BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2023

This paper is also taken for the relevant examination for the Associateship.

MATH70061

Commutative Algebra (Solutions)

Setter's signature	Checker's signature	Editor's signature
....	....	....

1. (a) It is clear that $f(a_i) = b_i$. Consider any other polynomial $h(X)$ such that $h(a_i) = b_i$. Now, the difference $f(X) - h(X)$ vanishes on a_i, hence we can write

$$f(X) - h(X) = (X - a_1)(X - a_2)\dots(X - a_n)g(X)$$

If both $f(X)$ and $h(X)$ are of degree $\leq n-1$, then $g(X) = 0$, hence $f(X) = h(X)$. $\boxed{6, \text{A}}$

(b) Consider $p(X,Y) \in \mathcal{I}(V)$ divide it by $Y - f(X)$ and $g(X)$ with remainder, we get

$$p(X,Y) = h_1(X,Y)(Y - f(X)) + h_2(X,Y)g(X) + r(X,Y)$$

Now, none of the terms of $r(X,Y)$ are divisible by $Y = LT(Y - f(X))$ and $X^n = LT(g(X)$ hence $r(X,Y) = r(X) \in k[X]$ a polynomial of degree $\leq n-1$. But, $r(a_i) = 0$ for all $i = 1, \dots, n$, hence $r(X) = 0$. $\boxed{7, \text{B}}$

(c) We compute the S-polynomial

$$S(Y - f(X), g(X)) = X^n(Y - f(X)) - Yg(X) = -X^n f(X) - Yh(X)$$

where $h(X) = g(X) - X^n$. Now, we divide by $Y - f(X), g(X)$, we get

$$\begin{aligned} S(Y - f(X), g(X)) &= (-h(X))(Y - f(X)) - X^n f(X) - h(X)f(X) \\ &= (-h(X))(Y - f(X)) - f(X)g(X) \end{aligned}$$

so the remainder is zero. $\boxed{7, \text{B}}$

2. (a) Write Γ for the set of ideals of A which are not finitely generated. If $\Gamma \neq \emptyset$, let $\mathcal{T} \subset \Gamma$ be a totally ordered set, then the ideal $\mathfrak{b} = \bigcup_{\lambda \in \mathcal{T}} I_\lambda$ is in Γ. Indeed, if $\mathfrak{b} = (x_1, \ldots, x_s)$, then $\{x_1, \ldots, x_s\} \subset I_\lambda$ for some λ, so that $\mathfrak{b} \subset I_\lambda$ which implies $\mathfrak{b} = I_\lambda$ is finitely generated, contradiction. Hence, $\mathfrak{b}$ is an upperbound for $\mathcal{T}$. By Zorn's lemma Γ contains a maximal element I. Then I is not a prime ideal, so there are elements $x, y \in A$ with $x \notin I, y \notin I$ but $xy \in I$. Now, $I + (y)$ is bigger than I, and hence is finitely generated, so that we can choose $u_1, \ldots, u_n \in I$ such that $I + (y) = (u_1, \ldots, u_n, y)$. Moreover $I : y = \{a \in A : ay \in I\}$ contains x, and is thus bigger than I, so it has a finite system of generators $v_1, \ldots, v_m$. Finally, it is easy to check that $I = (u_1, \ldots, u_n, v_1 y, v_2 y, \ldots, v_m y)$, hence $I \notin \Gamma$, which is a contradiction. 10, C

(b) Write Γ for the set of ideals of A that are not principal. Suppose Γ is non-empty. Let $\mathcal{T} \subset \Gamma$ be a totally ordered set, then the ideal $\mathfrak{b} = \bigcup_{\lambda \in \mathcal{T}} I_\lambda$ is in Γ. Indeed, if $\mathfrak{b} = (x)$ for some $x \in A$, then $x \in I_\lambda$ for some λ, so that $\mathfrak{b} \subset I_\lambda$ which implies $\mathfrak{b} = I_\lambda$ is principal, contradiction. Hence, $\mathfrak{b}$ is an upperbound for $\mathcal{T}$. By Zorn's lemma, the set Γ has a maximal element, call it I. By assumption I is not prime, so there exists $x, y \in A$ with $x \notin I, y \notin I$ but $xy \in I$. Now, $I + (y)$ is bigger than I and so it is principal, let $I + (y) = (a)$. Similarly, $I : y$ contains I and x hence is also principal, say $I : y = (b)$. We claim that $I = (ab)$. Indeed, let $c \in I \subset I + (y)$, then $c = am$ for some $m \in I$. Then, $m \in I : y$, hence $m = bn$ for some n, hence $c = abn$, which shows $I \subset (ab)$. Conversely, if $b \in I : y$ implies $by \in I$ so, $b(a) \subset I$, hence $ab \in I$. It follows that I is principal, which is a contradiction. 10, C

3. (a) For a ring A, the **Krull dimension** is defined to be

$$\dim A := \dim \mathrm{Spec}(A) = \sup\{n \geq 0 : \exists\ \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \ldots \subsetneq \mathfrak{p}_n \subsetneq A \text{ with } \mathfrak{p}_i \text{ prime ideal } \}$$

3, A

(b) $\mathbb{Z}[i]$ is isomorphic to $\mathbb{Z}[x]/(x^2+1)$ hence is an integral extension of $\mathbb{Z}$. Therefore, by Cohen-Seidenberg theorems, $\dim\mathbb{Z}[i] = \dim\mathbb{Z}$. In $\mathbb{Z}$ every non-zero prime ideal is maximal and is given by (p) for some prime p, hence the longest chain of prime ideals are all of the form

$$(0) \subsetneq (p)$$

for some prime number p. Therefore, $\dim\mathbb{Z}[i] = \dim\mathbb{Z} = 1$. 5, A

(c) Let $x_1, \ldots, x_n \in \mathfrak{m}$ such that $\overline{R} = R/(x_1, \ldots, x_n)$ is Artinian. Suppose that there exists a prime ideal $\mathfrak{p} \subset R$ such that

$$(x_1, \ldots, x_n) \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$$

Then, in $\overline{R}$, we would have a chain

$$\mathfrak{p}/(x_1, \ldots, x_n) \subsetneq \mathfrak{m}/(x_1, \ldots, x_n)$$

of prime ideals, which implies $\dim\overline{R} \geq 1$, hence $\overline{R}$ cannot be Artinian, contradiction. Therefore, $\mathfrak{m}$ is a minimal prime ideal containing $(x_1, \ldots, x_n)$ then it follows from Krull's Height Theorem that $\dim R = \mathrm{ht}\mathfrak{m} \leq n$. 5, B

(d) There are finitely many minimal prime ideals in a Noetherian ring, these are all the prime ideals of height 0 and we can view them as containing the empty set. Suppose now by induction that for $0 \leq k < n$, we have constructed elements $x_1, \ldots, x_k \in \mathfrak{m}$ such that every minimal prime ideal containing $x_1, \ldots, x_k$ has height k. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ be these minimal primes containing $(x_1, \ldots, x_k)$ with height k. We see that $\mathfrak{m} \nsubseteq \bigcup_{i=1}^{s} \mathfrak{q}_i$ since, by prime avoidance, this would imply $\mathfrak{m} \subset \mathfrak{q}_i$ for some i but height of $\mathfrak{q}_i$ is $k < n = h(\mathfrak{m})$ which is a contradiction. Hence, we can find $x_{k+1} \in \mathfrak{m} \setminus \bigcup_{i=1}^{s} \mathfrak{q}_i$. Now, let $\mathfrak{p}$ be a minimal prime ideal containing $(x_1, \ldots, x_k, x_{k+1})$. We have $h(\mathfrak{p}) \leq k+1$ by Krull's height theorem. On the other hand, since $\mathfrak{p} \supset (x_1, \ldots, x_k)$ and $\mathfrak{q}_i$ are minimal primes containing $(x_1, \ldots, x_k)$, we have $\mathfrak{p} \supset \sqrt{(x_1, \ldots, x_k)} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s \supset \mathfrak{q}_1 \ldots \mathfrak{q}_s$. Hence $\mathfrak{p} \supset \mathfrak{q}_i$ for some i, hence $h(\mathfrak{p}) = k+1$. Now, by induction, we can continue this until we produce elements $x_1, \ldots, x_n$. The minimal prime ideal containing $(x_1, \ldots, x_n)$ has height n, and thus has to coincide with $\mathfrak{m}$. 7, D

4. (a) Let $I = (X^2 + Y^2 - 1) \subset \mathbb{C}[X,Y]$. By Hilbert's Nullstellensatz, we have $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$. But, $X^2 + Y^2 - 1$ is an irreducible polynomial, hence I is prime, therefore $\sqrt{I} = I$. So, the polynomials vanishing on $\mathcal{V}(I)$ are exactly given by I, hence are of the form $P(X,Y)(X^2 + Y^2 - 1)$ where $P(X,Y) \in \mathbb{C}[X,Y]$ is an arbitrary polynomial. 2, A

(b) (i) Since k is not algebraically closed, we can find a non-trivial polynomial $p(X) \in k[X]$ with no zero. Write

$$p(X) = a_n X^n + a_{n-1} X^{n-1} + \ldots + a_1 X + a_0$$

Now, consider the homogenization $\phi_2(X,Y) \in k[X,Y]$ given by

$$\phi_2(X,Y) = a_n X^n + a_{n-1} X^{n-1} Y + \ldots + a_1 X Y^{n-1} + a_0 Y^n.$$

If $\phi_2(X,Y)$ has a root with $Y \neq 0$, then we would get a root of $\phi_2(X,1) = p(X)$ which is not the case. Therefore, the only zero of $\phi_2(X,Y)$ is at $(0,0)$. Now, we define recursively

$$\phi_m(X_1, \ldots, X_m) = \phi_2(\phi_{m-1}(X_1, \ldots, X_{m-1}), X_m)$$

It is clear that the only zero of ϕ_m is at $(0, \ldots, 0) \in k^m$ as required. 6, B

(ii) Consider the maximal ideal $(X,Y) \in \mathbb{C}[X,Y]$, the variety associated to this ideal is the point $(0,0) \in \mathbb{C}^2$. Suppose $(0,0) = \mathcal{V}((f))$ for $f \in \mathbb{C}[X,Y]$, then by Nullstellensatz, we would have $\sqrt{(X,Y)} = (X,Y) = \sqrt{(f)}$. Thus, there exists, n, m such that f divides X^n and Y^m, but this implies f has to be constant (since $\mathbb{C}[X,Y]$ is a UFD) which is a contradiction. 6, A

(c) By Zariski's lemma the field $\mathbb{R}[X,Y]/\mathfrak{m}$ is a finite field extension of $\mathbb{R}$. There are two such fields $\mathbb{R}$ and $\mathbb{C}$. One can take (X,Y) and (X^2+1, Y) as examples of maximal ideals such that $\mathbb{R}[X,Y]/\mathfrak{m}$ is $\mathbb{R}$ and $\mathbb{C}$. In the first case the isomorphism is given by sending X and Y to 0 and in the second case, X to i and Y to 0. 6, B

5. (a) (i) A valuation ring is an integral domain R such that for all $x \in K \setminus \{0\}$, where K is the field of fractions of R, either $x \in R$ or $x^{-1} \in R$. **2, A**

(ii) A valuation on K is given by map $\nu : K \to \Gamma \cup \{\infty\}$, where Γ is an ordered abelian group, satisfying
(1) $\nu(xy) = \nu(x) + \nu(y)$, for all $x, y \in K$,
(2) $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$,
(3) $\nu(x) = \infty$ if and only if $x = 0$.
The valuation ring associated to ν is defined by $R = \{x \in K : \nu(x) \geq 0\}$. R is called a discrete valuation ring (DVR) if $\Gamma = \mathbb{Z} \cup \{\infty\}$. **2, A**

(iii) If $x \in K$ satisfies an equation $x^n + a_1 x^{n-1} + \ldots + a_n = 0$ with $a_i \in R$. Thus, we have $\nu(a_i) \geq 0$. Now, if $\nu(x) < 0$, then we have $\nu(x^n) = n\nu(x) < \nu(a_i x^{n-i}) = (n-i)\nu(x) + \nu(a_i)$ for all $i = 1, \ldots, n$. Hence, this violates condition (2) of the valuation. **3, A**

(b) (i) Every element of $K = \mathbb{C}(X,Y)$ we can express is as $Y^n \frac{f}{g}$ for $f, g \in \mathbb{C}[X,Y]$ with $n \in \mathbb{Z}$ and Y not dividing f or g. The elements that are in R correspond precisely to the ones with $n \geq 0$. Hence, it is clear that if $x \in K \setminus R$, then $x^{-1} \in R$. Equivalently, one could define a valuation by letting $\nu(Y^n \frac{f}{g}) = n$. **3, A**

(ii) Both $\frac{X}{Y}$ and $\frac{Y}{X}$ are not in R, hence this is not a valuation ring. **3, A**

(c) Any ring R' with $R \subset R' \subset K$ is given by $R_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p} \subset R$. But, R is a local ring and an integral domain of dimension 1, hence the only prime ideals it has are (0) and $\mathfrak{m}$ (the maximal ideal in R), and localisation on these ideals gives R and K. So, R is maximal as a subring of K.

Conversely, suppose R is maximal proper subring of K. The integral closure of R is not the whole of K (as this would imply R is a field), hence R is integrally closed. On the other hand, let $x \in K \setminus R$. Then we have $R[x] = K$, so $x^{-1} \in R[x]$. Hence, we can write

$$x^{-1} = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0, \quad a_i \in R$$

which implies

$$x^{-n-1} - a_0 x^{-n} - \ldots - a_n = 0$$

Thus, x^{-1} is integral over R, hence $x^{-1} \in R$. Thus R is a valuation ring. It is of dimension 1, since otherwise we would have a prime ideal $\mathfrak{p} \neq (0), \mathfrak{m}$ and we would have $R \subsetneq R_{\mathfrak{p}} \subsetneq K$. **7, D**

Review of mark distribution:

Total A marks: 35 of 35 marks

Total B marks: 31 of 31 marks

Total C marks: 20 of 20 marks

Total D marks: 14 of 14 marks

Total marks: 100 of 100 marks

Total Mastery marks: 0 of 0 marks