Homework I

Due for 19 October 2022

This homework will constitute %2 of your final grade. I will choose 2 random ones and grade those ones. Thus, to guarantee that you get full marks you should submit correct solutions to all 14. No partial credit is given but minor mistakes will be tolerated. So, you get either %1 or %0 from each problem. Some problems may be harder and you may need to do some reading before being able to solve them. You are encouraged to work on the problem sets in groups but the final write-up should be yours.

1) If $I = (m)$, $J = (n)$ in $\mathbb{Z}$, then $IJ = (mn)$, $I \cap J = (\mathrm{lcm}(m, n))$ and $I + J = (\gcd(m, n))$.

2) Check that $I : J$ and $I : J^\infty$ are ideals.

3) Show that $\sqrt{I}$ is an ideal.

4) Show that $x \in \mathcal{J}(R)$ if and only if $1 - xy$ is a unit of R for all $y \in R$ (uses Zorn's lemma).

5) Show that $\sqrt{I \cdot J} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

6) Show that a maximal ideal is a prime ideal and a prime ideal is a radical ideal.

7) Show that if a prime ideal $\mathfrak{p} = I \cap J$ for ideals I and J, then $\mathfrak{p} = I$ or $\mathfrak{p} = J$.

8) Show that in a local ring R, the units in R are precisely the elements in $R - \mathfrak{m}$. (uses Zorn's lemma); conversely, a ring R whose non-units form an ideal is a local ring.

9) Show that if $M = R \cdot x$ is a cyclic module, then M is isomorphic as an R-module to a module of the form $R/\mathrm{ann}(M)$.

10) Show that $\mathbb{Z}[i]$ is an Euclidean domain using the norm $|a + bi| = a^2 + b^2$. (Use the fact that disks of unit norm centred at Gaussian integers cover all of the complex plane.)

11) Show that (a) and (ai) for $a \in \mathbb{Z}$ are prime ideals in $\mathbb{Z}[i]$ if only if a is prime number of the form $4n + 3$. $(a + bi)$ with $a, b \neq 0$ is a prime ideal if and only if $a^2 + b^2$ is a prime number.

12) Show that the ideal $(2, 1 + \sqrt{-3})$ is not principal in the ring $\mathbb{Z}[\sqrt{-3}]$.

13) Let k be an infinite field, and $f \in k[X_1, \ldots, X_n]$. Show that $f = 0$ if and only if $f(x_1, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_n) \in k^n$. What about if k is a finite field? Consider $X^2 - X$ in $\mathbb{F}_2[X]$.

14) Let $\mathfrak{p} \subset \mathbb{Z}[X]$ be a prime ideal. Suppose $\mathfrak{p} \neq \mathbb{Z}[X]$ or (0). Show that $\mathfrak{p} \cap \mathbb{Z}$ is a prime ideal in $\mathbb{Z}$. Thus, $\mathfrak{p} \cap \mathbb{Z} = (p)$ where $p = 0$ or p is a prime number.

(i) Suppose $\mathfrak{p} \cap \mathbb{Z} = \{0\}$, then $\mathfrak{p} = (f)$ where $f \in \mathbb{Z}[X]$ is an irreducible polynomial.

(ii) Suppose $\mathfrak{p} \cap \mathbb{Z} = (p)$ with p a prime number. Then, $\mathfrak{p} = (p, f)$ where $f = 0$ or $f \in \mathbb{Z}[X]$ is a monic polynomial (leading coefficient is one) and its mod p reduction $\overline{f} \in \mathbb{F}_p[X]$ is irreducible.