Homework IV

Due for 30 November 2022

This homework will constitute %2 of your final grade. I will choose 2 random ones and grade those ones. Thus, to guarantee that you get full marks you should submit correct solutions to all problems. No partial credit is given but minor mistakes will be tolerated. So, you get either %1 or %0 from each problem. Some problems may be harder and you may need to do some reading before being able to solve them. You are encouraged to work on the problem sets in groups but the final write-up should be yours.

1) A group G is called residually finite if for every non-identity element $g \in G$, there exists a group homomorphism $\pi : G \to F$ to a finite group F such that $\pi(g) \neq 1$. Let $GL(n, \mathbb{C})$ be the general linear group with $\mathbb{C}$ coefficients. Clearly, $GL(n, \mathbb{C})$ is not residually finite as it is connected. Show that every finitely generated subgroup of $GL(n, \mathbb{C})$ is residually finite (Hint: use the "arithmetic" version of Nullstellensatz).

2) Show that for any ring A, Spec(A) and mSpec(A) are topological spaces. Precisely, prove the following for $V = \mathcal{V}$ or $\mathscr{V}$:

(i) If $I_1, \ldots, I_r \subset A$ are ideals, then $V(I_1 \cap \ldots \cap I_r) = V(I_1) \cup \ldots \cup V(I_r)$.

(ii) $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals in A, then $V(\sum_{\lambda \in \Lambda} I_\lambda) = \bigcap_{\lambda \in \Lambda} V(I_\lambda)$.

(iii) $V(\{0\}) = \mathrm{Spec}(A)$ or $\mathrm{mSpec}(A)$, and $V(A) = \emptyset$.

3) i) For any ring A, show that Spec(A) is compact, that is each of its open covers has a finite subcover. (Sometimes, this is referred to as quasi-compact as Spec(A) is almost never Hausdorff.)

ii) If $f : A \to B$ is a ring homomorphism we get a map $f^* : \mathrm{Spec}B \to \mathrm{Spec}A$. Show that this map is continuous in the Zariski topology. Show that $A \to A/\sqrt{(0)}$ gives a homeomorphism between the spectrum of these rings.

iii) Suppose A is an integral domain. Prove that $\{(0)\}$ is a dense point in Spec(A) in the Zariski topology.

4) Let A be a finitely generated k-algebra (k a field), then show that A is an Artinian ring if and only if A is finite-dimensional as a k-vector space.

5) Show that for an Artinian ring A, the spectrum Spec(A) is discrete and finite.

6) Let $f \in A$ and $S := \{f^n : n = 1, 2, \ldots\} \cup \{1\}$ multiplicatively closed set. Then we write $A_f := S^{-1}A$. Prove that A_f is isomorphic to the ring $A[X]/(Xf - 1)$.

7) Suppose that A is a ring with the property that $A_\mathfrak{p}$ has no nilpotent elements for all $\mathfrak{p} \in$ Spec(A). Show that A has no nilpotent elements. If each $A_\mathfrak{p}$ is an integral domain, must A be an integral domain?