Homework V Solutions
Problems 3 and 4

3) Suppose that A is a finitely generated k-algebra and an integral domain. Show that $h(\mathfrak{m}) = \dim A$ for all maximal ideals $\mathfrak{m} \subset A$. (Hint: Show that you can reduce to the special case $A = k[X_1, \ldots, X_n]$.) Deduce that for a prime ideal $\mathfrak{p} \subset A$, one has $\dim A/\mathfrak{p} + \dim A_{\mathfrak{p}} = \dim A$.

By Noether normalisation theorem, we can find a polynomial algebra $k[X_1, \ldots, X_n] \subset A$ such that A is an integral extension of $k[X_1, \ldots, X_n]$, then by Cohen-Seidenberg theorem the intersection map gives a bijection between chains of prime ideals in A and chains of prime ideals in $k[X_1, \ldots, X_n]$, and maximal ideals in A are mapped to maximal ideals in $k[X_1, \ldots, X_n]$. Therefore, we can suppose that $A = k[X_1, \ldots, X_n]$.

We will next prove that $k[X_1, \ldots, X_n]$ is catenary: Let $\mathfrak{m}$ be any maximal ideal and suppose $(0) = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_d = \mathfrak{m}$ be a maximal chain of prime ideals in $k[X_1, \ldots, X_n]$ then $d = n$. Here by a maximal chain, it is meant that we cannot squeeze in more prime ideals in the chain.

We give a proof by induction on n. If $n = 1$, then $k[X]$ is a PID, and all the non-zero prime ideals are maximal, hence all the maximal chains of prime ideals are given by $(0) \subsetneq (f)$ for some irreducible polynomial f. Hence, $k[X]$ is catenary. Let $(0) = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_d = \mathfrak{m}$ be a maximal chain of prime ideals in $k[X_1, \ldots, X_n]$. Since $k[X_1, \ldots, X_n]$ is a UFD and $\mathfrak{p}_1$ has height 1, it follows that $\mathfrak{p}_1 = (f)$ for some irreducible polynomial (by the previous problem). As in the proof of Noether normalisation lemma, we can assume by a change of co-ordinates (which is an automorphism of the ring $k[X_1, \ldots, X_n]$) that f is monic in X_n. Now, consider the finitely generated k-algebra, $k[X_1, \ldots, X_n]/(f)$. We get the following chain of prime ideals in this ring

$$(0) \subsetneq \mathfrak{p}_2/(f) \subsetneq \ldots \subsetneq \mathfrak{p}_d/(f)$$

which is a maximal chain of prime ideals in this ring, since the original chain is a maximal chain of prime ideals. Now f is monic, hence $k[X_1, \ldots, X_n]/(f)$ is an integral extension of $k[X_1, \ldots, X_{n-1}]$ again by Cohen-Seidenberg, and by induction all the maximal chains of prime ideals has to have length $d - 1$. It follows that $d = n$, as required.

We now take A to be any finitely generated k-algebra and integral domain, and $\mathfrak{p}$ be prime ideal in A. Let $(0) = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_i = \mathfrak{p}$ be a chain of prime ideals computing $h(\mathfrak{p}) = \dim A_{\mathfrak{p}}$. On the other hand, let $\mathfrak{p} \subsetneq \mathfrak{p}_{i+1} \subsetneq \ldots \subsetneq \mathfrak{p}_d = \mathfrak{m}$ be a chain of prime ideals computing $\dim A/\mathfrak{p}$. Putting these together we get a maximal chain

$$(0) = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_d = \mathfrak{m}$$

of prime ideals in A with $\mathfrak{p}_i = \mathfrak{p}$. By the catenary property proved before, it follows that $d = \dim A$, hence $\dim A/\mathfrak{p} + \dim A_{\mathfrak{p}} = \dim A$ as required.

4) Let $I = (f_1, \ldots, f_r) \subset k[X_1, \ldots, X_n]$, then for any irreducible component W of $V = \mathscr{V}(I)$, one has $\dim(W) \geq n - r$.

In a Noetherian ring, irreducible components of $\mathscr{V}(I)$ correspond to (finitely many) prime ideals $\mathfrak{p} \supset I$ which are minimal among prime ideals containing I. Indeed, we have $\sqrt{I} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_r$ where $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$ are minimal prime ideals containing I and $\mathscr{V}(I) = \mathscr{V}(\mathfrak{p}_1) \cup \ldots \cup \mathscr{V}(\mathfrak{p}_r)$ (see lecture notes).

Now, let $\mathfrak{p}$ be one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, and $W = \mathscr{V}(\mathfrak{p})$. We have

$$\dim(W) \coloneqq \dim k[X_1, \ldots, X_n]/\mathfrak{p} = n - h(\mathfrak{p})$$

where the first equality is by definition, and the second equality is by the previous problem. Finally, by Krull's height theorem $h(\mathfrak{p}) \leq r$, hence it follows that $\dim W \geq n-r$ as required.

If we wanted to think about the affine varieties in the case k is algebraically closed, we can consider the (geometric) decomposition into irreducible affine varieties $\mathcal{V}(I) = \mathcal{V}(\mathfrak{p}_1) \cup \ldots \cup \mathcal{V}(\mathfrak{p}_r)$ and take $W = \mathcal{V}(\mathfrak{p})$. Then we appeal to Theorem 6.24 in the lecture notes, to say that $\dim(W) = \dim k[X_1, \ldots, X_n]/\mathfrak{p}$ and the rest of the proof is the same.