Homework III Solutions
Problems 7 and 9

7) Consider a 2-by-2 matrix M with entries X, Y, Z, W. If we then want to solve $M^2 = 0$, we get four equations and let's make that into an ideal $I = (X^2+YZ, XY+YW, XZ+WZ, W^2+YZ) \in k[X,Y,Z,W]$. Is I a radical ideal? Show that $\sqrt{I} = (X+W, XW-YZ)$.

Let us first give a proof under the assumption $k = \overline{k}$ algebraically closed. Recall that the characteristic polynomial of a 2-by-2 matrix M is given by

$$\chi_M(t) = t^2 - (\mathrm{tr}M)t + \det M$$

Now, since $M^2 = 0$ and $\chi_M(M) = 0$ (Cayley-Hamilton theorem), the minimal polynomial of M has to divide both t^2 and $\chi_M(t)$. This implies that $\chi_M(t) = t^2$, hence $M^2 = 0$ implies that $\mathrm{tr}M = X + W = 0$ and $\det M = XW - YZ = 0$.

Thus, it follows from Nullstellensatz (using the fact that k is algebraically closed) that $X + W, XW - YZ \in \mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$. Or equivalently,

$$J := (X+W, XW-YZ) \subset \sqrt{I}$$

Conversely, to prove that $\sqrt{I} \subset J$ it suffices to show that $I \subset J$ and J is prime. It is easy to see that $I \subset J$ because $X^2 + YZ = (X+W)X - (XW - YZ)$, $XY + YW = (X+W)Y$, $XZ + WZ = (X+W)Z$, $W^2 + YZ = (X+W)W - (XW - YZ)$. (Alternatively, observe that if $\mathrm{tr}M = 0 = \det M$, then from $\chi_M(M) = 0$, we conclude that $M^2 = 0$. Hence, $I \subset \sqrt{J}$ by Nullstellensatz, which implies $I \subset J = \sqrt{J}$ if we show that J is prime.)

It remains to show $J = (X+W, XW-YZ)$ is prime. Showing this is equivalent to proving that $k[X,Y,Z,W]/(X+W, XW-YZ)$ is an integral domain. Let us first observe the ring isomorphism

$$k[X,Y,Z,W]/(X+W, XW-YZ) \simeq k[X,Y,Z]/(X^2+YZ)$$

given by sending $(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) \to (\overline{X}, \overline{Y}, \overline{Z}, -\overline{X})$. We observe that this sends $(X+W, XW-YZ)$ to (X^2+YZ). It is obviously surjective. To see that it is injective, by using the relation $X+W=0$, any element of $k[X,Y,Z,W]/(X+W, XW-YZ)$ can be represented by $f(\overline{X}, \overline{Y}, \overline{Z})$ for some $f \in k[X,Y,Z]$. This element goes to zero in $k[X,Y,Z]/(X^2+YZ)$ if and only if $f(X,Y,Z)$ is divisible by X^2+YZ but then it is also zero $k[X,Y,Z,W]/(X+W, XW-YZ)$ since $X^2+YZ = X(X+W) - (XW-YZ) \in (X+W, XW-YZ)$. Thus, it suffices to see that $k[X,Y,Z]/(X^2+YZ)$ is an integral domain, or that $(X^2+YZ) \subset k[X,Y,Z]$ is prime but this holds because $k[X,Y,Z]$ is a UFD and $f = X^2+YZ$ is irreducible. (To see that f is irreducible, you can observe that it doesn't factor into linear components).

We now give a proof that works in general (without assuming k is algebraically closed). The only place where we used k is algebraically closed was when we argued that $X+W$ and $XW-YZ$ are in $\sqrt{I}$. Indeed, we can verify this directly as follows:

$$(X+W)^3 = (X+3W)(X^2+YZ)+(W+3X)(W^2+YZ)-4Z(XY+YW)$$
$$(XW-YZ)^2 = (X^2+YZ)(W^2+YZ)-XZ(XY+YW)-YW(XZ+WZ)$$

Alternatively, once we see that $X+W \in \sqrt{I}$ from the first equation, we can write $XW-YZ = (X+W)W-(W^2+YZ)$ to conclude that it too is in $\sqrt{I}$.

9) Show that if $f: R \to S$ is a ring homomorphism between finitely generated k-algebras R and S, then $f^{-1}(\mathfrak{m})$ is a maximal ideal of R for all maximal ideals $\mathfrak{m}$ of S.

Because S is a finitely generated k-algebra, its quotient $S/\mathfrak{m}$ by a maximal ideal $\mathfrak{m}$ is a finite field extension of k by Zariski's lemma. Therefore, the image of R in $S/\mathfrak{m}$ under the composition of ring homomorphisms $R \to S \to S/\mathfrak{m}$ is an integral domain which is an integral extension of k, hence is a field (see Corollary 4.22 in the lecture notes). But, the kernel of the map $R \to S/\mathfrak{m}$ is $f^{-1}(\mathfrak{m})$ hence the image is isomorphic to $R/f^{-1}(\mathfrak{m})$ which we have shown is a field, therefore $f^{-1}(\mathfrak{m})$ is a maximal ideal.