Lecture 16. The spectrum of a ring

6 Abstract commutative algebra

We now begin studying commutative rings via more abstract methods. In particular, we leave the constructive mindset behind for now and admit the consequences of Zorn's lemma. We generally have Noetherian rings in mind, but we will not assume this at the outset. The first question we address is the following: given a general commutative ring A, can we define a space on which A is the ring of algebraic functions?

6.1 $\mathrm{Spec}(A)$

Let A be a commutative ring with unity. The **maximal spectrum** of A is

$$\mathrm{mSpec}(A) \coloneqq \{\mathfrak{m} \subset A : \mathfrak{m} \text{ is a maximal ideal}\}.$$

The constructions $\mathcal{V}$ and $\mathcal{I}$ have the following analogues. For an ideal $I \subset A$, define

$$\mathcal{V}(I) \coloneqq \{\mathfrak{m} \in \mathrm{mSpec}(A) : I \subset \mathfrak{m}\}$$

and for $S \subset \mathrm{mSpec}(A)$

$$\mathcal{I}(S) \coloneqq \bigcap_{\mathfrak{m} \in S} \mathfrak{m}.$$

If $A = k[X_1, X_2, \ldots, X_n]$ and k is algebraically closed, then the weak Nullstellensatz (Theorem 4.29) identifies $\mathrm{mSpec}(A)$ with k^n, and these constructions coincide with the geometric ones defined before.

For a general commutative ring, it turns out that using prime ideals is more fruitful. The main problem with the maximal spectrum is that for a ring homomorphism $f : A \to B$ and a maximal ideal $\mathfrak{m} \subset B$, the ideal $f^{-1}(\mathfrak{m})$ need not be maximal: for the inclusion $\mathbb{Z} \to \mathbb{Q}$ and $\mathfrak{m} = (0)$ we get $f^{-1}(\mathfrak{m}) = (0)$. By Proposition 2.5, on the other hand, $f^{-1}(\mathfrak{p})$ is prime whenever $\mathfrak{p}$ is prime.

The **spectrum** of the ring A is

$$\mathrm{Spec}(A) \coloneqq \{\mathfrak{p} \subset A : \mathfrak{p} \text{ is a prime ideal}\}.$$

The analogues of the constructions $\mathcal{V}$ and $\mathcal{I}$ are as follows. For an ideal $I \subset A$, define

$$\mathscr{V}(I) \coloneqq \{\mathfrak{p} \in \mathrm{Spec}(A) : I \subset \mathfrak{p}\}$$

and for $S \subset \mathrm{Spec}(A)$

$$\mathscr{I}(S) \coloneqq \bigcap_{\mathfrak{p} \in S} \mathfrak{p}.$$

We changed the notation from $\mathcal{V}, \mathcal{I}$ to $\mathscr{V}, \mathscr{I}$ since the new constructions do not reproduce the previous ones when $A = k[X_1, X_2, \ldots, X_n]$, even when k is algebraically closed: by Lemma 4.39, $\mathscr{V}(I)$ is in bijection with the irreducible affine varieties contained in $\mathcal{V}(I)$, not with its points.

▶ If $I_1, \ldots, I_r \subset A$ are ideals, then $\mathscr{V}(I_1 \cap \ldots \cap I_r) = \mathscr{V}(I_1) \cup \ldots \cup \mathscr{V}(I_r)$: a prime ideal containing $I_1 \cap \ldots \cap I_r \supset I_1 \cdots I_r$ contains some I_i.

▶ If $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals in A, then $\mathscr{V}(\sum_{\lambda \in \Lambda} I_\lambda) = \bigcap_{\lambda \in \Lambda} \mathscr{V}(I_\lambda)$, since an ideal contains $\sum_\lambda I_\lambda$ if and only if it contains every I_λ.

▶ $\mathscr{V}(\{0\}) = \operatorname{Spec}(A)$ and $\mathscr{V}(A) = \emptyset$.

These properties imply that $\operatorname{Spec}(A)$ is a topological space with closed sets given by $\mathscr{V}(I)$ for I an ideal in A. This is called the (abstract) **Zariski topology**. Note that $\mathscr{V}(I) = \mathscr{V}(\sqrt{I})$, since prime ideals are radical. For $f \in A$, the open set

$$D(f) \coloneqq \operatorname{Spec}(A) \setminus \mathscr{V}((f)) = \{\mathfrak{p} \in \operatorname{Spec}(A) : f \notin \mathfrak{p}\}$$

is called a **basic open set**; since $\mathscr{V}(I) = \bigcap_{f \in I} \mathscr{V}((f))$, every open set is a union of basic open sets. The sets $\mathcal{V}(I) = \mathscr{V}(I) \cap \operatorname{mSpec}(A)$ are likewise the closed sets of a topology on $\operatorname{mSpec}(A)$, namely the subspace topology induced from $\operatorname{Spec}(A)$; for $A = k[X_1, \ldots, X_n]$ with k algebraically closed, this is the Zariski topology on k^n.

◁ Suppose M is a square matrix with entries in $\mathbb{C}$. Consider the $\mathbb{C}$-algebra $\mathbb{C}[M] \coloneqq \{f(M) : f \in \mathbb{C}[X]\}$, with matrix multiplication as the product. Show that $\operatorname{Spec}(\mathbb{C}[M])$ is in bijection with the set of eigenvalues of M. This is where the name "spectrum" comes from: the set of eigenvalues of a Hermitian operator is called its spectrum in quantum mechanics, a term that goes back to the spectrum of light.

◁ By Proposition 2.5, a ring homomorphism $f : A \to B$ induces a map $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$, $\mathfrak{q} \mapsto f^{-1}(\mathfrak{q})$. Show that f^* is continuous in the Zariski topology. (Hint: $(f^*)^{-1}(D(a)) = D(f(a))$ for $a \in A$.)

◁ Show that the quotient map $A \to A/\sqrt{(0)}$, where $\sqrt{(0)}$ is the ideal of nilpotent elements of A, called the **nilradical**, induces a homeomorphism between the prime spectra.

Lemma 6.1. *For $S \subset \operatorname{Spec}(A)$ we have $\mathscr{V}(\mathscr{I}(S)) = \overline{S}$.*

Proof. Recall that $\overline{S}$ is the smallest closed set containing S. Every $\mathfrak{p} \in S$ contains $\mathscr{I}(S)$, so $\mathscr{V}(\mathscr{I}(S))$ is a closed set containing S, and hence $\overline{S} \subset \mathscr{V}(\mathscr{I}(S))$. Conversely, write $\overline{S} = \mathscr{V}(I)$ for an ideal I. Then $I \subset \mathfrak{p}$ for all $\mathfrak{p} \in S$, so $I \subset \mathscr{I}(S)$ and $\mathscr{V}(\mathscr{I}(S)) \subset \mathscr{V}(I) = \overline{S}$. □

Lemma 6.2. *Let $I \subset A$ be an ideal. Then $\mathscr{I}(\mathscr{V}(I)) = \sqrt{I}$.*

Proof. This is Proposition 2.7. □

Together, the two lemmas say that $I \mapsto \mathscr{V}(I)$ is a bijection between the radical ideals of A and the closed subsets of $\mathrm{Spec}(A)$, with inverse $C \mapsto \mathscr{I}(C)$, for every commutative ring A. Lemma 6.2 is only a formal analogue of the Nullstellensatz, and much easier: the Nullstellensatz says that for $A = k[X_1, \ldots, X_n]$ with k algebraically closed it suffices to intersect the maximal ideals,

$$\mathcal{I}(\mathcal{V}(I)) = \bigcap_{I \subset \mathfrak{m}} \mathfrak{m} = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p} = \mathscr{I}(\mathscr{V}(I)).$$

The two intersections coincide for Jacobson rings (by definition) but not in general. By Lemma 6.1, the closure in $\mathrm{Spec}(A)$ of the set of maximal ideals containing I is $\mathscr{V}(\bigcap_{I \subset \mathfrak{m}} \mathfrak{m})$. So A is Jacobson if and only if every closed subset of $\mathrm{Spec}(A)$ is the closure of the maximal ideals it contains, and then $C \mapsto C \cap \mathrm{mSpec}(A)$ is a bijection between the closed subsets of $\mathrm{Spec}(A)$ and of $\mathrm{mSpec}(A)$. For $A = k[[X]]$ this fails: the closed sets $\{(X)\}$ and $\mathrm{Spec}(A) = \{(0), (X)\}$ have the same trace on $\mathrm{mSpec}(A) = \{(X)\}$.

Question: How can A be viewed as functions on $\mathrm{Spec}(A)$?

For $f \in A$ and $\mathfrak{p} \in \mathrm{Spec}(A)$ we define the "value" of f at $\mathfrak{p}$ as follows. Since $\mathfrak{p}$ is a prime ideal, $A/\mathfrak{p}$ is an integral domain, and we let $k(\mathfrak{p}) = \mathrm{Frac}(A/\mathfrak{p})$ be its fraction field, called the **residue field** at $\mathfrak{p}$. Then $f(\mathfrak{p})$ is the image of f under the composition of the ring homomorphisms

$$A \to A/\mathfrak{p} \to k(\mathfrak{p}).$$

Thus f can be "evaluated" at the point $\mathfrak{p}$, and $f(\mathfrak{p}) = 0$ if and only if $f \in \mathfrak{p}$, that is, if and only if $\mathfrak{p} \in \mathscr{V}((f))$. Note that $f(\mathfrak{p}) = 0$ for all $\mathfrak{p}$ if and only if $f \in \sqrt{(0)}$ by Proposition 2.7: for instance, in $A = k[X]/(X^2)$ the class of X is a non-zero element whose values are all zero. So an element of A is not determined by its values.

Conversely, the prime ideals of A are exactly the kernels of ring homomorphisms $A \to K$ to fields K: such a kernel is prime, since K is an integral domain, and $\mathfrak{p}$ is the kernel of $A \to k(\mathfrak{p})$. Thus $\mathrm{Spec}(A)$ has a point for every possible evaluation homomorphism from A to a field.

In the geometric case $A = k[X_1, \ldots, X_n]/I$ with k algebraically closed, the maximal ideals are the images $\mathfrak{m}$ of the ideals $(X_1 - a_1, \ldots, X_n - a_n)$ with $(a_1, \ldots, a_n) \in \mathcal{V}(I)$, and $k(\mathfrak{m}) = A/\mathfrak{m} = k$, with $A \to k(\mathfrak{m})$ the evaluation homomorphism $f \mapsto f(a_1, \ldots, a_n)$. So on the maximal ideals we recover the usual functions on $\mathcal{V}(I)$.

In general, an element of A is a generalised function on $\mathrm{Spec}(A)$, whose values lie in fields $k(\mathfrak{p})$ that vary with the point $\mathfrak{p}$. This is the starting point of Grothendieck's scheme theory. You should take a course on Algebraic Geometry to see how this idea develops further.

▶ Consider the case $A = \mathbb{Z}$. The points of $\mathrm{Spec}(\mathbb{Z})$ are the maximal ideals (p), for p a prime number, and the prime ideal (0), which is not maximal. An element $n \in \mathbb{Z}$ is the function

$$(0) \mapsto n \in \mathbb{Q}, \qquad (p) \mapsto n \bmod p \in \mathbb{F}_p,$$

see Figure 9. The closed sets are $\mathscr{V}((n)) = \{(p) : p \mid n\}$ for $n \neq 0$, which are the finite sets of maximal ideals, and $\mathscr{V}((0)) = \mathrm{Spec}(\mathbb{Z})$. In particular, the only closed set containing the point (0) is the whole space.

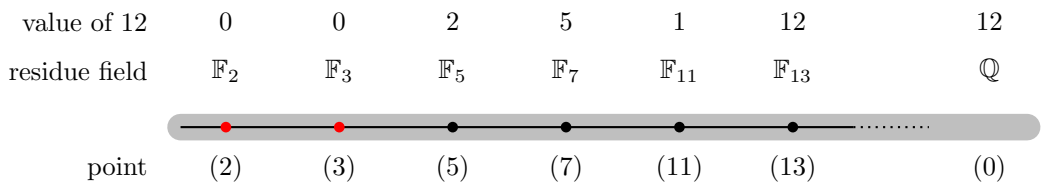


Figure 9: Spec($\mathbb{Z}$): the closed points (p) and the point (0), drawn as a grey band because its closure is the whole space. The element 12 takes values in varying fields and vanishes exactly on $\mathscr{V}((12)) = \{(2), (3)\}$, in red.

◁ Let A be an integral domain. Prove that the closure of the point (0) is all of Spec(A). The point (0) is called the **generic point** of Spec(A).

◁ Prove that Spec($\mathbb{Z}$) and Spec($\overline{\mathbb{Q}}[X]$) are homeomorphic.

Definition 6.3. *Let X be a topological space.*

(a) We say that X is **Noetherian** *if every chain $Y_1 \supset Y_2 \supset \dots$ of closed subsets of X stabilises: there exists N such that $Y_N = Y_{N+i}$ for all $i > 0$.*

(b) We say that X is **reducible** *if $X = X_1 \cup X_2$ with X_1, X_2 proper closed subsets of X, and* **irreducible** *if it is non-empty and not reducible. A subset of X is called irreducible if it is so in the subspace topology.*

The following statements are proved in exactly the same way as Corollary 4.37 and Lemma 4.39.

▶ If A is a Noetherian ring, then Spec(A) is a Noetherian topological space: by Lemma 6.2, a chain $\mathscr{V}(I_1) \supset \mathscr{V}(I_2) \supset \dots$ of closed subsets gives a chain $\sqrt{I_1} \subset \sqrt{I_2} \subset \dots$ of ideals.

▶ A closed subset $Y \subset$ Spec(A) is irreducible if and only if $\mathscr{I}(Y)$ is prime. Hence, by Lemma 6.1, the irreducible closed subsets of Spec(A) are exactly the closures $\overline{\{\mathfrak{p}\}} = \mathscr{V}(\mathfrak{p})$ of its points; the point $\mathfrak{p}$ is the generic point of $\mathscr{V}(\mathfrak{p})$, see Figure 10.

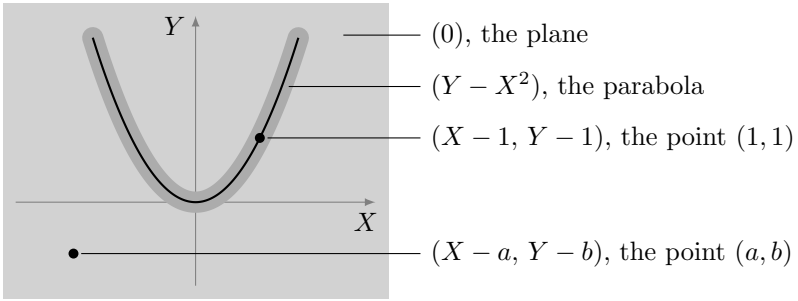


Figure 10: Some points of Spec($k[X, Y]$) for k algebraically closed, drawn as the irreducible affine varieties they correspond to: closed points, the point $(Y - X^2)$, whose closure is itself together with the points of the parabola, and the point (0), whose closure is the whole space.

► If X is a Noetherian topological space, then $X = Y_1 \cup Y_2 \cup \ldots \cup Y_r$ with Y_i closed and irreducible subsets.

Proposition 6.4. *Let A be a Noetherian ring and $I \subset A$ a proper ideal. Then there exist finitely many prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$ with $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for $i \neq j$ such that $\sqrt{I} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_r$. Moreover, if $\mathfrak{q} \in \mathrm{Spec}(A)$ with $I \subset \mathfrak{q}$, then there exists i such that $\mathfrak{p}_i \subset \mathfrak{q}$. In particular, A has only finitely many minimal prime ideals.*