
Lecture 9. Dimension of a variety; towards the Nullstellensatz

The Hilbert polynomial provides one computable way of defining the dimension of a variety:

Definition 4.17. Let $V = \mathcal{V}(I)$ be an affine variety defined by an ideal $I \subset k[X_1, \dots, X_n]$. If $V \neq \emptyset$, then $\mathcal{I}(V) \neq k[X_1, \dots, X_n]$ and $HP_{\mathcal{I}(V)} \neq 0$. We define

$$\dim(V) := \deg HP_{\mathcal{I}(V)}(t).$$

$\triangleleft$ Suppose that $V \subset k^n$ is a linear subspace, that is,

$$V = \mathcal{V}(\{f_j = \sum_{i=1}^n a_{ij}X_i : 1 \leq j \leq m\})$$

for some $a_{ij} \in k$. Show that $\dim_k V = \deg HP_I(t)$, where $I = (f_1, f_2, \dots, f_m)$.

The previous exercise gives a justification for the definition of the dimension as it generalizes the notion of dimension for linear spaces. Here is a more geometric justification that applies in general.

Corollary 4.18. Let $I \subset k[X_1, \dots, X_n]$ be a proper ideal. Then $\deg HP_I(t)$ is the largest $d \geq 0$ such that

$$I \cap k[X_{i_1}, \dots, X_{i_d}] = \{0\} \quad \text{for some } 1 \leq i_1 < i_2 < \dots < i_d \leq n.$$

(For $d = 0$ the condition reads $I \cap k = \{0\}$, which holds since I is proper.)

In other words, $\deg HP_I(t)$ is the largest number of variables $X_{i_1}, \dots, X_{i_d}$ that are “algebraically independent” modulo I , that is, satisfy no non-trivial polynomial relation in $k[X_1, \dots, X_n]/I$. Geometrically, these are d independent directions in which to project $\mathcal{V}(I) \subset k^n$ to k^d . After a linear change of co-ordinates (for k infinite) one can further show that the projection is onto with finite fibres; this is the Noether normalisation theorem, which we will see later.

Proof. In the proof of Theorem 4.16, we have seen that

$$\deg HP_I(t) = \max\{n - |J| : \exists (J, \tau) \text{ with } C(J, \tau) \subset C(I)\}.$$

We show first that if $C(J, \tau) \subset C(I)$ then $I \cap k[X_j : j \notin J] = \{0\}$. Suppose that $0 \neq f \in I \cap k[X_j : j \notin J]$ and write $LM(f) = X^\alpha$. Then $\alpha_j = 0$ for all $j \in J$, so $\gamma + \alpha \in C(J, \tau) \subset C(I)$ for all $\gamma \in C(J, \tau)$. On the other hand, $X^{\gamma+\alpha} = X^\gamma LM(f) \in \mathfrak{in}(I)$, which contradicts $\gamma + \alpha \in C(I)$. Taking (J, τ) with $n - |J| = \deg HP_I(t)$ and $\{i_1, \dots, i_d\} = \{1, \dots, n\} \setminus J$, we see that the maximum in the corollary is at least $\deg HP_I(t)$.

Conversely, suppose that $1 \leq i_1 < \dots < i_d \leq n$ and $I \cap k[X_{i_1}, \dots, X_{i_d}] = \{0\}$. We will show that $d \leq \deg HP_I(t)$. Consider the ring homomorphisms

$$k[X_{i_1}, X_{i_2}, \dots, X_{i_d}] \rightarrow k[X_1, \dots, X_n] \rightarrow k[X_1, \dots, X_n]/I.$$

By assumption, the kernel of the composition is trivial. It follows that for each $s \geq 0$, the composition of the k -linear maps

$$k[X_{i_1}, X_{i_2}, \dots, X_{i_d}]_{\leq s} \rightarrow k[X_1, \dots, X_n]_{\leq s} \rightarrow k[X_1, \dots, X_n]_{\leq s} / I_{\leq s}$$

is also injective. Hence

$$HF_I(s) \geq \dim_k k[X_{i_1}, \dots, X_{i_d}]_{\leq s} = \binom{d+s}{s} = \frac{1}{d!}(s^d + \dots).$$

Since $HF_I(s) = HP_I(s)$ for s sufficiently large, it follows that $d \leq \deg HP_I(t)$. $\square$

► We have seen before that the initial ideal of $I = (Y - X^2, Z - X^3) \subset k[X, Y, Z]$ with respect to $\leq_{grlex}$ is given by $\mathbf{in}(I) = (X^2, XY, XZ, Y^3)$. Now, it is clear that $\mathbf{in}(I) \cap k[Z] = \{0\}$ and $\mathbf{in}(I) \cap k[X, Z]$, $\mathbf{in}(I) \cap k[Y, Z]$ and $\mathbf{in}(I) \cap k[X, Y]$ are all non-trivial. Hence $\dim V(I) = 1$.

◁ For $I = (Y - X^2, Z - X^3) \subset k[X, Y, Z]$. Show directly that $I \cap k[X, Y]$, $I \cap k[X, Z]$ and $I \cap k[Y, Z]$ are non-trivial.

Here is how you may compute the Hilbert polynomial in Macaulay2:

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R = QQ[x,y,z,w];
I = ideal(y-x^2,z-x^3);
inI = ideal leadTerm I;
hilbertPolynomial(inI, Projective=>false)
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This code computes the Hilbert polynomial of $I = (Y - X^2, Z - X^3) \subset \mathbb{Q}[X, Y, Z]$. However, we added one more variable to the ring in the code. This is because by default Macaulay2 computes projective Hilbert polynomial of a homogeneous ideal. Our trick of adding one more variable “homogenises” the initial ideal. It is easy to see that computing the projective Hilbert polynomial of the homogenised ideal gives the same answer as the Hilbert polynomial of the original ideal.

4.5 Nullstellensatz: theorem of zeros

Recall that $k[X]$ is a PID, hence any ideal is of the form $I = (f)$ for some $f \in k[X]$. The affine variety $\mathcal{V}(I)$ is thus simply the set of roots of f . If k is algebraically closed, then every non-constant polynomial f has a root (this is the definition of algebraically closed; for $k = \mathbb{C}$ it is the fundamental theorem of algebra). Therefore $\mathcal{V}(I) = \emptyset$ if and only if $I = k[X]$: any proper ideal corresponds to a non-empty affine variety. The assumption that k is algebraically closed is necessary, as the example $I = (X^2 + 1) \subset \mathbb{R}[X]$ shows.

An important result of Hilbert, known by the German name **Nullstellensatz** (literally “theorem of the zero places”), says that the same holds for polynomial rings in several variables.

Theorem 4.19. (*Weak Nullstellensatz*) *If k is algebraically closed and $I \subset k[X_1, \dots, X_n]$ is a proper ideal, then $\mathcal{V}(I) \neq \emptyset$.*

There are various equivalent formulations (and proofs) of the Weak Nullstellensatz, as well as a Strong Nullstellensatz. We will cover some of these. The approach that I find most worthwhile goes via the Noether normalisation lemma, which is an important structural result for finitely generated algebras over k . We will spend some time developing the notion of “integral dependence” before giving a proof of the Nullstellensatz.