Homework II Solutions
Problems 4 and 9

4) Take $I = (f_1, f_2)$ with $f_1 = X^3 - 2XY$ and $f_2 = X^2Y - 2Y^2 + X$ and use $\leq_{grlex}$. Show that $(LT(f_1), LT(f_2))$ is strictly contained in $\mathfrak{in}(I)$. In other words, show that $(LT(f_1), LT(f_2)) \subsetneq \mathfrak{in}(I)$.

We observe that $X(X^2Y - 2Y^2 + X) - Y(X^3 - 2XY) = X^2$. Hence, $X^2 \in I$ which gives $X^2 = LT(X^2) \in \mathfrak{in}(I)$. On the other hand, $LT(f_1) = X^3$ and $LT(f_2) = X^2Y$. Since X^2 is not divisible by either X^3 or X^2Y, it cannot be an element of the monomial ideal (X^3, X^2Y). In other words, $X^2 \in \mathfrak{in}(I) \setminus (LT(f_1), LT(f_2))$.

9) Suppose that V is a linear space, that is

$$V = \mathcal{V}(\{f_j = \sum_{i=1}^{n} a_{ij}X_i : 1 \leq j \leq m\})$$

Show that $\dim_k V = \deg HP_I(t)$, where $I = (f_1, f_2, \ldots, f_m)$.

Solution 1 Let $A = (a_{ij})$ be the $m \times n$ matrix with entries a_{ij} and let $B = (b_{ij})$ be its reduced row echelon form (obtained after the process of Gauss elimination). Define linear forms $g_j = \sum_{i=1}^{n} b_{ij}X_i$ for $j = 1, \ldots, r$ corresponding to non-zero rows of B, where we note that r is equal to $\mathrm{rank}A$. It is clear that $I = (g_1, \ldots, g_r)$ since the linear forms $\{g_i\}_{i=1}^{r}$ are k-linear combinations of the linear forms $\{f_i\}_{i=1}^{m}$ and vice versa.

Note that $LT(g_i) = X_{\sigma(i)}$ for where $\sigma : \{1, \ldots, r\} \to \{1, \ldots n\}$ is defined such that the first non-zero entry of the i^{th}-row in the reduced row echelon form appears in column $\sigma(i)$.

Claim: $g_1, \ldots, g_r$ is a Gröbner basis for I with respect to the lexicographic monomial order with $X_1 > X_2 > \ldots > X_n$.

Computing S-polynomials, we get

$$S(g_i, g_j) = X_j g_i - X_i g_j = X_j(g_i - LT(g_i)) - X_i(g_j - LT(g_j))$$

Now all the terms that appear in $g_i - LT(g_i)$ and $g_j - LT(g_j)$ are not divisible by any of $LT(g_k)$ for $k = 1, \ldots, r$. Hence, performing division with $\{g_1, \ldots, g_r\}$ gives

$$S(g_i, g_j) = (g_i - LT(g_i))g_j - (g_j - LT(g_j))g_i$$

which shows that the remainder is zero for all $1 \leq i, j \leq r$. Hence, we proved our claim.

Now, we can use the Gröbner basis to deduce that the initial ideal is given by

$$\mathfrak{in}(I) = (X_{\sigma(1)}, X_{\sigma(2)}, \dots, X_{\sigma(r)})$$

Thus, the complement $C(\mathfrak{in}(I))$ is generated as a k-vector space by monomials X^α for $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i = 0$ if $i \in \mathrm{Im}(\sigma)$.

Using the graded lexicographic monomial order (with the above ordering of variables), and appealing to Macaulay's lemma, we deduce that

$$HF_I(s) = |C(\mathfrak{in}(I))_{\leq s}| = \binom{n-r+s}{s} = \frac{1}{(n-r)!}s^{n-r} + \dots$$

from which we find that

$$\mathrm{deg} HP_I = n - r.$$

On the other hand, as a linear space V is given by the solutions to the linear equation

$$A \cdot X = 0$$

which by the rank-nullity theorem has dimension given by

$$\dim_k V = \dim_k \ker A = n - \mathrm{rank} A = n - r$$

Solution 2 This solution instead uses the column reduction. However, we have to be careful and remember that column operations do not preserve the null space, or the ideal I. Indeed, column reduction modifies a matrix A by multiplying from the right by a sequence of elementary matrices. Thus, the matrix A is related to its reduced column echelon form C via

$$C = A \cdot E$$

where E is some invertible matrix given as a product of matrices that perform the column operations. On the other hand, we can re-write the equation $A \cdot X = 0$ as follows:

$$A \cdot X = (A \cdot E) \cdot (E^{-1} X) = 0$$

Therefore, let us define new variables Y by

$$E^{-1} \cdot \begin{pmatrix} X_1 \\ \cdot \\ \cdot \\ X_n \end{pmatrix} = \begin{pmatrix} Y_1 \\ \cdot \\ \cdot \\ Y_n \end{pmatrix}$$

Since this is a linear change of variables (using an invertible matrix), it induces a ring isomorphism

$$k[X_1, \dots, X_n] \to k[Y_1, \dots, Y_n]$$

sending the ideal $I = (f_1, \dots, f_m)$ in $k[X_1, \dots, X_n]$ to the ideal $J = (h_1, h_2, \dots, h_m)$ in $k[Y_1, \dots, Y_n]$ given by the rows of C.

Now, we can observe that from the shape of the reduced column echelon form, it is easy to deduce that $J = (Y_1, \dots, Y_r)$ where r is the number of non-zero columns in C which is again equal to the rank of A since column rank and row rank of a matrix are equal. Hence, we conclude that

$$HP_J(s) = \frac{1}{(n-r)!}s^{n-r} + \dots$$

Alternatively, we can observe that since none of the h_i involve the variables $Y_{r+1}, \dots, Y_n$, we conclude that $J \cap k[Y_{r+1}, \dots, Y_n] = \{0\}$, and we also have to show that this is the maximal number of variables that J avoids but this again follows from inspecting the reduced column echelon matrix. In either way, we deduce that

$$\deg HP_J = n - r$$

It looks as if we are done, but now we have to pay back our debt since column operations changed the ideal I (as well as the null space of A). However, since we made a linear change of variables there is a k-linear bijection $C(I)_{\leq s}$ and $C(J)_{\leq s}$. Therefore, it follows that

$$HF_I(s) = HF_J(s)$$

Hence, we finally conclude that

$$\deg HP_I = n - r$$

Bonus question: Let us consider a non-linear isomorphism between polynomial rings. For example, consider the ring isomorphism $k[X_1, X_2] \to k[Y_1, Y_2]$ given by $X_1 \to Y_1 + Y_2^r$ and $X_2 \to Y_2$ for some $r \in \mathbb{N}$. Since this is clearly a ring isomorphism, it sends any ideal I of $k[X_1, X_2]$ to an ideal J of $k[Y_1, Y_2]$ and induces an isomorphism of rings

$$k[X_1, X_2]/I \simeq k[Y_1, Y_2]/J$$

However, it is not a degree preserving isomorphism. Hence, $C(I)_{\leq s}$ may very well be not isomorphic to $C(J)_{\leq s}$. Can you construct an example of an ideal such that under such an isomorphism

$$HF_I(s) \neq HF_J(s)$$

What about $\deg HP_I(s)$ and $\deg HP_J(s)$?