Homework IV Solutions
Problems 5 and 7

5) Show that for an Artinian ring A, the spectrum $\mathrm{Spec}(A)$ is discrete and finite.

We have seen in the lectures that (i) an Artinian ring has finitely many maximal ideals, (ii) every prime ideal in an Artinian ring is maximal.

Thus, from (i) and (ii), we see that $\mathrm{Spec}(A)$ is finite. Moreover, each point $\mathfrak{m} \in \mathrm{Spec}A$ is closed, since $\mathfrak{m} = \mathscr{V}(\mathfrak{m})$. The complement of each point is also closed, since it is a finite union of closed sets. Hence, $\mathrm{Spec}(A)$ has discrete topology.

Proof of (ii) is explicitly stated as Proposition 6.6 in the lecture notes. Proof of (i) appears implicitly in the proof of Theorem 6.7 but let's spell out a slight variation of that proof here (which does not use the Noetherian assumption):

Consider the set of all finite intersections $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r$ with $\mathfrak{m}_i$ maximal ideals in A. As this is an Artinian ring, this set has to have a minimal element (otherwise, we can form an infinite descending chain). Denote the minimal element as $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_s$. Now, given any maximal ideal $\mathfrak{m}$, we have

$$\mathfrak{m} \cap \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_s = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_s$$

by minimality. But, this means $\mathfrak{m} \supset \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_s$. This implies $\mathfrak{m} \supset \mathfrak{m}_i$ for some i, as otherwise, we can find $x_i \in \mathfrak{m}_i \setminus \mathfrak{m}$ for all $i = 1, \ldots, s$ and then the element $x = x_1 x_2 \ldots x_s \in \mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_s = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_s$ but not in $\mathfrak{m}$ since $\mathfrak{m}$ is prime, a contradiction. Thus, $\mathfrak{m} \supset \mathfrak{m}_i$ for some i, but then since $\mathfrak{m}_i$ is maximal, $\mathfrak{m} = \mathfrak{m}_i$.

7) Suppose that A is a ring with the property that $A_\mathfrak{p}$ has no nilpotent elements for all $\mathfrak{p} \in \mathrm{Spec}(A)$. Show that A has no nilpotent elements. If each $A_\mathfrak{p}$ is an integral domain, must A be an integral domain?

Suppose that A has a nilpotent element, that is $x \in A$ such that $x^n = 0$ for som $n > 1$. Let $\mathrm{ann}(x) = \{a \in A : ax = 0\}$ be the annihilator ideal of x. In fact, it is a proper ideal since $1 \notin \mathrm{ann}(x)$. Let $\mathfrak{m} \supset \mathrm{ann}(x)$ be a maximal ideal containing $\mathrm{ann}(x)$. Consider the localisation $A_\mathfrak{m}$. We claim that $\frac{x}{1} \neq 0 \in A_\mathfrak{m}$. Indeed, if $\frac{x}{1} = 0 \in A_\mathfrak{m}$, there exists $s \in A \setminus \mathfrak{m}$ such that $sx = 0$, but such $s \in \mathrm{ann}(x) \subset \mathfrak{m}$. Therefore, it follows that $\frac{x}{1} \neq 0 \in A_\mathfrak{m}$ but $(\frac{x}{1}) = \frac{x^n}{1} = 0$, hence $\frac{x}{1}$ is a nilpotent element in $A_\mathfrak{m}$, which is a contradiction to the hypothesis of the problem. Hence, A has no nilpotent elements.

Take any field k (for example $k = \mathbb{F}_2$) and consider $A = k \times k$. Then, $(1,0) \cdot (0,1) = 0$ hence A is not an integral domain, but the only prime ideals in A are $k \times (0)$ and $(0) \times k$ (since for any ideal $I \subset k \times k$, $I \cap k \times \{0\}$ and $I \cap \{0\} \times k$ are ideals), and their localisations are easily seen to be isomorphic to k, hence are integral domains (in fact, they are isomorphic to fields).