(S^n, S^n - \{q\}) \cong H_n(S^n) \\
\bigoplus H_n(S^n) \arrow[r] & \bigoplus_i H_n(S^n, f_i^{-1}(q)) \arrow[r, "\cong"] & \bigoplus_i H_n(U_i, U_i - f_i^{-1}(q)) \arrow[r, "f_*"] \arrow[u, "\{(\iota_i')_*\}"] & \bigoplus H_n(S^n, S^n - \{q\}) \cong \bigoplus H_n(S^n) \arrow[u, "\{id\}"']
\end{tikzcd}$$

where $\{id\}$ is a map all of whose components are identity, ι_i, ι_i' are the obvious inclusions. □

Remark 5.36. *Assuming a little knowledge of differential topology, the degree of a map $f : S^n \to S^n$ can be computed succinctly as follows: Let $q \in S^n$ be a regular value of f (which exists by Sard's theorem), and $f^{-1}(q) = \{p_1, \ldots, p_k\}$. Then the local degree at p_i is the sign of the Jacobian, $det(df_{p_i})$. Hence,*

$$deg(f) = \sum_i sign(det(df_{p_i}))$$

5.8 Cellular homology

Cellular homology provides an alternative construction of a homology theory for CW-complexes. As we shall see, cellular homology is much more amenable to direct computation, and it agrees with singular homology.

Let $X = \bigcup_n X^n$ be CW-complex. Recall that X^n is obtained from X^{n-1} by attaching n-cells D_i^n for $i \in I_n$ via the attaching maps $j_i^n : \partial D_i^n \to X^{n-1}$. Since X^{n-1} is a CW-subcomplex of X^n by a previous homework problem, we know that:

$$H_i(X^n, X^{n-1}) = H_i(X^n/X^{n-1}) \text{ for } i > 0$$

Now, we exploit the fact that X^n/X^{n-1} is a really simple space. Namely,

$$X^n/X^{n-1} = \bigvee_{i \in I_n} S_i^n$$

Hence, we have proved part i) of the following:
Lemma 5.37. *i) $H_n(X^n, X^{n-1})$ is isomorphic to the free R-module generated by n-cells of X, and $H_i(X^n, X^{n-1}) = 0$ for $i \neq n$.*
ii) $H_i(X^n) = 0$ for $i > n$ and the inclusion of the subcomplex $X^n \to X$ induces an isomorphism $H_i(X^n) \to H_i(X)$ for all $i < n$.

Proof. To see part ii). Consider the long exact sequence of the pair (X^n, X^{n-1}):

$$\cdots \to H_{i+1}(X^n, X^{n-1}) \to H_i(X^{n-1}) \to H_i(X^n) \to H_i(X^n, X^{n-1}) \to \cdots$$

So, for $i > n$, we have an isomorphism $H_i(X^n) \cong H_i(X^{n-1})$. Inductively, we deduce that $H_i(X^n) \cong H_i(X^0) = 0$.

On the other hand if $i < n$, we get $H_i(X^n) \cong H_i(X^{n+1}) \cong H_i(X^{n+2}) \cong \ldots \cong H_i(X^N)$ for any $N > n$, and taking $i = n$, we see that $H_n(X^n) \to H_n(X^N)$ is surjective. In particular, this implies that $H_i(X^N, X^n) = 0$ for $N > n$ and $i \leq n$ by another application of long exact sequence associated with the pair (X^N, X^n).

Now, to show that $H_i(X^n) \to H_i(X)$ is an isomorphism for $i < n$, from the long exact sequence of the pair (X, X^n), it suffices to show $H_i(X, X_n) = 0$ for $i \leq n$. Since $X = \bigcup_n X^n$, for any cycle z, representing a class $[z] \in H_i(X, X^n)$, there exists a large $N > n$ such that $[z] \in \mathrm{im}[H_i(X^N, X^n) \to H_i(X, X^n)]$ but $H_i(X^N, X^n) = 0$ as we have seen above for any $N > n$. □

Now, from the long exact sequence of the pair (X^{n+1}, X^n) we have the boundary homomorphism:

$$\partial_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})$$

On the other hand, we have the natural quotient map:

$$j_{n-1} : H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$$

The composite gives a map:

$$d_n = j_{n-1} \circ \partial_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$$

Theorem 5.38. *Let $C_n = H_n(X^n, X^{n-1}) = R^{\{n-cells\}}$ and $d_n : C_n \to C_{n-1}$ be the composite $j_{n-1} \circ \partial_n$. Then, (C_*, d_*) is a chain complex and its homology is canonically isomorphic to singular homology $H_*(X)$.*

Proof. $d_{n-1} \circ d_n = j_{n-2} \circ \partial_{n-1} \circ j_{n-1} \circ \partial_n = 0$ because the middle piece $\partial_{n-1} \circ j_{n-1} = 0$ as these are consecutive maps in the long exact sequence of the pair (X^{n-1}, X^{n-2}).

From the previous lemma and long exact sequence of the pairs, we have an injection

$$0 \to H_n(X^n) \xrightarrow{j} H_n(X^n, X^{n-1})$$

and

$$H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial} H_n(X^n) \to H_n(X^{n+1}) \cong H_n(X) \to 0$$

So, composing with the injection j one has:

$$H_n(X) = \frac{\mathrm{im}[j : H_n(X^n) \to H_n(X^n, X^{n-1})]}{\mathrm{im}[j \circ \partial : (H_n(X^{n+1}, X^n) \to H_n(X^n, X^{n-1})]}$$

On the other hand, by long exact sequence of the pair (X^n, X^{n-1}), we have:

$$\mathrm{im}[j : H_n(X^n) \to H_n(X^n, X^{n-1}) = \ker[\partial : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})]$$

Finally, since $H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ is injective, we obtain that

$$H_n(X) = \frac{\ker[j \circ \partial : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-1})]}{\mathrm{im}[j \circ \partial : (H_n(X^{n+1}, X^n) \to H_n(X^n, X^{n-1})]}$$

as required. $\square$

The next proposition gives a geometric way of computing the boundary map on cellular chain complex via degree theory. We have the cellular boundary map

$$d_n : R^{\{n-cells\}} \to R^{\{(n-1)-cells\}}$$

We can describe this with integer valued matrix coefficients a_{ij} as follows:

$$d_n[\alpha] = \sum_{\beta} a_{\alpha\beta}[\beta]$$

where $[\alpha]$ is an $n - cell$ and the index β runs through $(n-1)$-cells.

On the other hand, given an n-cell D^n_α. We have its attaching map $j_\alpha : S^{n-1} \cong \partial D^n \to X^{n-1}$, and given a $n-1$ cell D^{n-1}_β, we have a projection map $\pi_\beta : X^{n-1}/X^{n-2} \to S^{n-1}$. We can form the composite:

$$c_{\alpha\beta} : S^{n-1} \xrightarrow{j_\alpha} X^{n-1} \to X^{n-1}/X^{n-2} \xrightarrow{\pi_\beta} S^{n-1}$$

The following is very useful in doing computations of cellular homology :

Proposition 5.39. *$a_{\alpha\beta} = deg(c_{\alpha\beta})$.*

Proof. We first need to identify an explicit chain representative of the group $H_n(D^n, S^{n-1}) \cong H_{n-1}(S^{n-1}) = \mathbb{Z}$ (for $n > 0$).

Exercise: After identifying the pair (D^n, S^{n-1}) with $(\Delta_n, \partial\Delta_{n-1})$ via a homeomorphism, the identity map $\iota_n : \Delta_n \to \Delta_n$ gives a cycle $C_n(\Delta_n, \partial\Delta_n)$. Show that this is a generator of $H_n(\Delta_n, \partial\Delta_n) = \mathbb{Z}$.

Now, it follows that $\partial\iota \in C_{n-1}(S^{n-1})$ gives an explicit chain representative of the generator of $H_{n-1}(S^{n-1})$ via our calculation of homology of spheres using the long exact sequence of the pair (D^n, S^{n-1}) and excision.

It follows that the α^{th} summand of $H_n(X^n, X^{n-1})$ is represented by the characteristic map $f^n_\alpha : (\Delta^n, \partial\Delta_n) \to (X^n, X^{n-1})$. Hence, $\partial_n f^n_\alpha$ is represented by the attaching map $(j^n_\alpha)_*[S^{n-1}]$ where $[S^{n-1}] \in H_{n-1}(S^{n-1}) = H_{n-1}(\partial\Delta_n)$ is a notation for a generator.

So, if e^n_α is the corresponding cell, we have $d_n(e^n_\alpha) = (q_{n-1})_* \circ (j^n_\alpha)_*[S^{n-1}] \in H_{n-1}(X^{n-1}, X^{n-2})$, where $q_{n-1} : X^{n-1} \to X^{n-1}/X^{n-2}$ is the collapsing map.

On the other hand, the projection from $H_{n-1}(X^{n-1}, X^{n-2}) \to \mathbb{Z}$ corresponding to an $(n-1)$-cell β is induced by the map $\pi_\beta : X^{n-1}/X^{n-2} \to S^{n-1}$.

Therefore, we get the equality for the matrix element $a_{\alpha\beta}$. $a_{\alpha\beta}[S^{n-1}] = (\pi_\beta)_* \circ (q_{n-1})_* \circ (j^n_\alpha)_*[S^{n-1}]$ but this is by definition the degree of the map $c_{\alpha\beta} = \pi_\beta \circ q_{n-1} \circ j^n_\alpha$. $\square$

5.9 Interlude

I will be away on Monday, Oct. 20th. Ben Antieau kindly accepted to take over my duty for the day. I expect that the lecture will consist of a selection of classical applications of singular homology based on Chapter 2.B of Hatcher.

5.10 Euler Characteristic

Let X be a finite CW complex. Then define the *Euler characteristic* to be :

$$\chi(X) = \sum_n (-1)^n c_n$$

where c_n is the number of n-cells. It is by no means obvious that $\chi(X)$ does not depend on the cell decomposition of X. We see this as follows:

Proposition 5.40. *$\chi(X) = \sum_n (-1)^n$ rank $H_n(X)$.*

Proof. This will follow from a purely algebraic statement. Suppose

$$0 \to C_k \xrightarrow{d_k} C_{k-1} \xrightarrow{d_{k-1}} \dots \xrightarrow{d_1} C_0 \to 0$$

is a chain complex consisting of *finitely generated* abelian groups, and let us write H_n for the corresponding homology groups. Then, we prove that

$$\sum_n (-1)^n \text{ rank } C_n = \sum_n (-1)^n \text{ rank } H_n$$

Write $Z_n = \ker(d_n)$ and $B_n = \mathrm{im}(d_{n+1})$. We have the equalities:

$$\text{rank } H_n = \text{rank } Z_n - \text{rank } B_n$$

$$\text{rank } C_n = \text{rank } Z_n + \text{rank } B_{n-1}$$

The desired equality follows from these by multiplying with $(-1)^n$ and summing over n. $\square$

Example: $\chi(\Sigma_g) = 2 - 2g$, $\chi(\mathbb{C}P^n) = n + 1$.

The following generalizes the Euler characteristic:

Definition 5.41. *Suppose X is a topological space with such that $H_k(X) \neq 0$ for finitely many k and is of finite rank (for ex. a finite CW complex). Then define the Lefschetz number*

$$L(f) = \sum_n (-1)^n Tr(f_* : H_n(X) \to H_n(X))$$

where $Tr(f_* : H_n(X) \to H_n(X))$ is by definition the trace of the linear map

$$f_* \otimes \mathbb{Q} : H_n(X) \otimes \mathbb{Q} \to H_n(X) \otimes \mathbb{Q}$$

Note that $L(id) = \chi(X)$. The Lefschetz-Hopf fixed point theorem states that for X an ENR (Euclidean Neighborhood Retract, for ex., every finite CW complex), if $L(f) \neq 0$, then f has a fixed point.

5.11 Eilenberg-Steenrod axioms

The singular homology theory that we have constructed so far gives a functor:

$$h\mathscr{T}^2 \to h\mathscr{C}h$$

, sending a pair (X, A) to $C_*(X, A)$. We can then take the homology an get abelian groups $\{H_n(X, A)\}_{n\in\mathbb{N}}$.

For simplicity, let us restrict our discussion to CW-pairs (X, A), where X is a CW-complex and A is a CW-subcomplex.

Definition 5.42. *(Eilenberg-Steenrod) A homology theory is an sequence of functors h_n, for $n \in \mathbb{Z}$, from the homotopy category of CW-pairs (X, A) to the category of R-modules together with natural transformations $\partial : h_n(X, A) \to h_{n-1}(A, \emptyset) =: h_{n-1}(A)$ that satisfy the following axioms:*

- *DIMENSION : $h_0(pt.) = R$ and $h_n(pt.) = 0$ for $n \neq 0$.*
- *EXACTNESS : There is a long exact sequence of a pair (X, A) induced by inclusions $i : A \to X$ and $p : (X, \emptyset) \to (X, A)$:*

$$\cdots \xrightarrow{i_*} h_n(A) \to h_n(X) \xrightarrow{p_*} h_n(X, A) \xrightarrow{\partial} h_{n-1}(A)$$

- *EXCISION : If $X = A \cup B$ where A and B are CW subcomplexes, then the inclusion $(A, A \cap B) \to (X, B)$ induces an isomorphism:*

$$h_*(A, A \cap B) \to h_*(X, B)$$

- *ADDITIVITY: If (X, A) is a disjoint union of a set of pairs (X_i, A_i) then the inclusions $(X_i, A_i) \to (X, A)$ induce an isomorphism:*

$$\bigoplus_i h_*(X_i, A_i) \to h_*(X, A)$$

We have seen that singular homology groups $H_*(X, A)$ satisfy all the axioms above, hence they define a homology theory. In fact, all homology theories are isomorphic to singular homology:

Theorem 5.43. *For any homology theory h_n on CW-pairs, there exists a unique natural transformation $h_* \to H_*$, inducing an isomorphisms $h_n(X, A) \cong H_n(X, A)$ for all n, extending a given isomorphism from $h_*(pt.) \to H_*(pt.)$.*

Proof. (Sketch) Given a homology theory, we can use the axioms to construct a cellular homology theory which gives isomorphic R-modules. Namely, we set:

$$C_* = h_n(X^n, X^{n-1})$$

and define the differential $d_n : C_n \to C_{n-1}$ as before via $d_n = j_{n-1} \circ \partial_n$, where $\partial_n : h_n(X^n, X^{n-1}) \to h_{n-1}(X^{n-1})$ and $j_{n-1} : h_{n-1}(X^{n-1}) \to h_{n-1}(X^{n-1}, X^{n-2})$ are obtained from the axioms. Furthermore, we can compute using the excision axiom that $h_n(X^n, X^{n-1}) = R^{n-cells}$.

The proof of the isomorphism between H_n and the cellular homology groups used only consequences of the axioms, therefore we can apply it to any homology theory h_n.

Now, we have noted that the cellular chain complex for H_n and h_n have the same generators, it remains to prove that they have the same differential as well. Recall that the differential can be computed via a degree calculation. So, we need the following:

If $f : S^n \to S^n$ is a map, then the $deg(f)$ defined via the induced map $f_* : H_n(S^n) \to H_n(S^n)$ coincides with the $deg(f)$ defined via the induced map $f_* : h_n(S^n) \to h_n(S^n)$

This is easy for $n = 0$ and since we know that any map $S^1 \to S^1$ is homotopic to a map $e^{i\theta} \to e^{ik\theta}$ for some $k \in \mathbb{Z}$, it is also easy to check this for $n = 1$. For higher n one uses induction using the suspension operation. Recall that the suspension is the operation:

$$SX = (X \times I)/(X \times \{0\} \cup X \times \{1\})$$

If $f : X \to Y$, then there is an induced map $Sf : SX \to SY$. Now, $S(S^n) = S^{n+1}$ (suggestively!). Via a Mayer-Vietoris argument (as a consequence of the axioms) as in Lemma 5.29 , one shows that :

$$\begin{tikzcd}
h_n(S^n) \arrow[r, "\cong"] \arrow[d, "Sf_*"'] & h_{n-1}(S^{n-1}) \arrow[d, "f_*"] \\
h_n(S^n) \arrow[r, "\cong"'] & h_{n-1}(S^{n-1})
\end{tikzcd}$$

This shows that the degree of $g : S^n \to S^n$ can be computed using h_n or H_n for maps g of the form $g = Sf$. The fact that, for $n \geq 2$, any map $g : S^n \to S^n$ is homotopic to a suspension of $f : S^{n-1} \to S^{n-1}$ requires a bit more homotopy theory (cf. Freudenthal suspension, Hurewicz theorem) and this is where our sketch falls short of a proof.

Remark 5.44. *A generalized (or extraordinary) homology theory is one without the DIMENSION axiom. It turns out that there is a zoo of generalized homology theories. Bordism, K-theory,.... You've got a lot to learn!*

6 Cohomology

6.1 Cochain complexes

Recall that we have seen that singular chain complex gives a functor

$$h\mathcal{T} \to h\mathcal{C}h$$

It was a consequence of our construction that the singular chain complex is supported in non-negative degrees. That is, given a topological space X, we set $C_i(X) = 0$ for $i < 0$.

A general complex need not have this special property. In the other extreme, if the chain complex is supported in nonpositive degrees, that is, if $C_i = 0$ for $i > 0$, via an identification

$$C_{-i} = C^i$$

we usually write this as a *cochain complex* which is supported in nonnegative degrees. Hence, a general cochain complex looks like:

$$\cdots \to C^i \xrightarrow{d^i} C^{i+1} \xrightarrow{d^{i+1}} C^{i+2} \to \cdots$$

such that $d^{i+1} \circ d^i = 0$. Note, in particular, the change in the direction of arrows. To distinguish between chain complexes and cochain complexes one usually uses subscripts and superscripts respectively. But, really, a cochain complex is just a chain complex supported in nonpositive degrees, in particular, it is an object of the category $h\mathcal{C}h$.

6.2 Two functors from $h\mathcal{C}h \to h\mathcal{C}h$

Given an R-module M, we have two natural functors $h\mathcal{C}h \to h\mathcal{C}h$ given by prolongations of two natural functors mod-$R \to$ mod-R :

$$C_i \to C_i \otimes_R M$$
$$C^i \to hom_R(C_i, M)$$

(Caution : We carefully defined C^* by $C^i = hom_R(C_i, M)$ for each i. The alternative, definition $C^* = hom_R(C_*, M)$ is usually bigger, cf. the difference between direct sum and direct product of R-modules)

In principle, composing the singular chain complex with these functors and then taking the homology of the resulting chain complex may lead to new topological invariants.

In the first case, the differential is defined by

$$\sigma \otimes_R m \to d(\sigma) \otimes m, \text{for } \sigma \in C_* \text{ and } m \in M$$

In the second case, the differential will be denoted by δ and is defined via

$$\delta(f)(\sigma) = f(d\sigma) \text{for } \sigma \in C_* \text{ and } f \in hom_R(C_*, M)$$

In this second case, $\delta : C^i \to C^{i+1}$, hence we get a cochain complex.

Definition 6.1. *Let X be a topological space, M be an R-module, then we define*

$$C_*(X; M) = C_*(X) \otimes_R M$$

to be the singular chain complex with coefficients in M*, and we define*

$$C^*(X; M) = hom_R(C_*(X), M)$$

to be the singular cochain complex with coefficients in M*.*

As usual, one drops M from the notation, if $R = \mathbb{Z}$ and $M = \mathbb{Z}$. The homology of the cochain complex $C^*(X; M)$ is called the cohomology of X (with coefficients in M), and is denoted by $H^*(X; M)$.

All the axiomatic properties of homology (long exact sequences of the pair, homotopy invariance, excision, Mayer-Vietoris, etc.) have dual versions in cohomology, with essentially identical proofs. Note that the connecting maps in long exact sequences go up not down in degree.

It turns out that both $H_*(X; M)$ and $H^*(X; M)$ are determined by the homology groups $H_*(X)$ (at least for reasonable rings R, such as a PID). Of course, the dual statement is also true, that is, $H^*(X; M)$ and $H_*(X; M)$ are determined by $H^*(X)$.

Statements of this form are called "universal coefficients theorems". To state these precisely, we need to have a digression in homological algebra.

Digression : Tor and Ext

In this digression and in the statement of universal coefficients theorem, for simplicity, we will restrict to the case $R = \mathbb{Z}$ and M an abelian group. The statements can be generalized quite a bit. At first instance, one can replace R with any hereditary ring. These rings have the property that every submodule of a free R-module is free, which is the what we use. For example, commutative hereditary rings are the same as PIDs.

Before going on deeper in homological algebra, it is useful to illustrate the difficulty that we are facing via the following example. Consider the chain complex:

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

The resulting homology groups are : $H_0 = \mathbb{Z}, H_1 = \mathbb{Z}_2, H_2 = 0, H_3 = \mathbb{Z}$. If we dualize this using $hom(\cdot, \mathbb{Z})$, we get the cochain complex:

$$0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

The resulting cohomology groups are: $H^0 = \mathbb{Z}, H^1 = 0, H^2 = \mathbb{Z}_2, H^3 = \mathbb{Z}$. In particular, we observe that it is NOT true that $H^* = hom(H_*, \mathbb{Z})$.

As we shall see, the main trouble comes from the fact that dualization is not an exact functor, that is, it does not preserve short exact sequences. Similar problem occurs, with the tensor product functor $\cdot \otimes M$. To address this issue, we will need to study derived functors of $\otimes$ and *hom* functors. These are called Tor and Ext respectively.

Given an abelian group H, a free resolution of H is an exact sequence:

$$\cdots \to F_2 \to F_1 \to F_0 \to H \to 0$$

such that F_i are free abelian groups.

The idea of a free resolution is simply to replace a possibly complicated object H of the category of abelian groups with a sequence of simple (free) objects, namely the sequence, $\cdots \to F_2 \to F_1 \to F_0$. As it turns out, one does not need more than just 2 simple objects.

Lemma 6.2. *Every abelian group has a "short" free resolution:*

$$0 \to F_1 \to F_0 \to H \to 0$$

To see this, choose a set of generators of H, and let F_0 be the free abelian group with basis in one-to-one correspondence with these generators. Thus, we get a map $f_0 : F_0 \to H$ sending the basis of F_0 these generators. Now, let $F_1 = Ker(f_0)$, since this is a submodule of free abelian group, it is free. Defining $f_1 : F_1 \to F_0$ to be the inclusion map, we get a free resolution.

One can form a homotopy category of resolutions, whose objects are chain complexes

$$\cdots F_2 \to F_1 \to F_0 \to H \to 0$$

which are exact, where F_i are free abelian groups and H is an arbitrary abelian group, and whose morphisms are chain homotopy classes of chain maps.

The idea of replacing H with a free resolution can then be made precise by proving that the functor

$$\{\cdots F_2 \to F_1 \to F_0 \to H \to 0\} \to H$$

is an equivalence of categories.

Proving this equivalence is not hard. It is really an exercise in repeatedly using the fact that F_i are free. In particular, we have the following, which we leave it as an exercise. (See Hatcher Lemma 3.1 for a proof).

Lemma 6.3. *Given two free resolutions $F_* \to H$ and $F'_* \to H$. There is a chain map $\alpha : F_* \to F'_*$ extending the identity map $H \to H$. Moreover, α is unique up to chain homotopy.*

Diagramatically, this looks like:

$$\begin{tikzcd}
\cdots \arrow[r, "f_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "id"] & 0 \\
\cdots \arrow[r, "f'_2"'] & F'_1 \arrow[r, "f'_1"] & F'_0 \arrow[r, "f'_0"] & H \arrow[r] & 0
\end{tikzcd}$$

Now, let A and B be abelian group, take a free resolution of A of the form:

$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} A \to 0$$

We then can apply $\otimes B$ to this exact sequence and get a chain complex:

$$0 \to F_1 \otimes B \xrightarrow{f_1 \otimes id} F_0 \otimes B \xrightarrow{f_0 \otimes id} A \otimes B \to 0$$

Lemma 6.4. *This complex may not be exact only at the left (i.e. tensor product is a right exact functor).*

proof Surjectivity of $f_0 \otimes id$ is obvious from the surjectivity of f_0. To see exactness in the middle. Let $I = Im(f_1 \otimes id)$. We have an induced map $(F_0 \otimes B)/I \to A \otimes B$. It suffices to show that this is an isomorphism. To define an inverse, given $a \otimes b$ let $c \in F_0$ such that $f_0(c) = a$. Define the inverse $s : A \otimes B \to (F_0 \otimes B)/I$ by $s(a \otimes b) = c \otimes b$. By using the exactness of the free resolution, it is straightforward to check that this is well-defined and an isomorphism. $\square$

In view of the above lemma, we define the homology there to be $Tor(A, B) = \mathrm{Ker}(f_1 \otimes id)$.

Similarly, we can apply $hom(\cdot, B)$ to the free resolution and get a chain complex:

$$0 \to hom(A, B) \xrightarrow{f_0^*} hom(F_0, B) \xrightarrow{f_1^*} hom(F_1, B) \to 0$$

An easy exercise shows that this complex is not exact only at the right (i.e. $hom(\cdot, B)$ is a left exact functor). So, we define $Ext(A, B) = \mathrm{Coker}(f_1^*)$.

Theorem 6.5. *$Tor(A, B)$ and $Ext(A, B)$ are independent of the free resolutions.*

Proof Let us just give the proof for $Tor(A, B)$; the proof goes through along the same lines in the dual case.

Let $0 \to F_1 \to F_0 \to A \to 0$ and $0 \to F_1' \to F_0' \to A \to 0$ be two different resolutions of A. Because of the extension lemma, we have $\alpha_* : F_* \to F_*'$ and $\beta_* : F_*' \to F_*$ extending the identity maps of A. Furthermore, by uniqueness, we have that $\alpha_* \circ \beta_*$ and $\beta_* \circ \alpha_*$ are chain homotopic to the identity maps.

Applying, $\otimes B$ to get, we get induced maps $\alpha_* \otimes id : F \otimes B \to F' \otimes B$ and $\beta_* \otimes id : F' \otimes B \to F \otimes B$ which compose to maps chain homotopic to identity. $\square$

Example computations and properties: Let A, B be abelian groups.

- $Tor(A, B) = Tor(B, A)$
- $Tor(A, B) = 0$ if A or B is free.
- $Tor(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_{gcd(m,n)}$
- $Tor(\oplus A_i, B) = \oplus_i (Tor(A_i, B))$

- $Ext(A, B) = 0$ if A is free.
- $Ext(\mathbb{Z}_n, B) = B/nB$

All of these are straightforward exercises. Let's do at least one.

Observe that $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}_n \to 0$ is a free resolution of $\mathbb{Z}_n$. Applying $hom(\cdot, B)$ to this resolution gives:

$$0 \leftarrow hom(\mathbb{Z}, B) \xleftarrow{\times n} hom(\mathbb{Z}, B) \leftarrow hom(\mathbb{Z}_n, B) \leftarrow 0$$

So, $Ext(\mathbb{Z}_n, B) = Coker(B \xleftarrow{\times n} B) = B/nB$.

Theorem 6.6. *(Universal coefficients for homology) Let (C_*, d_*) be a chain complex of free abelian group and M an abelian group, then there are natural short exact sequences:*

$$0 \to H_n(C_*) \otimes M \to H_n(C_*; M) \to Tor(H_{n-1}(C_*), M) \to 0$$

for all n. Furthermore, there is a non-canonical splitting.

Proof: Let us write, as usual, $Z_n = \ker(d_n)$ and $B_n = \mathrm{im}(d_{n+1})$. Let us also write $\iota_n : B_n \to Z_n$ for the inclusion map.

Firstly, we can make Z_* and B_* subcomplexes of C_* by the restriction of the differential d. Note that these restrictions are just the zero map. Now, observe that because of the commutativity of the following diagram, we get a short exact sequence of *chain complexes*.

$$\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "d_n"] & C_n \arrow[r, "d_n"] \arrow[d, "d_n"] & B_{n-1} \arrow[r] \arrow[d, "d_{n-1}"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] & C_{n-1} \arrow[r, "d_{n-1}"] & B_{n-2} \arrow[r] & 0
\end{tikzcd}$$

Now, since $0 \to Z_n \to C_n \to B_{n-1} \to 0$ is a short exact sequence of *free* abelian groups, tensoring it with M will still be exact. Indeed, tensoring with M we get a chain complex:

$$0 \to Z_n \otimes M \to C_n \otimes M \to B_{n-1} \otimes M \to 0$$

which is chain homotopy equivalent to the complex

$$0 \to B_{n-1} \otimes M \to B_{n-1} \otimes M \to 0$$

Therefore, by tensoring with M, we get a short exact sequence of chain complexes:

$$0 \to Z_* \otimes M \to C_* \otimes M \to B_{*-1} \otimes M \to 0$$

which is exact.

The associated long exact sequence looks like:

$$\cdots \to B_n \otimes M \to Z_n \otimes M \to H_n(C_*; M) \to B_{n-1} \otimes M \xrightarrow{\partial} Z_{n-1} \otimes M \to \cdots$$

The boundary homomorphism $\partial : B_{n-1} \otimes M \to Z_{n-1} \otimes M$ can be identified to be the inclusion $\iota_{n-1} \otimes id$ by tracing through its definition given by the Snake lemma.

So, these long exact sequences, can be broken up into short exact sequences:

$$0 \to Coker(\iota_n \otimes id) \to H_n(C_*; M) \to Ker(\iota_{n-1} \otimes id) \to 0$$

Next, let us observe that:

$$0 \to B_n \xrightarrow{\iota_n} Z_n \to H_n(C_*) \to 0$$

is a short exact sequence, hence a free resolution of $H_n(C_*)$ as Z_n and B_n are free abelian groups. Tensoring with M gives a chain complex:

$$0 \to B_n \otimes M \xrightarrow{\iota_n \otimes id} Z_n \otimes M \to H_n(C_*) \otimes M \to 0$$

Hence, we conclude that $Tor(H_n(C), M) = Ker(\iota_n \otimes id)$. Furthermore, by exactness in the middle, we get $H_n(C_* \otimes M) = Coker(\iota_n \otimes id)$.

With these identifications, we have the short exact sequence:

$$0 \to H_n(C_* \otimes M) \to H_n(C_*; M) \to Tor(H_{n-1}(C_*), M) \to 0$$

as required. It is left as an exercise to see how a splitting arises from choosing splitting of the short exact sequences $0 \to Z_n \to C_n \to B_{n-1} \to 0$. (I will assign a homework problem which shows that these splittings cannot be natural). $\square$

Theorem 6.7. *(Universal coefficients for cohomology) Let (C_*, d_*) be a chain complex of free abelian group and M an abelian group, then there are natural short exact sequences:*

$$0 \to Ext(H_{n-1}(C_*), M) \to H^n(C_*; M) \to Hom(H_n(C_*), M) \to 0$$

for all n. Furthermore, there is a non-canonical splitting.

As a particular case, when $R = M = \mathbb{Z}$ we get short exact sequences:

$$0 \to H_{n-1}(C_*)_{tors} \to H^n(C_*) \to Hom(H_n(C_*), \mathbb{Z}) \to 0$$

Proof The proof of this is formally similar to the one for homology. Consider the same exact sequence of chain complexes given by $0 \to Z_* \to C_* \to B_{*-1} \to 0$. Again, since these are free abelian groups, applying the *hom* functor will preserve exactness, hence we get a short exact sequence of abelian groups:

$$0 \to hom(B_{*-1}, M) \to hom(C_*, M) \to hom(Z_*, M) \to 0$$

The associated long exact sequence of abelian groups is given as:

$$\cdots \to hom(Z_{n-1}, M) \xrightarrow{\iota_{n-1}^*} hom(B_{n-1}, M) \to H^n(C_*; M) \to hom(Z_n, M) \xrightarrow{\iota_n^*} hom(B_n, M) \to \dots$$

where the boundary homomorphism is induced by the inclusion $\iota_n : B_n \to Z_n$ as before. Hence, we have the short exact sequences:

$$0 \to Coker(\iota_{n-1}^*) \to H^n(C_*; M) \to Ker(\iota_n^*) \to 0$$

Next, we have the free resolutions:

$$0 \to B_n \xrightarrow{\iota_n} Z_n \to H_n(C_*) \to 0$$

Applying, $hom(\cdot, M)$ to this gives a chain complex:

$$0 \to hom(H_n(C_*), M) \to hom(Z_n, M) \xrightarrow{\iota_n} hom(B_n, M) \to 0$$

which is not exact only at the right end where the homology is by definition the group $Ext(H_n(C_*), M)$. Hence, we conclude that:

$$Ker(\iota_n^*) = hom(H_n(C_*), M)$$

and

$$Coker(\iota_n^*) = Ext(H_n(C_*), M)$$

With these identifications, we have the desired result. Again, we leave as an exercise to see that splitting of $0 \to Z_n \to C_n \to B_{n-1} \to 0$ induce non-canonical splittings.

Remark 6.8. *Note that $H^n(C_*; M) \to Hom(H_n(C_*), M)$ is just the map induced from the Kronecker product:*

$$H^n(C_*; M) \otimes H_n(C_*) \to M$$

given by $\langle [f], [\sigma] \rangle \to f(\sigma)$.

Remark 6.9. *Now, you know how to compute $H^*(X; M)$ and $H_*(X; M)$ of any (finite) CW-complex! Use cellular homology to compute homology, then use universal coefficients. Of course, this is not the most practical way in general but I think this is a moment to pause and appreciate the power (and perhaps, the purpose) of homological algebra.*

7 Product Structures

There are many products in (co-)homology of topological spaces. The most important product is the "cup product" and one could begin by introducing that first using an artificial looking formula (as in Hatcher). On the other hand, the product structures naturally appear when one studies the problem of determining $H_*(X \times Y)$ from $H_*(X)$ and $H_*(Y)$ (and the dual problem in cohomology). Therefore, we study this problem first. (We follow mostly Chapter VI of Bredon's Topology and Geometry).

7.1 The Cross product and the Künneth Theorem

Given topological spaces X and Y, we would like to relate $C_*(X)$, $C_*(Y)$ and $C_*(X \times Y)$. The basic intuition coming from cellular homology is that the product of a p-cell in X and q-cell in Y is a $p+q$-cell in $X \times Y$.

Definition 7.1. *Given chain complexes (C_*, d) and (D_*, d) over R, define their tensor product to be the sequence of R-modules*

$$(C_* \otimes_R D_*)_n = \bigoplus_{i+j=n} C_i \otimes_R D_j$$

with the differential $d \otimes_R 1 + 1 \otimes_R d$ acting by:

$$(d \otimes 1 + 1 \otimes d)(\sigma \otimes \tau) = d\sigma \otimes \tau + (-1)^{|\sigma|}\sigma \otimes d\tau$$

When it is clear, we will drop the R from $\otimes_R$.

Koszul signs: In homological algebra, one introduces a certain sign convention called the Koszul signs. This is a convention. It amounts to modifying signs in various formulas. You can come up with your own convention if you like, but the Koszul convention has been widely adopted and most people seem to believe that overall it makes the formulas appear more "logical".

The first time this modifies the naive sign conventions is in the definition of maps between tensor products. Namely, if $f : C_* \to C'_*$ and $g : D_* \to D'_*$ are chain maps then one defines $f \otimes g : C_* \otimes D_* \to C'_* \otimes D'_*$ by

$$(f \otimes g)(\sigma \otimes \tau) = (-1)^{|g||\sigma|} f(\sigma) \otimes g(\tau)$$

A useful rule for memorizing signs is that whenever two objects a,b are permuted to which degrees $|a|$ and $|b|$ are attached, then a sign $(-1)^{|a||b|}$ should be introduced.

In the above definition of the differential on the tensor product complex, we have adopted this convention, which explain the sign that appears in the formula for the differential.

The following theorem is what we are aiming for:
Theorem 7.2. *(Eilenberg-Zilber Theorem) There are natural chain maps:*

$$\times : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$$

and

$$\theta : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$$

that are homotopy equivalences and are naturally homotopy inverses of one another. One has that in degree 0, $x \times y = (x, y)$ and $\theta(x, y) = x \otimes y$.

We would like to define a bilinear chain maps:

$$\times : C_*(X) \times C_*(Y) \to C_*(X \times Y)$$

Bilinearity then induces the maps $C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ which we again denote by $\times$ and call cross-product. It is already the obvious map in degree 0, and we would like to extend to higher degrees. We have seen that the most natural approach to this kind of constructions is the "method of acyclic models".

Proposition 7.3. *There exists natural bilinear maps*

$$\times : C_*(X) \times C_*(Y) \to C_*(X \times Y)$$

such that $x \times y = (x, y)$ for $x \in C_0(X)$ and $y \in C_0(Y)$, and one has:

$$d(\sigma \times \tau) = d\sigma \times \tau + (-1)^{|\sigma|}\sigma \times d\tau$$

Naturality means that if $f : X \to X'$ and $g : Y \to Y'$ are continuous maps, then

$$(f, g)_*(\sigma \times \tau) = f_*(\sigma) \times g_*(\tau)$$

Proof. By naturalily, it suffices to define $\iota_p \times \iota_q$ if $\iota_p \in C_*(\Delta_p)$ and $\iota_q \in C_*(\Delta_q)$ are the singular chains corresponding to the identity maps of Δ_p and Δ_q. For general chains $\sigma : \Delta_p \to X$ and $\tau : \Delta_q \to X$, we will then define:

$$(\sigma \times \tau) = (\sigma, \tau)_*(\iota_p \times \iota_q)$$

and extend it to all chains using bilinearity.

It remains to define $\iota_p \times \iota_q$. If $p = 0$, then we can define $\iota_0 \times \iota_q = \iota_q$. Similarly, if $q = 0$, we can set $\iota_p \times \iota_0 = \iota_p$. So, let us assume $p > 0$ and $q > 0$. Then, we are required to satisfy the relation:

$$d(\iota_p \times \iota_q) = d\iota_p \times \iota_q + (-1)^{|p|}\iota_p \times d\iota_q \in C_{p+q-1}(\Delta_p \times \Delta_q)$$

By induction, assume that the right hand side is defined. Then, using the fact that $\Delta_p \times \Delta_q$ is contractible and $p + q - 1 > 0$, to find a chain for $\iota_p \times \iota_q$, all we need is to show that the right-hand-side is a cycle. We can use the induction hypothesis to compute:

$$d(d\iota_p \times \iota_q) + (-1)^p d(\iota_q \times d\iota_p) = (-1)^{p-1} d\iota_p \times d\iota_q + (-1)^p d\iota_q \times d\iota_p = 0$$

So, picking any chain whose boundary is this cycle defines a chain for $\iota_p \times \iota_q$.

□

To define the inverse we need the fact that the complex $C_*(\Delta_p) \otimes C_*(\Delta_q)$ has no homology in nonzero grading. (This is one of your homework problems.)

Proposition 7.4. *There exists a natural chain map*

$$\theta : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$$

such that $\theta(x \otimes y) = (x, y)$ for $(x, y) \in C_0(X \times Y)$.

Proof Suppose θ is defined in degree less than p and such that $d\theta = \theta d$ in those degrees. For $p = 1$, we have already defined θ in the obvious way.

Consider the case $X = \Delta_p$ and $Y = \Delta_p$. Let $d_p : \Delta_p \to \Delta_p \times \Delta_p$ be the diagonal map, which we view as a chain in $C_p(\Delta_p \times \Delta_p)$. To define $\theta(d_p)$, let us consider the equation:

$$d(\theta(d_p)) = \theta(d(d_p))$$

By acyclicity of $C_p(\Delta_p \times \Delta_p)$, all we need to check is that the right hand side is a cycle. We do this using induction hypothesis:

$$d(\theta(d(d_p))) = \theta(dd(d_p)) = 0$$

Having defined $\theta(d_p)$, let us see that the rest is determined by naturality. Indeed, let $\sigma : \Delta_p \to X \times Y$ is a p-simplex, and $\pi_X(\sigma)$ and $\pi_Y(\sigma)$ be the projections to the components. Then, we have the equality:

$$\sigma = ((\pi_X \sigma) \times (\pi_Y \sigma)) \circ d_p$$

Hence, naturality forces us to define:

$$\theta(\sigma) = ((\pi_X \sigma) \otimes (\pi_Y \sigma))_*(\theta(d_p))$$

It is then routine to check that this definition gives a chain map, which we omit. □

Proof of Theorem 7.2 We have constructed chain maps

$$\times : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$$

and

$$\theta : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$$

such that their composition in either direction is the identity map in degree 0. One is now in a position to repeat the argument given in Lemma 5.22 to construct natural chain homotopies. □

Recall that we obtained the universal coefficients theorem for homology by studying the functor given by $\cdot \otimes M$ for an R-modules M. There is a generalization of this, known as the Künneth formula, where one instead tensors with a chain complex.

Theorem 7.5. *(Algebraic Künneth theorem) If C_* and D_* are chain complexes of free abelian groups, then there is a natural (split) short exact sequence:*

$$0 \to \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(D_*) \xrightarrow{\times} H_n(C_* \otimes D_*) \to \bigoplus_{i+j=n-1} Tor(H_i(C_*), H_j(D_*) \to 0$$

Proof As before, consider the short exact sequence

$$0 \to Z_n \xrightarrow{j_n} C_n \xrightarrow{d_n} B_{n-1} \to 0$$

and tensor this with D_* to get a short exact sequence of chain complexes:

$$0 \to Z_* \otimes D_* \xrightarrow{j_* \otimes id} C_* \otimes D_* \xrightarrow{d_* \otimes id} B_{*-1} \otimes D_* \to 0$$

Get a long exact sequence that has the form:

$$\to \bigoplus_{i+j=n} B_i \otimes H_j(D_*) \to \bigoplus_{i+j=n} Z_i \otimes H_j(D_*) \to H_n(C_* \otimes D_*) \to \bigoplus_{i+j=n-1} B_i \otimes H_j(D_*) \to \bigoplus_{i+j=n-1} Z_i \otimes H_j(D_*) \to$$

,where the boundary homomorpshims can be identified with $\oplus_{i+j=n}(\iota_i \otimes 1)$ where $\iota_i : B_i \to Z_i$ is the inclusion. Therefore, one has the short exact sequence:

$$0 \to \bigoplus_{i+j=n} Coker(\iota_i \otimes 1) \to H_n(C_* \otimes D_*) \to \bigoplus_{i+j=n-1} Ker(\iota_i \otimes 1) \to 0$$

Finally, using the short exact sequence:

$$0 \to \bigoplus_{i+j=n} B_i \otimes H_j(D_*) \to \bigoplus_{i+j=n} Z_i \otimes H_j(D_*) \to \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(D_*) \to 0$$

one obtains:

$$\bigoplus_{i+j=n} Coker(\iota_i \otimes 1) = \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(D_*)$$

and

$$\bigoplus_{i+j=n-1} Ker(\iota_i \otimes 1) = \bigoplus_{i+j=n-1} Tor(H_i(C_*), H_j(D_*))$$

. This gives the desired result. $\square$

An immediate corollary to this is the Goemetric Künneth theorem:

Theorem 7.6. *Let X and Y be topological spaces, there is a natural exact sequence:*

$$0 \to \bigoplus_{i+j} = nH_i(X) \otimes H_j(Y) \to H_n(X \times Y) \to \bigoplus_{i+j=n-1} Tor(H_i(X), H_j(Y)) \to 0$$

As an example, let us compute:

$$\begin{aligned} H_2(\mathbb{R}P^2 \times \mathbb{R}P^2) &= (H_0(\mathbb{R}P^2) \otimes H_2(\mathbb{R}P^2)) \oplus (H_1(\mathbb{R}P^2) \otimes H_1(\mathbb{R}P^2)) \oplus (H_2(\mathbb{R}P^2) \otimes H_0(\mathbb{R}P^2)) \\ &\oplus Tor(H_0(\mathbb{R}P^2), H_1(\mathbb{R}P^2)) \oplus Tor(H_1(\mathbb{R}P^2), H_0(\mathbb{R}P^2) \\ &= (\mathbb{Z} \otimes 0) \oplus (\mathbb{Z}_2 \otimes \mathbb{Z}_2) \oplus (0 \otimes \mathbb{Z}) \oplus Tor(\mathbb{Z}, \mathbb{Z}_2) \oplus Tor(\mathbb{Z}_2, \mathbb{Z}) \\ &= \mathbb{Z}_2 \otimes \mathbb{Z}_2 \end{aligned}$$

Next, we define the cross product on cohomology. Recall that we have a chain equivalence:

$$\theta : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$$

Definition 7.7. *Working over the ring R. Let $f \in C^*(X) = hom(C_*(X), R)$ and $g \in C^*(Y) = hom(C_*(Y), R)$, then we have $f \otimes g : C_*(X) \otimes C_*(Y) \to R$ defined via the composition:*

$$C_*(X) \otimes C_*(Y) \to R \otimes R \to R$$

where the map on the right is the multiplication in the ring R. We define:

$$f \times g = (f \otimes g) \circ \theta$$

Koszul signs: Previously, we have defined differential on cohomology via the formula, $\delta(f) = f \circ d$. The Koszul convention dictates that we modify this via the signs:

$$\delta(f) = (-1)^{|f|+1} f \circ d$$

With that in mind, the following properties of the cross product on cohomology are routine exercises in keeping track of the signs:

- $\delta(f \times g) = (\delta f \times g) + (-1)^{|f|} f \times \delta g$
- $(f \times g)(\sigma \times \tau) = (-1)^{|g||\sigma|} f(\sigma) f(\tau)$

In particular, from the first identity it follows that $\times$ induces a product:

$$H^p(X; R) \otimes H^q(Y; R) \to H^{p+q}(X \times Y; R)$$

We next study the commutativity of the cross product.

Proposition 7.8. *Let $T : X \times Y \to Y \times X$ be the map given by $T(x, y) = (y, x)$. For $\alpha \in H^p(X)$ and $\beta \in H^q(Y)$ we have :*

$$\alpha \times \beta = (-1)^{pq} T^*(\beta \times \alpha)$$

Proof. Consider the following non-commutative diagram

$$\begin{tikzcd}
C_*(X \times Y) \arrow[r, "\theta_{X,Y}"] \arrow[d, "T"'] & C_*(X) \otimes C_*(Y) \arrow[d, "\tau"] \\
C_*(Y \times X) \arrow[r, "\theta_{Y,X}"'] & C_*(Y) \otimes C_*(X)
\end{tikzcd}$$

where τ is the chain map given by $\tau(b \otimes a) = (-1)^{|a||b|} a \otimes b$. Even though, the diagram does not commute, it commutes up to homotopy. Meaning that, $\tau \circ \theta_{Y,X} \circ T \cong \theta_{X,Y}$. In other words, we have a chain homotopy $s_n : C_n(X \times Y) \to \bigoplus_{i+j=n+1}(C_i(X) \otimes C_j(Y))$ satisfying :

$$\tau \circ \theta_{Y,X} \circ T - \theta_{X,Y} = s_n d + d s_{n-1}$$

We then compute:

$$\begin{aligned} T^*([g] \times [f]) = T^*[g \times f] = T^*[(g \otimes f) \circ \theta_{Y,X}] = [(g \otimes f) \circ \theta_{Y,X} \circ T] = (-1)^{|f||g|}[(f \otimes g) \circ \tau \circ \theta_{Y,X} \circ T] \\ = (-1)^{|f||g|}[(f \otimes g) \circ \theta_{X,Y}] = (-1)^{|f||g|}[f \times g] = (-1)^{|f||g|}[f] \times [g] \end{aligned}$$

$\square$

7.2 Cup product

The most important product is the "cup product". In the case, $X = Y$, cross-product on cohomology provides us with a map:

$$\times : C^p(X) \otimes C^p(Y) \to C^{p+q}(X \times X)$$

Note that the coefficients should be taken on any commutative ring R. (That is, we don't have $\times$ on cohomology with arbitrary module coefficients.)

Now, there is a natural canonical map called the diagonal map $\mathscr{D} : X \to X \times X$ given by $\mathscr{D}(x) = (x, x)$. This gives us a map on cochains:

$$\mathscr{D}^* : C^*(X \times X) \to C^*(X)$$

Composing the two maps, we get a "product" on the cochain complex of a single topological space, which is well-defined up to chain homotopy, hence is well-defined on homology.
Definition 7.9. *The homomorphism*

$$\cup : H^p(X) \otimes H^q(X) \to H^{p+q}(X)$$

defined by $\alpha \cup \beta = \mathscr{D}^*(\alpha \times \beta)$ is called the cup product. (The coefficients are taken to be on any commutative ring R with unit).

The following properties of the cup product are deduced easily from the corresponding properties of the cross product that we have established:

- The cup product equips $H^*(X)$ with a graded ring structure.
- The ring has a unit element $1 \in H^0(X)$. It is the class of the cocahin $\epsilon : C_0(X) \to R$ taking each 0-simplex to 1.
- The product is associative.
- The product is commutative in the graded sense:

$$\alpha \cup \beta = (-1)^{|\alpha||\beta|} \beta \cup \alpha$$

- The product is natural: If $f : X \to Y$ is map then:

$$f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta$$

We do not give detailed proofs of these properties. In effect, the only non-trivial property is the graded commutativity and that can be proved after the graded commutativity of $\times$-product which we proved.

As a result of the last property, we observe that, the cohomology as a ring is an invariant of homotopy type.

At the chain level, we can unwind the definition of the cup product as:

$$(f \cup g)(\sigma) = (f \otimes g) \circ \theta \circ \mathscr{D}(\sigma)$$

By the naturality of this formula, if $A \subset X$ and f is a cochain in $C^*(X)$ such that it vanishes on any singular chain contained entirely in A, then so does $f \cup g$.

Therefore, if f vanishes on simplices contained in A and g vanishes on simplices contained in B, then $f \cup g$ vanishes on simplices contained in A and in B but not generally in $A \cup B$.

On the other hand if A and B are such that $Int(A) \cup Int(B) = A \cup B$, then we had an isomorphism via subdivision to the effect that:

$$C_*(A) + C_*(B) \to C_*(A \cup B)$$

is an isomorphism. Therefore, the complex

$$\{f \in C^*(X) : f(\sigma) = 0 \text{ if either } im(\sigma) \subset A \text{ or } im(\sigma) \subset B\}$$

can be used to compute the group $H^*(X, A \cup B)$. Therefore, in this case, we have a well-defined cup product map on relative groups:

$$H^p(X, A) \otimes H^q(X, B) \to H^{p+q}(X, A \cup B)$$

This holds in particular if both A and B are open, or if one of them is empty.

Remark 7.10. *The cross product and cup product satisfy the following natural formula: If $\alpha_1, \alpha_2 \in H^*(X)$ and $\beta_1, \beta_2 \in H^*(Y)$, then :*

$$(\alpha_1 \times \beta_1) \cup (\alpha_2 \times \beta_2) = (-1)^{|\alpha_2||\beta_1|}(\alpha_1 \cup \alpha_2) \times (\beta_1 \cup \beta_2)$$

We omit the proof of this, though it is a straightforward computation. This means that the cross-product map : $H^*(X; R) \otimes H^*(Y; R) \xrightarrow{\times} H^*(X \times Y)$ is a ring homomorphism.

From this, it is possible to recover the cross-product from cup product. Namely, let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be projections, then for $\alpha \in H^p(X)$ and $\beta \in H^q(Y)$, one has:

$$p_X^*(\alpha) \cup p_Y^*(\beta) = (\alpha \times 1) \cup (1 \times \beta) = (\alpha \cup 1) \times (1 \cup \beta) = \alpha \times \beta \in H^{p+q}(X \times Y)$$

An example: Computation of the ring $(H^*(\mathbb{R}P^2), \cup)$.

We begin with some $\cup$-product computations in the Euclidean space. For $k \leq n$, consider $\mathbb{R}^k$ as a subspace of $\mathbb{R}^n$, namely:

$$\mathbb{R}^k = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n | x_i = 0 \text{ for } i > k\}$$

and dually put

$$\widehat{\mathbb{R}}^{n-k} = \{x \in \mathbb{R}^n | x_i = 0 \text{ for } i \leq k\}$$

Clearly, we have:

$$(\mathbb{R}^n - \mathbb{R}^k) \cup (\mathbb{R}^n - \widehat{\mathbb{R}}^{n-k}) = \mathbb{R}^n - \{0\}$$

Lemma 7.11. *The cup product map:*

$$\cup : H^k(\mathbb{R}^n, \mathbb{R}^n - \widehat{\mathbb{R}}^{n-k}) \otimes H^{n-k}(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^k) \to H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$$

is an isomorphism.

Proof. Consider the following diagram:

$$\begin{tikzcd}
H^k(\mathbb{R}^k, \mathbb{R}^k - \{0\}) \otimes H^{n-k}(\mathbb{R}^{n-k}, \mathbb{R}^{n-k} - \{0\}) \arrow[r, "\times"] \arrow[d, "p^* \times q^*"] & H^n(\mathbb{R}^k \times \mathbb{R}^{n-k}, \mathbb{R}^k \times \mathbb{R}^{n-k} - \{0\}) \arrow[d, "="] \\
H^k((\mathbb{R}^k, \mathbb{R}^k - \{0\}) \times \mathbb{R}^{n-k}) \otimes H^{n-k}(\mathbb{R}^k \times (\mathbb{R}^{n-k}, \mathbb{R}^{n-k} - \{0\})) \arrow[r, "\cup"] & H^n(\mathbb{R}^k \times \mathbb{R}^{n-k}, \mathbb{R}^k \times \mathbb{R}^{n-k} - \{0\})
\end{tikzcd}$$

The two rows can be identified via the formula $p^*\alpha \cup q^*\beta = \alpha \times \beta$, where $p : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ and $q : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ are projections.

Since the first map is an isomorphism by a relative version of Künneth theorem for cohomology, we get that the second map is also an isomorphism.

□

We now give a computation of cup-product for projective spaces $\mathbb{R}P^n$ over $R = \mathbb{Z}_2$. We assume that its known to the reader that $H^k(\mathbb{R}P^2; \mathbb{Z}_2) = \mathbb{Z}_2$ for $0 \leq k \leq n$ and zero otherwise. (You have already computed $H_*(\mathbb{R}P^2; \mathbb{Z})$ additively as a homework problem, you can deduce from that the cohomology with coefficients in $R = \mathbb{Z}_2$ using universal coefficients, or you can also do the computation from scratch using cellular cochain complex.)

Here our focus is on understanding the ring structure.

Theorem 7.12. *There exists an isomorphism of graded rings $H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ where the grading is given by $|\alpha| = 1$*

For fixed $k \leq n$ recall that the k-skeleton of $\mathbb{R}P^n$ in its usual cell decomposition can be identified with

$$X^k = \mathbb{R}P^k = \{[x_0; x_1; \ldots; x_n] : x_j = 0 \text{ for all } j > k\}$$

Dually, we have another cell decomposition such that its $(n-k)$-skeleton is given by:

$$Y^{n-k} = \mathbb{R}P^{n-k} = \{[x_0; x_1; \ldots; x_n] : x_j = 0 \text{ for all } j < k\}$$

We have that $\mathbb{R}P^n \backslash Y^{n-k} \cong X^{k-1} = \mathbb{R}P^{k-1}$ via the deformation retraction :

$$x \to [x_0 : x_1 : \ldots x_{k-1} : tx_k : \ldots : tx_n], 0 \leq t \leq 1$$

Furthermore, let us identify $\{x \in \mathbb{R}P^n | x_k \neq 0\}$ with $\mathbb{R}^n$ and let $\mathbb{R}^k = \mathbb{R}^n \cap X^k$ and $\mathbb{R}^{n-k} = \mathbb{R}^n \cap Y^{n-k}$.

Consider the following diagram (where the coefficients are always taken to be in $\mathbb{Z}_2$):

$$
\begin{tikzcd}
H^k(\mathbb{R}P^n) \arrow[d, "\rho"] & H^k(\mathbb{R}P^n, \mathbb{R}P^n - Y^{n-k}) \arrow[l] \arrow[r] \arrow[d, "\phi"] & H^k(\mathbb{R}^n, \mathbb{R}^n - \widehat{\mathbb{R}}^{n-k}) \arrow[d, "\rho"] \\
H^k(\mathbb{R}P^k) & H^k(\mathbb{R}P^k, \mathbb{R}P^k - (\mathbb{R}P^k \cap Y^{n-k})) \arrow[l, "\rho"'] \arrow[r, "\rho"] & H^k(\mathbb{R}^k, \mathbb{R}^k - \{0\})
\end{tikzcd}
$$

where all the maps are induced by inclusion.From cellular cochain complex, we can deduce that $H^i(\mathbb{R}P^k) = H^i(\mathbb{R}P^n)$ for $i \leq k$. We know that all maps ρ are isomorphisms. The map ϕ is also an isomorphism because $\mathbb{R}^n - Y^{n-k} \cong \mathbb{R}P^{k-1} \cong \mathbb{R}P^k - (\mathbb{R}P^k \cap Y^{n-k})$. Hence, we conclude that all the maps are isomorphisms.

There is a similar diagram where the role of $X^k = \mathbb{R}P^k$ and $Y^{n-k} \cong \mathbb{R}P^{n-k}$ are interchanged. Now, consider the diagram:

$$
\begin{tikzcd}
H^k(\mathbb{R}P^n) \times H^{n-k}(\mathbb{R}P^n) \arrow[r, "\cup"] & H^n(\mathbb{R}P^n) \\
H^k(\mathbb{R}P^n, \mathbb{R}P^n - Y^{n-k}) \times H^{n-k}(\mathbb{R}P^n, \mathbb{R}P^n - X^k) \arrow[u] \arrow[d] \arrow[r, "\cup"] & H^n(\mathbb{R}P^n, \mathbb{R}P^n - (Y^{n-k} \cap X^k)) \arrow[u] \arrow[d] \\
H^k(\mathbb{R}^n, \mathbb{R}^n - \widehat{\mathbb{R}}^{n-k}) \times H^{n-k}(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^k) \arrow[r, "\cup"] & H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\})
\end{tikzcd}
$$

We have seen that all the vertical maps are isomorphisms, and the last row is an isomorphism. Hence, every row is an isomorphism. From this, we deduce that

$$H^k(\mathbb{R}P^n) \times H^{n-k}(\mathbb{R}P^n) \xrightarrow{\cup} H^n(\mathbb{R}P^n)$$

are isomorphisms. The theorem follows because for $i+j \leq n$, $\iota^* : H^*(\mathbb{R}P^n) \to H^*(\mathbb{R}P^{i+j})$ is an isomorphism of rings up to dimension $(i+j)$. $\square$

Similarly, one can prove that

$$H^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[\beta]/(\beta^{n+1}) \text{ with } |\beta| = 2$$

$$H^*(\mathbb{H}P^n; \mathbb{Z}) = \mathbb{Z}[\gamma]/(\gamma^{n+1}) \text{ with } |\gamma| = 4$$

Alexander-Whitney diagonal approximation:

The composition
$$\Delta = \theta \circ \mathscr{D}_* : C_*(X) \to C_*(X) \otimes C_*(X)$$
is called a diagonal approximation.

Definition 7.13. *A diagonal approximation is a* natural *chain map:*

$$\Delta : C_*(X) \to C_*(X) \otimes C_*(X)$$

such that on 0-chains $\Delta(x) = x \otimes x$.

As we have seen, arguments similar to the one given in 5.22, implies that any two diagonal approximations are chain homotopic.

Note that cup product was defined as $f \cup g = (f \otimes g) \circ \Delta$. Choosing different diagonal approximations will give the same operation on cohomology (but may differ at the chain level!).

A particularly popular diagonal approximation seems to be the Alexander-Whitney diagonal approximation. Defined by the formula:

$$\Delta(\sigma) = \sum_{p+q=n} \sigma|_{[t_0,\ldots,t_p,0,\ldots 0]} \otimes \sigma|_{[0,\ldots,0,t_p,\ldots,t_n]}$$

Using this, one may recover the formula that Hatcher uses to define cup product (up to sign).

Remark 7.14. *Additively, it is easy to show that the reduced cohomology groups* $\tilde{H}^*(X)$ *are "stable" in the sense that there are isomorphisms* $\tilde{H}^p(X) = \tilde{H}^{p+1}(\Sigma X)$*, where* $\Sigma X = X \wedge S^1$ *is the suspension of* X*. On the other hand, cup products are unstable. For* $Y = X \wedge S^1$*, the cup product:*

$$\tilde{H}^p(Y) \otimes \tilde{H}^q(Y) \to \tilde{H}^{p+q}(Y)$$

is the zero homomorphism (see the series of exercises in Chapter 19 of May). This gives a sense of how much more information does the cup product carry than the mere additive groups.

7.3 Cap product

Choose a diagonal approximation $\Delta : C_*(X) \to C_*(X) \otimes C_*(X)$ such as $\theta \circ \mathscr{D}_*$. Define the *cap product* on the chain-cochain level via:

$$\cap : C^p(X) \otimes C_n(X) \to C_{n-p}(X)$$

by

$$\phi \cap \sigma = (1 \otimes \phi)\Delta\sigma$$

Here are properties of the $\cap$-product. We omit the straightforward verification of these.

Firstly, we have for $\phi \in C^p(X)$ and $\sigma \in C_n(X)$,

$$d(\phi \cap \sigma) = \delta\phi \cap \sigma + (-1)^p \phi \cap d\sigma$$

This implies that, we get a bilinear product

$$H^p(X) \otimes H_n(X) \to H_{n-p}(X)$$

As usual, this is well-defined; it is independent of the choice of the chain level map θ.

This bilinear product satisfies the following properties. Let $\epsilon : C_0(X) \to R$ be the augmentation taking 0-simplices to 1, and let $\langle\ ,\ \rangle : H^*(X) \otimes H_*(X) \to R$ be the Kronecker pairing.

- $\epsilon \cap \sigma = \sigma$
- If $\phi \in H^p(X), \psi \in H^q(X), \sigma \in H_{p+q}(X)$ then

$$\langle \phi, \psi \cap \sigma \rangle = \langle \phi \cup \psi, \sigma \rangle$$

- $(\phi \cup \psi) \cap \sigma = \phi \cap (\psi \cap \sigma)$
- If $f : X \to Y$, then $f_*(\sigma) \cap \phi = f_*(\sigma \cap f^*(\phi))$

The first and third listed property means that, $H_*(X)$ is a graded unital $H^*(X)$-module. The fourth property shows that homology, as a graded module over cohomology, is an invariant of homotopy type. All of these properties

One can easily deduce from the properties above that cap product generalizes to relative cap products:

$$\cap : H^p(X, A) \otimes H_n(X, A) \to H_{n-p}(X)$$

and

$$\cap : H^p(X) \otimes H_n(X, A) \to H_{n-p}(X, A)$$

To really appreciate the important role of $\cap$-product, we must now turn to manifolds.

8 Orientations and duality on manifolds

8.1 Local homology and fundamental classes

Here, we follow Chapter 20 of May's book.

Let M be a topological n-manifold. Recall that this is a topological space which is locally homeomorphic to $\mathbb{R}^n$. To avoid pathologies, one often imposes that the topology is Hausdorff and second countable.

Let $x \in M$ be a point. Then $H_*(M, M - \{x\})$ is called the local homology group at x. Using excision, exactness and homotopy invariance, one proves that:

$$H_i(M, M - x) \cong H_i(U, U - x) \cong \tilde{H}_{i-1}(U - x) \cong \tilde{H}_{i-1}(S^{n-1})$$

From this, one deduces:

Lemma 8.1. *If M is an n-manifold, then $H_*(M, M - \{x\}) = R$ at $* = n$ and 0 otherwise.*

Thus $H_n(M, M - x)$ is a free R-module with one generator, but the generator is not specified. An R-orientation of M will be defined to be a consistent choice of orientations as we vary the point x. More precisely,

Definition 8.2. *An R-fundamental class of M at a subspace X is an element $z \in H_n(M, M-X)$ such that for each $x \in X$, the image of z under the map*

$$H_n(M, M - X) \to H_n(M, M - x)$$

induced by inclusion $(M, M - X) \to (M, M - x)$ is a generator. If $X = M$, then we refer to $z \in H_n(M)$ as a fundamental class of M and use the notation $z = [M]$. An R-orientation is an open cover $\{U_i\}$ of M and R-fundamental classes z_i of M at U_i such that if $U_i \cap U_j$ is non-empty, then u_i and u_j are sent to the same element of $H_n(M, M - U_i \cap U_j)$.

We say that M is R-orientable if it admits an R-orientation.

A nice way to re-package this is to construct a covering space :

$$H_R \to M$$

with fibres over x, $H_n(M, M - x; R)$.

If $[M] \in H_n(M)$ is a R-fundamental class, then we can use the restrictions: $H_n(M) \to H_n(M, M - U_i)$ to define R-fundamental classes z_i as the image of $[M]$. Clearly, this gives an R-orientation.

If M is compact, then one has a converse to this statement. To prove this, we will need the following vanishing theorem. The proof of this is somewhat complicated. We will get back to it in the next subsection.

Theorem 8.3. *(Vanishing theorems) Let M be an n-manifold. For any abelian group G, $H_i(M; G) = 0$ if $i > n$ and $\tilde{H}_n(M; G) = 0$ if M is connected and non-compact.*

Let us assume this theorem and prove that R-orientations determine an R-fundamental class.

Theorem 8.4. *Let K be a compact subset of an n-manifold M. Then, for any abelian group G, one has $H_i(M, M - K; G) = 0$ if $i > n$, and an R-orientation of M determines and R-fundamental class at K.*

Proof. First suppose that $K \subset U$ for some coordinate chart $U \cong \mathbb{R}^n$. Then by excision and exactness, we have:

$$H_i(M, M - K; G) = H_i(U, U - K; G) = \tilde{H}_{i-1}(U - K; G)$$

hence for $i > n$ the last group vanishes by the previous vanishing theorem, as it is a non-compact manifold of dimension n. Also notice that by excision and a deformation retraction, one has: $H_n(M, M - U) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$. Hence, an orientation determines a fundamental class for M at U, and hence at K, by the restriction map : $H_n(M, M - U) \to H_n(M, M - K)$.

For a general compact set $K \subset M$. One can divide it into finitely many compact subsets, each of which is contained in a coordinate chart. Therefore, by induction it suffices to prove the theorem for $K \cup L$, assuming it hold for K, L and $K \cap L$.

We apply a relative form of the Mayer-Vieoris to get the following long exact sequence:

$$\cdots \to H_{i+1}(M, M-(K\cap L)) \to H_i(M, M-(K\cup L)) \to H_i(M, M-K)) \oplus H_i(M, M-L) \to \cdots$$

The vanishing of $H_i(M, M-(K\cup L))$ for $i > n$ follows immediately. For $i = n$, we have fundamental classes $z_K \in H_n(M, M-K)$ and $z_L \in H_n(M, M-L)$ determined by the R-orientation. Their difference in $H_n(M, M-(K\cap L))$ is 0, and the map from $H_i(M, M-(K\cup L)) \to H_i(M, M-K)) \oplus H_i(M, M-L)$ is injetive, therefore $z_K + z_L$ comes from a unique class $z_{K\cup L}$ which is an R-fundamental class of M at $K \cup L$. □

Corollary 8.5. *Let M be a connected compact n-manifold, $n > 0$. Then either M is not orientable and $H_n(M;\mathbb{Z}) = 0$ or M is orientable and the map :*

$$H_n(M;\mathbb{Z}) \to H_n(M, M-x;\mathbb{Z}) \cong \mathbb{Z}$$

is an isomorphism for every $x \in M$.

Proof. Since $M - x$ is a connected and non-compact manifold, by the vanishing theorem, we have $H_n(M-x;G) = 0$ for any abelian group G. Therefore, by the long exact sequence of the pair, we have:

$$H_n(M;G) \to H_n(M, M-x;G) = G$$

is injective for all coefficient groups G. Now, universal coefficients theorem for homology implies that:

$$H_n(M;\mathbb{Z}) \otimes \mathbb{Z}_p \to H_n(M, M-x;\mathbb{Z}) \otimes \mathbb{Z}_p \cong \mathbb{Z}_p$$

is injective for all positive integers p. Now, if $H_n(M;\mathbb{Z}) \neq 0$, then by testing the injectivity property for all p, we conclude that $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ and the map to $H_n(M, M-x;\mathbb{Z})$ should be ± 1.

8.2 Vanishing theorems

Theorem 8.6. *Let M be an n-manifold, then for any abelian group G, $H_i(M;G) = 0$ for $i > n$.*

Proof. The case $n = 0$ is trivial. So, let us assume $n > 0$. We shall first prove the following special case:

Lemma 8.7. *Let $U \subset \mathbb{R}^n$ be an open set. Then $H_i(U) = 0$ for $i \geq n$.*

Proof. Let $s = \sum a_k[\sigma_k] \in H_i(U)$ be a homology class $i \geq n$. Since, the domain of σ_k is compact, it follows that, there exists a compact subspace $K \subset U$ such that $\mathrm{im}(\sigma_k) \subset K$ for all k. Therefore, there exist a homology class $h \in H_i(K)$ that maps to s.

By taking a cubical grid with small enough mesh, we can find a CW decomposition of $\mathbb{R}^n$ consisting of small n-cubes in such a way that there is a finite subcomplex L of $\mathbb{R}^n$ with $K \subset$

$L \subset U$. Now, for $i > 0$, the connecting morpshims ∂ are isomorphisms in the commutative diagram.

$$\begin{array}{ccc} H_{i+1}(\mathbb{R}^n, L) & \longrightarrow & H_{i+1}(\mathbb{R}^n, U) \\ \partial \big\downarrow & & \big\downarrow \partial \\ H_i(L) & \longrightarrow & H_i(U) \end{array}$$

Since $(\mathbb{R}^n, L)$ has no relative q-cells for $q > n$, using cellular homology, we see that the groups on the left are zero for $i \geq n$. Since s is the image of the h via the composition $H_i(K) \to H_i(L) \to H_i(U)$, $s = 0$ as required. $\square$

Now, for an n-manifold M, let $s = \sum_k a_k \sigma_k$ be an i-cycle. Let $K = \bigcup \mathrm{im}(\sigma_k)$. Since K is compact, it can be covered by finitely many coordinate charts $V_1, \ldots, V_N$ each homeomorphic to $\mathbb{R}^n$. We apply induction to prove that if s is a cycle contained in the union of N open sets each homeomorphic to $\mathbb{R}^n$, then it is a boundary in the $\bigcup_{j=1}^N V_j$, this then implies the result for M. For $N = 1$, the conclusion follows from the previous lemma. Let $U = V_1 \cup \cdots \cup V_{N-1}$. Mayer-Vietoris gives an exact sequence:

$$H_i(U) \oplus H_i(V_N) \to H_i(U \cup V_N) \to H_{i-1}(U \cap V_N)$$

By induction, the left term vanishes. The previous lemma implies that the right term also vanishes. This implies the desired result. $\square$

Before proving the next result, we prove a lemma that we give a lemma that will be used in the proof.

Lemma 8.8. *Let V be an open in $\mathbb{R}^n$. Suppose that $s \in H_n(\mathbb{R}^n, V)$ maps to zero in $H_n(\mathbb{R}^n, \mathbb{R}^n - x)$ for all $x \in \mathbb{R}^n - V$. Then $s = 0$.*

Proof. We prove the equivalent statement that if $s \in \tilde{H}_{n-1}(V)$ maps to zero in $\tilde{H}_{n-1}(\mathbb{R}^n - x)$ for all $x \in \mathbb{R}^n - V$, then $s = 0$. Let K be a compact set containing the image of a chain representing s, let U be an open set such that

$$K \subset U \subset \overline{U} \subset V$$

Let T be a large open cube such that $\overline{U} \subset T$. It suffices to show that image of s viewed as an element in $\tilde{H}_{n-1}(T \cap V)$ is zero. We know that s maps to zero in $\tilde{H}_{n-1}(T - x)$ for all $x \in T - (T \cap V)$. Now, for any point $x \in T - (T \cap V)$, choose a small cube that contains x an disjoint from U. A finite set $\{D_1, \ldots D_q\}$ of these cubes covers $T - (T \cap V)$. Let $C_i = D_i \cap T$. We claim that s maps to zero in $\tilde{H}_{n-1}(T - \cup_{i=1}^p D_i)$ for all $i \leq q$. This is clear for $i = 0$. Next, observe that for $p > 0$ we have

$$T - \cup_{i=1}^p D_i = (T - \cup_{i=1}^{p-1} D_i) \cap (\mathbb{R}^n - D_p)$$

Mayer-Vietoris gives

$$H_n((T - \cup_{i=1}^{p-1} D_i) \cup (\mathbb{R}^n - D_p)) = 0 \to \tilde{H}_{n-1}(T - \cup_{i=1}^p D_i) \to \tilde{H}_{n-1}(T - \cup_{i=1}^{p-1} D_i) \oplus \tilde{H}_{n-1}(\mathbb{R}^n - D_p)$$

where the group on the left vanishes by the previous lemma. Since s maps to zero in $\tilde{H}_{n-1}(T - \cup_{i=1}^{p-1} D_i) \oplus \tilde{H}_{n-1}(\mathbb{R}^n - \cup_{i=1}^p D_i)$ it must then map to zero in $\tilde{H}_{n-1}(T - \cup_{i=1}^p D_i)$. $\square$

Theorem 8.9. *Let M be a connected non-compact n-manifold, then $\tilde{H}_n(M;G) = 0$ for all abelian groups G.*

Proof. We have already established this for an open subset of $\mathbb{R}^n$. As in the proof of the previous theorem, we can assume that M is a finite union of coordinate charts. As part of the inductive step, suppose U is a coordinate chart (homeomorphic to $\mathbb{R}^n$), and V is an open set such that $H_n(V) = 0$. We also have $H_n(U) = 0$ and $\tilde{H}_{n-1}(U) = 0$. Mayer-Vietoris sequence reads:

$$H_i(U) \oplus H_i(V) \to H_i(U \cup V) \to \tilde{H}_{i-1}(U \cap V) \to \tilde{H}_{i-1}(U) \oplus \tilde{H}_{i-1}(V)$$

It follows that $H_n(U \cup V) = 0$ if and only if the inclusion $i : U \cap V \to V$ induces an injection $i_* : \tilde{H}_{n-1}(U \cap V) \to \tilde{H}_{n-1}(V)$.

Let $r \in \ker(i_*)$. Consider the following diagram, where $y \in U - U \cap V$:

$$\begin{tikzcd}
 & & H_n(U \cup V) \arrow[r] \arrow[dl] & H_n(M) \arrow[d, "0"] \\
H_n(V, U \cap V) \arrow[r] \arrow[d, "\partial"] & H_n(U \cup V, U \cap V) \arrow[rr] \arrow[dl, "\partial"] & & H_n(M, M - y) \\
\tilde{H}_{n-1}(U \cap V) \arrow[d, "i_*"] & H_n(U, U \cap V) \arrow[l, "\partial"] \arrow[u] \arrow[rr] & & H_n(U, U - y) \arrow[u, "\cong"] \\
\tilde{H}_{n-1}(V) & & &
\end{tikzcd}$$

We explain why the vertical map $H_n(M) \to H_n(M, M - y)$ vanishes: Let $\nu \in H_n(M)$, since the image of ν lies on a compact set, we can take a point $x \in M$, which avoids that compact set. Then, it follows that ν goes to zero, under the map, $H_n(M) \to H_n(M, M - x)$. Now, since M is connected choose a path $\gamma : x \to y$. Then we have that $H_n(M, M - x) \cong H_n(M, M - \gamma) \cong H_n(M, M - y)$. On the other hand, we have the factorization $H_n(M) \to H_n(M, M - \gamma) \to H_n(M, M - x)$, therefore, it follows that $H_n(M) \to H_n(M, M - \gamma)$ sends ν to zero, thus $H_n(M) \to H_n(M, M - y)$ sends ν to zero.

Now, since $\tilde{H}_{n-1}(U) = 0$, the bottom map ∂ is surjective and there exists and $s \in H_n(U, U \cap V)$ such that $\partial s = r$. We claim that s maps to zero in $H_n(U, U - y)$ for every $y \in U - (U \cap V)$. By the previous lemma, this will imply that $s = 0$, and thus $r = 0$, so that i_* is indeed an injection. Since $i_*(r) = 0$, there exists a $t \in H_n(V, V \cap U)$ such that $\partial(t) = r$. Let s' and t' be images of s and t in $H_n(U \cup V, U \cap V)$. Then $\partial(s' - t') = 0$, hence there exists $w \in H_n(U \cup V)$ that maps to $s' - t'$. Since w maps to zero in $H_n(M, M - y)$, so does $s' - t'$. Since the map $(V, U \cap V) \to (M, M - y)$ factors through $(M - y, M - y)$, t and thus t' maps to zero in $H_n(M, M - y)$. Therefore, s' maps to zero in $H_n(M, M - y)$ and thus s maps to zero in $H_n(U, U - y)$, as required. $\square$

8.3 Poincaré Duality

The standard version of the Poincaré duality theorem is the following statement:

Theorem 8.10. *Let R be a commutative ring. Let M be a compact, connected and R-oriented manifolds, with R-fundamental class $[M] \in H_n(M; R)$. Then, the duality map:*

$$D : H^p(M; R) \to H_{n-p}(M; R) \quad , \quad c \to c \cap [M]$$

is an isomorphism for all p.

In fact, for any M-module G, one can show that cap product with $[M] \in H_n(M; R)$ defines a duality map $H^p(M; G) \to H_{n-p}(M; G)$, which is an isomorphism. (It is easy to see from the definition that cap product extends to this setting.)

Before embarking on the proof of this theorem, let us discuss some immediate corollaries. In practice, one works with the following corollary:

Corollary 8.11. *Let M be an oriented, compact, connected, n-manifold. Let $T_p \subset H^p(M)$ be the torsion subgroup. The cup product pairing*

$$\alpha \otimes \beta \to \langle \alpha \cup \beta, [M] \rangle$$

induces a non-degenerate pairing:

$$H^p(M)/T_p \otimes H^{n-p}(M)/T_{n-p} \to \mathbb{Z}$$

Proof If $\alpha \in T_p$, then some multiple $m\alpha = 0$, therefore $m(\alpha \cup \beta) = m(\alpha \cup \beta) = 0$. But, since $H^n(M) = \mathbb{Z}$, this means that $\alpha \cup \beta = 0$. Thus the pairing vanishes on torsion elements. On the other hand, recall that for a compact n-manifold, $H_*(M)$ is finitely generated, and we also have $Ext(\mathbb{Z}_r, \mathbb{Z}) = \mathbb{Z}_r$, hence by the universal coefficients theorem, we get:

$$H^p(M)/T_p = Hom(H_p(M), \mathbb{Z})$$

Therefore, for $\alpha \in H^p(M)$ that projects to a generator of the free abelian group $Hom(H_p(M), \mathbb{Z})$, there exists an element $a \in H_p(M)$, such that $\langle \alpha, a \rangle = 1$. Using the Poincaré duality, we can find a $\beta \in H^{n-p}(M)$ such that $\beta \cap [M] = a$. Now, by the properties of cap and cup products, we have:

$$\langle \alpha \cup \beta, [M] \rangle = \langle \alpha, \beta \cap [M] \rangle = 1$$

□

The following can also be proved very similar to above.

Corollary 8.12. *Let R be a field. Let M be an R-oriented, compact, connected, n-manifold. The cup product pairing*

$$\alpha \otimes \beta \to \langle \alpha \cup \beta, [M] \rangle$$

induces a non-degenerate pairing:

$$H^p(M; R) \otimes_R H^{n-p}(M; R) \to \mathbb{R}$$

Note that if R is a field, then universal coefficients theorem gives an isomorphism:

$$H^p(M;R) \cong Hom_R(H_p(M;R), R)$$

We will prove Poincaré duality via the strategy that we have been applying in the proofs of the past few theorems. Namely, first prove it for any coordinate chart, next we prove it for any open set in a coordinate chart, then we prove it for any union of these open sets to conclude. To do this, we need a formulation of Poincaré duality for open sets (as stated, the duality map uses the fact that M is compact so that one has a fundamental class).

For this purpose, we need to study cohomology with compact support.

Definition 8.13. *Let M be an n-manifold, then we define:*

$$H^q_c(M;R) := \operatorname{colim} H^q(M, M-K; R)$$

where the colimit is taken with respect to the homomorphisms $H^q(M, M-K) \to H^q(M, M-L)$ induced by the inclusions $(M, M-L) \subset (M, M-K)$ for $K \subset L$.

By definition of colimit (also known as direct limit), $H^q_c(M;R)$ is disjoint union of $H^q(M, M-K;R)$ over all compact subsets $K \subset M$, with the equivalence relation that two elements $c_K \in H^q(M, M-K;R)$ and $c_L \in H^q(M, M-L;R)$ are declared to be equivalent in $H^q_c(M;R)$, if there exists some compact set C containing $K \cup L$ such that the image of c_K and c_L agrees in $H^q(M, M-C;R)$.

Intuitively, one should think of $H^*_c(M)$ as the cohomology built from singular cochains c for which there is some compact subset K so that c annihilates all chains in $X-K$.

Of course, there are canonical maps : $H^q(M, M-K;R) \to H^q_c(M)$ and these commute with the maps of the direct system induced by the inclusions $K \subset L$. Indeed, $H^q_c(M)$ is universal with respect to this property (similar to the push-out construction that we have seen before).

In practice, one only needs to take the colimit over some compact exhaustion, that is a family of compact set K_i whose union is M. One can easily see that this gives the same colimit by applying the universal property. In particular, for M compact, we have $H^q_c(M) = H^q(M)$.

Remark 8.14. *In general, one should never take limits or colimits of (co)homology groups. The right way of taking (co)limits is to first take the (co)limit of the chain complex and then take homology. However, in the current context, this doesn't matter.*

To give an example, let us compute the groups $H^p_c(\mathbb{R}^n)$. As our compact exhaustion, we can take the family of compact cubes in $\mathbb{R}^n$. For any compact cube K, we have:

$$H^p(\mathbb{R}^n, \mathbb{R}^n - K) \cong H^p(\mathbb{R}^n, \mathbb{R}^n - x) \cong \tilde{H}^{p-1}(S^{n-1}) \cong \tilde{H}^p(S^n)$$

Hence, this immediately implies that $H^p_c(\mathbb{R}^n) = 0$ for $p \neq n$. On the other hand, $H^n(\mathbb{R}^n, \mathbb{R}^n - K) = R$ and if $K \subset L$ are cubes, the induced maps are clearly isomorphisms. Therefore, we have $H^n_c(\mathbb{R}^n) = R$.

We will need the following (semi-)functorial property. Let U be an open subspace of M, then if K is a compact subspace of U, then we get excision isomorphisms:

$$H^q(U, U - K) \to H^q(M, M - K)$$

These commute with the maps in the direct system, therefore we get an induced map as the composite:

$$H^q_c(U) = \mathrm{colim}_{K \subset U} H^q(U, U - K) \cong \mathrm{colim}_{K \subset U} H^q(M, M - K) \to H^q_c(M)$$

Now for an R-oriented, connected n-manifold M and a compact subset $K \subset M$, we have a R-fundamental class $[M_K] \in H_n(M, M - K; R)$. Therefore, we can define a duality map, using the relative cap product:

$$D_K : H^p(M, M - K; R) \to H_{n-p}(M; R) \quad , \quad c \to c \cap [M_K]$$

If $K \subset L$, these commute with the maps in the directed system, i.e., the following diagram commutes:

$$\begin{tikzcd} H^p(M, M-K) \arrow[rr] \arrow[dr, "D_K"'] & & H^p(M, M-L) \arrow[dl, "D_L"] \\ & H_{n-p}(M) & \end{tikzcd}$$

Therefore, passing to colimits, we obtain a duality homomorphism:

$$D : H^p_c(M) \to H_{n-p}(M)$$

The following is a generalization of the previously stated Poincaré duality theorem to possibly non-compact manifolds.

Theorem 8.15. *For an R-oriented, connected, n-manifold M, the duality homomorphism $D : H^p_c(M) \to H_{n-p}(M)$ is an isomorphism.*

Proof. Following May's Chapter 20, we prove this in steps:

Step 1: The theorem is true for a coordinate chart $U = \mathbb{R}^n$.

We have seen that $H *_c (\mathbb{R}^n) \cong H_{n-*}(\mathbb{R}^n)$. Indeed, for any compact cube K, we have a fundamental class $[\mathbb{R}^n_K] \in H_n(\mathbb{R}^n, \mathbb{R}^n - K)$ and the cap product of $[\mathbb{R}^n_K]$ with a generator of $H^n(\mathbb{R}^n, \mathbb{R}^n - K) \cong R$ is (up to sign) just the point class in $H_0(\mathbb{R}^n)$ by the relation between cup, cap and Kronecker products. Passing to colimit over compact cubes gives that D is an isomorphism.

Step 2: If the result holds for open subspaces U, V and $U \cap V$, it holds for $U \cup V$.

This step is technically the most complicated one. It involves a Mayer-Vietoris construction for compactly supported cohomology and the naturality of the duality homomorphism. Indeed, there exists a commutative diagram with exact rows as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & H^p_c(U) \oplus H^p_c(V) \arrow[r] \arrow[d, "D"] & H^p_c(U \cup V) \arrow[r] \arrow[d, "D"] & H^{p+1}_c(U \cap V) \arrow[r] \arrow[d, "D"] & \cdots \\
\cdots \arrow[r] & H_{n-p}(U) \oplus H_{n-p}(V) \arrow[r] & H_{n-p}(U \cup V) \arrow[r] & H_{n-p-1}(U \cap V) \arrow[r] & \cdots
\end{tikzcd}$$

Once this is established the conclusion follows from the five-lemma in homological algebra (this purely algebraic lemma states that in a commutative diagram as above, in a sequence of 5 vertical maps if all but the middle map is known to be an isomorphism then the middle map is also an isomorphism).

To establish the above one considers the corresponding diagram before passing to colimits. Namely, let $K \subset U$ and $L \subset V$ be compact sets. Call $Z = U \cup V$. Then we have a commutative diagram:

$$\begin{tikzcd}
H^p(Z, Z-K) \oplus H^p(Z, Z-L) \arrow[r] \arrow[d, "\cong"] & H^p(Z, Z-(K \cup L)) \arrow[r, "\partial"] \arrow[d, "="] & H^{p+1}(Z, Z-(K \cap V)) \arrow[d, "\cong"] \\
H^p(U, U-K) \oplus H^p(V, V-L) \arrow[r] \arrow[d, "D \oplus D"] & H^p(Z, Z-(K \cup L)) \arrow[r, "\partial"] \arrow[d, "D"] & H^{p+1}(U \cap V, U \cap V - (K \cap L)) \arrow[d, "D"] \\
H_{n-p}(U) \oplus H_{n-p}(V) \arrow[r] & H_{n-p}(U \cup V) \arrow[r, "\partial"] & H_{n-p-1}(U \cap V)
\end{tikzcd}$$

The top and bottom rows are various versions of Mayer-Vietoris sequences. The middle row is isomorphic to the top row by excision isomorphisms. The only non-trivial verification needed is the commutativity of the squares that involve the boundary homomorphism. This involves a diagram chase that we do not reproduce here. (See Hatcher Lemma 3.36 for details).

To obtain the previous diagram from this, one passes to colimits over K, L. Note that we have required $K \subset U$ and $L \subset V$. This set of compacts in $U \cup V$ is enough to compute the group $H^p_c(U \cup V)$ as they form a compact exhaustion.

Step 3: If the result holds for each U_i in a directed system $U_1 \subset U_2 \subset \ldots$, then it holds for the union U of all the U_i.

Since any compact set must be contained in one of the U_i, it follows that

$$\mathrm{colim}_i H_{n-p}(U_i) = H_{n-p}(U)$$

Indeed, singular chain complex of an expanding union is the colimit of the singular chain complexes, and taking homology commutes with colimits.

On the cohomology side, again since any compact is contained in some U_i,

$$H^p_c(U) = \text{colim}_i \text{colim}_{K\subset U_i} H^p(U_i, U_i - K) = \text{colim}_i H^p_c(U_i)$$

Now, since $D : H^p_c(U_i) \to H_{n-p}(U_i)$ is an isomorphism and it commutes with the maps induced by inclusion of the open set U_i in U_{i+1}, it follows that $D : H^p_c(U) \to H_{n-p}(U)$ is an isomorphism.

Step 4: The theorem holds if U is an open subset of a coordinate neighborhood.

The result holds for an open convex subset of $\mathbb{R}^n$, since then it is homeomorphic to $\mathbb{R}^n$. Since the intersection of any two convex sets is convex, by Step 2, inducting on the number of open balls, the result holds for any finite union of open balls. Every open set $U \subset \mathbb{R}^n$ is a countable union of convex sets (for example, open balls). Now, by ordering a countable union and letting U_i to be the union of first i, we see that the result holds for any open set U.

Step 5: The result holds for any open subset of M.

We may as well take $M = U$. By Step 3, we may apply Zorn's lemma to conclude that there is a maximal open set V of M for which the result holds. If V is not all of M, then there exists $x \in M - V$. We may choose a coordinate chart U such that $x \in U$. By, steps 2 and 4 the result holds for $U \cup V$, contradicting the maximality of V. □

Remark 8.16. *There is a lucid proof of Poincaré duality for* smooth *manifolds which uses Morse theory. However, there are topological manifolds which are not smoothable. In fact, there are also more general topological spaces whose (co)homology groups satisfy Poincaré duality. These are called Poincaré duality spaces and they are not necessarily homotopy equivalent to a topological manifold. (You need to learn surgery theory à la Browder, Novikov, Sullivan, Wall to understand these spaces.)*

Here is an application:

Theorem 8.17. *There exists a compact 3-manifold having the homology groups of S^3 but which is not simply connected.*

Proof. Consider the group I of rotational symmetries of a regular icosahedron. We have $I \in SO(3)$ and it is well-known that I is isomorphic to the alternating group A_5 on five letters. Also, well-known is the fact that A_5 is simple.

On the other hand $SO(3)$ is homeomorphic to $\mathbb{R}P^3$. To see this, think of $\mathbb{R}P^3$ as D^3 with the antipodal points on the boundary identified. Now, $SO(3)$ is the group of rotations of $\mathbb{R}^3$. A rotation is given by a choice of an axis of rotation and an angle θ with $-\pi \leq \theta \leq \pi$, the angle of rotation. So via this map, one can see $SO(3)$ as the closed ball in $\mathbb{R}^3$ of radius π, with its antipodal points identified. (Since rotation of angle π and $-\pi$ are the same).

Also, observe that S^3 has a natural group structure as it can be identified with $SU(2)$ via the description

$$\{(a, b) \in \mathbb{C}^2 : |a|^2 + |b|^2 = 1\} \to \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2)$$

Furthermore, the double cover of $\mathbb{R}P^3$ by S^3 can be identified with the double covering of topological groups $SU(2) \to SO(3)$. A nice way to understand this map is via Möbius transformations. Namely, we have :

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \to \left\{ z \to \frac{az+b}{-\bar{b}z+\bar{a}} \right\}$$

Preimage of I under this cover is a group of order 120. This is called the binary icosahedral group $2I$. I is the quotient of $2I$ by its center $\{\pm 1\}$.

The binary icosahedral group $2I$ is a finite subgroup. The three-manifold required is the quotient:

$$\Sigma = S^3/2I$$

Since S^3 is the universal cover, we have $\pi_1(\Sigma) = 2I$. Under the double cover $S^3 \to \mathbb{R}P^3$, the commutator subgroup $[2I, 2I]$ goes to $[I, I] = I$. The latter equality holds because $I \cong A_5$ is simple. We claim that -1 is also in the commutator subgroup $[2I, 2I]$ hence it follows that $[2I, 2I] = 2I$.

Indeed, consider the Pauli matrices in $SU(2)$.

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

These correspond to the Möbius transformations $z \to -z$, $z \to -1/z$ and $z \to 1/z$ respectively, which are all rotations around an axis by an angle π. If our icosahedron is chosen approporiately (as suggested by the Figure 8.3), it is invariant under these transformations.

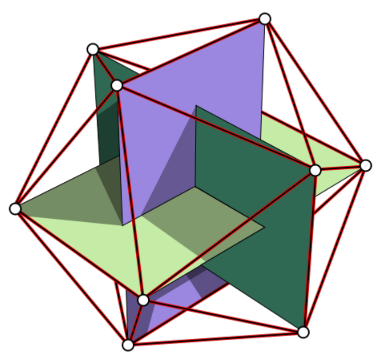


Figure 1: Picture from Wikipedia

Therefore, these matrices are in $2I$. On the other hand, we can compute that the commutator of any two of these is -1. For example:

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Therefore, $H_1(\Sigma) = \pi_1/[\pi_1, \pi_1] = 2I/[2I, 2I] = 0$. By Poincaré duality, we get $H_2(\Sigma) = 0$. Therefore, Σ has the same homology groups as S^3. $\square$

Poincaré duality has version for relative version. This is an easy consequence of what we have proved and naturality of cap product. It is useful to at least know the statement.

An n-manifold with boundary is a Hausdorff topological space M in which each point has an open neighborhood homeomorphic to either $\mathbb{R}^n$ or to the half space $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$.

Let M be compact n-manifold with boundary ∂M. Suppose M is connected and $R-$orientable by which we mean that its interior $M - \partial M$ is R-orientable. Then, one can construct an R-fundamental class $[M] \in H_n(M, \partial M; R) \cong R$ as before.

The relative version of Poincaré duality is the following statement:

Theorem 8.18. *Let G be an R-module. Let M be a compact, connected, R-oriented n-manifold with boundary and R-fundamental class $[M] \in H_n(M, \partial M; G)$, then capping with $[M]$ specifies duality isomorphisms:*

$$D : H^p(M, \partial M) \to H_{n-p}(M) \quad \textit{and} \quad D : H^p(M) \to H_{n-p}(M, \partial M)$$

8.4 Signature of manifolds

Suppose M is an $n = 2m$ (even) dimensional oriented manifold. Then, the middle dimensional homology $H^m(M)$ acquires a bilinear pairing. Let us take $R = \mathbb{R}$. We have:

$$H^m(M) \otimes H^m(M) \to \mathbb{R}$$

given by $\alpha \otimes \beta \to \langle \alpha \cup \beta, [M] \rangle$. We have seen before that this is a non-degenerate pairing as a consequence of Poincaré duality.

We will now study an application of cohomology and duality to the question of which n-manifolds M can be the boundary of an $(n+1)$-manifold V.

Since $\alpha \cup \beta = (-1)^{m^2} \beta \cup \alpha$. This pairing is skew-symmetric if m is odd, and symmetric if m is even.

A recourse to elementary linear algebra informs us that non-degenerate symmetric and skew-symmetric bilinear forms over $\mathbb{R}$ can be classified as follows:

Lemma 8.19. *Let V be a finite-dimensional vector space and $\phi : V \times V \to \mathbb{R}$ be a non-singular bilinear pairing:*

- *If ϕ is skew-symmetric, then there exists a basis $\{p_1, \dots, p_r, q_1 \dots, q_r\}$ such that $\phi(p_i, q_i) = 1$ for $1 \leq i \leq r$, and $\phi(z, w) = 0$ for all other basis element (z, w). In particular, dimension of V is even.*

- *if ϕ is symmetric, then there exists basis elements $\{x_1, \ldots x_r, y_1, \ldots y_s\}$ such that $\phi(x_i, x_i) = 1$ for $1 \leq i \leq r$, $\phi(y_j, y_j) = -1$ for $j = 1, \ldots, s$ and $\phi(z, w) = 0$ for all other basis elements (z, w). The number $\sigma = r - s$ is an invariant of ϕ called the signature of ϕ.*

As a result, we conclude that if M is an $n = 2m$ dimensional manifold, and m is odd, then $H^m(M)$ is even-dimensional. On the other hand, if m is even, then we obtain numerical invariant $\sigma(M)$ defined as the signature of the symmetric bilinear form on $H^m(M)$. (This definition is due to H. Weyl.) Another term used for the signature is "index".

Now, we remark that all orientable one-, two- and three-manifolds compact manifolds are boundaries. This breaks down in dimension 4 :

Theorem 8.20. *(Thom) If $M^{4n} = \partial V^{4n+1}$ is connected with V compact and orientable, then $\sigma(M) = 0$*

As an easy example of a 4-manifold, with $\sigma \neq 0$, one can take $\mathbb{C}P^2 \# \mathbb{C}P^2$. The signature is ± 2 according to how you choose to orient it.

We will deduce the theorem from the following:
Theorem 8.21. *Let $M = \partial B$, where B is compact oriented $(4k+1)$-manifold, and $i : M \to B$ be the inclusion. Then the image $i^* : H^{2k}(B) \to H^{2k}(M)$ is a subspace of half the dimension of $H^{2k}(M)$ on which the cup product pairing vanishes identically.*

Proof. Let $\phi : H^{2k}(M) \otimes H^{2k}(M) \to \mathbb{R}$ denote the cup product pairing.

Consider the following commutative diagram of duality homomorphisms associated to long-exact sequence of the pair (B, M).

$$\begin{tikzcd}
\cdots \arrow[r] & H^{2k}(B) \arrow[r, "i^*"] \arrow[d, "D"] & H^{2k}(M) \arrow[r, "\partial^*"] \arrow[d, "D"] & H^{2k+1}(B, M) \arrow[r] \arrow[d, "D"] & \cdots \\
\cdots \arrow[r] & H_{2k+1}(B, M) \arrow[r, "\partial"] & H_{2k}(M) \arrow[r, "i_*"] & H_{2k}(B) \arrow[r] & \cdots
\end{tikzcd}$$

Now $H^{2k}(M) \cong \mathrm{im} i^* \oplus \mathrm{im} \partial^* \cong \mathrm{im} i^* \oplus \mathrm{im} i_*$. But i^* and i_* are dual homomorphisms, hence $\mathrm{im} i^*$ and $\mathrm{im} i_*$ have the same dimension.

Furthermore, if $\alpha, \beta \in H^{2k}(B)$, then $\partial^*(i^*\alpha \cup i^*\beta) = \partial^* i^*(\alpha \cup \beta) = 0$ by exactness. Moreover, $\partial^* : H^{4k}(M) \to H^{4k+1}(B, M)$ is an injection, since it is dual to $i_* : H_0(M) \to H_0(B)$. Hence, $i^*\alpha \cup i^*\beta = 0$. $\square$

Proof of Thom's theorem: Let $V = H^{2k}(M; \mathbb{R})$ and let $dim(V) = 2s$. The bilinear form has r positive squares and $2s - r$ negative squares. Call the space of the positive basis V^+ and the span of the negative basis V^-. On the other hand, we have subspace $U = i^*(H^{2k}(B)$ of dimension s, on which the bilinear form vanishes identically. Clearly $U \cap V^+ = \{0\}$ hence, $dim(U + V^+) = s + r$. Similarly, $U \cap V^- = \{0\}$ hence, $dim(U + V^-) = 3s - r$. Since these are both contained in V, we must have the inequalities $s + r \leq 2s$ and $3s - r \leq 2s$, which forces $s = r$. Hence $\sigma(M) = 0$, as required. $\square$

8.5 Thom Isomorphism

A fibre bundle $\pi : E \to B$ is a map which is locally trivial, that is, for all points in $b \in B$, there exists a neighborhood U of b such that there are *local trivializations*, that is, homeomorphisms:

$$\pi^{-1}(U) \cong U \times F$$

by a map taking $\pi^{-1}(q) \to \{q\} \times F$ for all $q \in U$ and F is the *fibre* of the bundle.

For example, a covering space is a fibration with fibre F a discrete set. A vector bundle is a fibre bundle with $F \cong \mathbb{R}^n$ and the local trivializations are linear on the fibres. Given a vector bundle over a manifold, one can associate a (unit) disk and sphere bundles by choosing a fibrewise metric.

A k-disk bundle is a fibre bundle with fibre $F \cong D^k$, where D^k is the closed unit ball in $\mathbb{R}^k$. If $\pi : E \to B$ is a disk bundle, then restriction gives a k-sphere bundle $\pi_{|\partial E} : \partial E \to B$. We require these to have linear trivializations (like the ones that are obtained from a vector bundle).

Theorem 8.22. *Let $\pi : E \to M$ be a k-disk bundle over a connected closed oriented n-manifold. One has $H^r(E, \partial E) \cong 0$ for $r < k$. There exist a generator $\tau \in H^k(E, \partial E) \cong \mathbb{Z}$ called the Thom class and the map $\alpha \to \pi^*(\alpha) \cup \tau$ from $H^r(M) \to H^{r+k}(E)$ is an isomorphism for all $r \geq k$.*

Proof. Because the disk bundle has linear transition functions, there is a well-defined origin on each fibre. This gives rise to a *section*, $i : M \to E$ which is called the zero-section. By definition,

$$\tau = D(i_*([M]))$$

where $D : H_n(E) \to H^k(E, \partial E)$ is the inverse of the relative Poincaré duality isomorphism.

We will show that for $\alpha \in H^r(M)$, one has

$$\pi^*(\alpha) \cup \tau = (D_E) \circ i_* \circ (D_M)^{-1}(\alpha) \tag{4}$$

where $D_M : H_{n-r}(M) \to H^r(M)$ and $D_E : H_{n-r}(E) \to H^{r+k}(E, \partial E)$ are inverses of Poincaré duality isomorphisms. Note that D_M, D_E and $i_* : H_{n-r}(M) \to H_{n-r}(E)$ are isomorphisms, so the result follows from this.

Let $\beta = \pi^*\alpha$. To see equation 4, we just compute using properties of cup and cap products:

$$\begin{aligned} D_E \circ i_* \circ (D_M)^{-1}(\alpha) &= D_E(i_*(i^*\beta \cap [M])) = D_E(\beta \cap i_*([M]) \\ &= D_E(\beta \cap (\tau \cap [E])) = D_E((\beta \cup \tau) \cap [E]) = \beta \cup \tau = \pi^*(\alpha) \cup \tau \end{aligned}$$

□

If E is a rank n-bundle over an n-dimensional manifold, then we write the Thom class of the associated disk bundle is written as $\tau(E) \in H^n(E, \partial E)$. Its restriction $e(E) = i^*(\tau(E)) \in H^n(M)$ is called the *Euler class*. This has the property that if $f : M \to N$ is a map of spaces $\pi : E \to N$ is a vector bundle, and $f^*(E) \to M$ is the pull-back vector bundle, then

$e(f^*(E)) = f^*(e(E)) \in H^n(M)$. An assignment of a vector bundle $E \to c(E) \in H^*(X)$ for bundle $E \to X$ with this property is called a *characteristic class*; there is a beautiful book of Milnor and Stasheff on this topic that you should read.

The relative exact sequence of the pair $(E, \partial E)$ for a k-disk bundle over an n-manifold M gives rise to the *Gysin sequence* via Thom isomorphism. Namely, we have :

$$\begin{tikzcd}
\cdots \arrow[r] & H^{k+n}(E,\partial E) \arrow[r] \arrow[d, "\pi^*(\cdot)\cup\tau"] & H^{k+n}(E) \arrow[r] \arrow[d, "\cong"] & H^{k+n}(\partial E) \arrow[r] \arrow[d, "="] & H^{k+n+1}(E,\partial E) \arrow[r] \arrow[d, "\pi^*(\cdot)\cup\tau"] & \cdots \\
\cdots \arrow[r] & H^k(M) \arrow[r] & H^{k+n}(M) \arrow[r] & H^{k+n}(\partial E) \arrow[r] & H^{k+1}(M) \arrow[r] & \cdots
\end{tikzcd}$$

The lower line is called the Gysin sqeuence. One can use this to compute cohomology of sphere bundles. Furthermore, this gives us a way to find out information about cup products in $H^*(M)$ as it enters into the exact sequence.

Next, we will give a little bit of a discussion of intersection theory. The discussion here assumes rudimentary level manifold theory (you may come back to understanding the discussion here after next term's manifolds class).

Let us assume that $i : N^n \to M^m$ is a smooth embedding of oriented compact smooth manifolds (possibly with boundary in which case we assume that N meets ∂M transversely in ∂N.)

One then defines the Thom class $\tau_N^M = D_M(i_*[N]) \in H^{m-n}(M)$, where as before $D_M : H_n(M, \partial M) \to H^{m-n}(M)$ is the inverse of the Poincaré duality isomorphism. One can see this as the image of the Thom class of the normal $(m-n)$-disk bundle of N in M.

The key property of this class is that it relates cup product to an intersection product. Namely, if K, N are oriented submanifolds of compact oriented connected manifold M, then one defines:

$$[K] \cdot [N] = D^{-1}(D(N) \cup D(K)) = D(N) \cap K$$

where $D : H_i(M) \to H^{n-i}(M)$ is inverse to the Poincaré duality isomorphism. By definition, one has:

$$[K] \cdot [N] = (\tau_K^M \cup \tau_N^M) \cap [M]$$

Hence, since Thom classes are associated to the normal bundles of the submanifolds, this intersection number only depends on a local computation near each submanifold. This gives us an efficient way of computing the cup product geometrically.

Indeed, suppose that K and N are submanifolds of M that intersect transversely at a submanifold $K \cap N$. We assume that all of these are oriented in a consistent manner (we omit a sign discussion here). Then the normal bundle to K in M restricts to the normal bundle of $K \cap N$ in N (resp. $K \cap N$ in K). This implies that

$$\tau^N_{K\cap N} = i^*_N \tau^M_K \quad \text{and} \quad \tau^K_{K\cap N} = i^*_K \tau^M_N$$

where $i_N : N \to M$ and $i_K : K \to M$ are inclusion maps.

From this, we deduce the following theorem:

Theorem 8.23. *Suppose $K, N \subset M$ as before. Then we have :*

$$\tau^M_{K\cap N} = \tau^M_K \cup \tau^N_K$$

, and equivalently,

$$[K] \cdot [N] = [K \cap N]$$

Proof. After all we said above, the proof is a routine computation. Let $i_{K\cap N} : K \cap N \to N$ be the inclusion map. We have,

$$\begin{aligned}[K \cap N] &= i_{N*} i_{K\cap N*}[K \cap N] = i_{N*}(\tau^N_{K\cap N} \cap [N]) = i_{N*}(i^*_N \tau^M_K \cap [N]) = \tau^M_K \cap i_{N*}[N] \\ &= \tau^M_K \cap (\tau^N_K \cap [M]) = (\tau^M_K \cup \tau^M_N) \cap [M] = [N] \cdot [M]\end{aligned}$$

□

This is a very interesting result: In the case of manifolds, intersections of submanifolds gives a geometric way of understanding what $\cup$-product is. Of course, the next question you should ask is which cohomology classes can be realized as Thom classes of submanifolds? Not surprisingly, Thom has also explored this question. But,

time to take a break. . . .