Lecture 16. The spectrum of a ring

6 Abstract commutative algebra

We now begin studying commutative rings via more abstract methods. In particular, we leave the constructive mindset behind for now and admit the consequences of Zorn's lemma. We generally have Noetherian rings in mind, but we will not assume this at the outset. The first question we address is the following: given a general commutative ring A, can we define a space on which A is the ring of algebraic functions?

6.1 $\mathrm{Spec}(A)$

Let A be a commutative ring with unity. The **maximal spectrum** of A is

$$\mathrm{mSpec}(A) \coloneqq \{\mathfrak{m} \subset A : \mathfrak{m} \text{ is a maximal ideal}\}.$$

The constructions $\mathcal{V}$ and $\mathcal{I}$ have the following analogues. For an ideal $I \subset A$, define

$$\mathcal{V}(I) \coloneqq \{\mathfrak{m} \in \mathrm{mSpec}(A) : I \subset \mathfrak{m}\}$$

and for $S \subset \mathrm{mSpec}(A)$

$$\mathcal{I}(S) \coloneqq \bigcap_{\mathfrak{m} \in S} \mathfrak{m}.$$

If $A = k[X_1, X_2, \ldots, X_n]$ and k is algebraically closed, then the weak Nullstellensatz (Theorem 4.29) identifies $\mathrm{mSpec}(A)$ with k^n, and these constructions coincide with the geometric ones defined before.

For a general commutative ring, it turns out that using prime ideals is more fruitful. The main problem with the maximal spectrum is that for a ring homomorphism $f : A \to B$ and a maximal ideal $\mathfrak{m} \subset B$, the ideal $f^{-1}(\mathfrak{m})$ need not be maximal: for the inclusion $\mathbb{Z} \to \mathbb{Q}$ and $\mathfrak{m} = (0)$ we get $f^{-1}(\mathfrak{m}) = (0)$. By Proposition 2.5, on the other hand, $f^{-1}(\mathfrak{p})$ is prime whenever $\mathfrak{p}$ is prime.

The **spectrum** of the ring A is

$$\mathrm{Spec}(A) \coloneqq \{\mathfrak{p} \subset A : \mathfrak{p} \text{ is a prime ideal}\}.$$

The analogues of the constructions $\mathcal{V}$ and $\mathcal{I}$ are as follows. For an ideal $I \subset A$, define

$$\mathscr{V}(I) \coloneqq \{\mathfrak{p} \in \mathrm{Spec}(A) : I \subset \mathfrak{p}\}$$

and for $S \subset \mathrm{Spec}(A)$

$$\mathscr{I}(S) \coloneqq \bigcap_{\mathfrak{p} \in S} \mathfrak{p}.$$

We changed the notation from $\mathcal{V}, \mathcal{I}$ to $\mathscr{V}, \mathscr{I}$ since the new constructions do not reproduce the previous ones when $A = k[X_1, X_2, \ldots, X_n]$, even when k is algebraically closed: by Lemma 4.39, $\mathscr{V}(I)$ is in bijection with the irreducible affine varieties contained in $\mathcal{V}(I)$, not with its points.

▶ If $I_1, \ldots, I_r \subset A$ are ideals, then $\mathscr{V}(I_1 \cap \ldots \cap I_r) = \mathscr{V}(I_1) \cup \ldots \cup \mathscr{V}(I_r)$: a prime ideal containing $I_1 \cap \ldots \cap I_r \supset I_1 \cdots I_r$ contains some I_i.

▶ If $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals in A, then $\mathscr{V}(\sum_{\lambda \in \Lambda} I_\lambda) = \bigcap_{\lambda \in \Lambda} \mathscr{V}(I_\lambda)$, since an ideal contains $\sum_\lambda I_\lambda$ if and only if it contains every I_λ.

▶ $\mathscr{V}(\{0\}) = \operatorname{Spec}(A)$ and $\mathscr{V}(A) = \emptyset$.

These properties imply that $\operatorname{Spec}(A)$ is a topological space with closed sets given by $\mathscr{V}(I)$ for I an ideal in A. This is called the (abstract) **Zariski topology**. Note that $\mathscr{V}(I) = \mathscr{V}(\sqrt{I})$, since prime ideals are radical. For $f \in A$, the open set

$$D(f) \coloneqq \operatorname{Spec}(A) \setminus \mathscr{V}((f)) = \{\mathfrak{p} \in \operatorname{Spec}(A) : f \notin \mathfrak{p}\}$$

is called a **basic open set**; since $\mathscr{V}(I) = \bigcap_{f \in I} \mathscr{V}((f))$, every open set is a union of basic open sets.

◁ Suppose M is a square matrix with entries in $\mathbb{C}$. Consider the $\mathbb{C}$-algebra $\mathbb{C}[M] \coloneqq \{f(M) : f \in \mathbb{C}[X]\}$, with matrix multiplication as the product. Show that $\operatorname{Spec}(\mathbb{C}[M])$ is in bijection with the set of eigenvalues of M. This is where the name "spectrum" comes from: the set of eigenvalues of a Hermitian operator is called its spectrum in quantum mechanics, a term that goes back to the spectrum of light.

◁ By Proposition 2.5, a ring homomorphism $f : A \to B$ induces a map $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$, $\mathfrak{q} \mapsto f^{-1}(\mathfrak{q})$. Show that f^* is continuous in the Zariski topology. (Hint: $(f^*)^{-1}(D(a)) = D(f(a))$ for $a \in A$.)

◁ Show that the quotient map $A \to A/\sqrt{(0)}$, where $\sqrt{(0)}$ is the ideal of nilpotent elements of A, called the **nilradical**, induces a homeomorphism between the prime spectra.

Lemma 6.1. *For $S \subset \operatorname{Spec}(A)$ we have $\mathscr{V}(\mathscr{I}(S)) = \overline{S}$.*

Proof. Recall that $\overline{S}$ is the smallest closed set containing S. Every $\mathfrak{p} \in S$ contains $\mathscr{I}(S)$, so $\mathscr{V}(\mathscr{I}(S))$ is a closed set containing S, and hence $\overline{S} \subset \mathscr{V}(\mathscr{I}(S))$. Conversely, write $\overline{S} = \mathscr{V}(I)$ for an ideal I. Then $I \subset \mathfrak{p}$ for all $\mathfrak{p} \in S$, so $I \subset \mathscr{I}(S)$ and $\mathscr{V}(\mathscr{I}(S)) \subset \mathscr{V}(I) = \overline{S}$. □

Lemma 6.2. *Let $I \subset A$ be an ideal. Then $\mathscr{I}(\mathscr{V}(I)) = \sqrt{I}$.*

Proof. This is Proposition 2.7. □

Together, the two lemmas say that $I \mapsto \mathscr{V}(I)$ is a bijection between the radical ideals of A and the closed subsets of $\operatorname{Spec}(A)$, with inverse $C \mapsto \mathscr{I}(C)$, for every commutative ring A. Lemma 6.2 is only a formal analogue of the Nullstellensatz, and much easier: the Nullstellensatz

says that for $A = k[X_1, \ldots, X_n]$ with k algebraically closed it suffices to intersect the maximal ideals,

$$\mathcal{I}(\mathcal{V}(I)) = \bigcap_{I \subset \mathfrak{m}} \mathfrak{m} = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p} = \mathscr{I}(\mathscr{V}(I)).$$

The two intersections coincide for Jacobson rings (by definition) but not in general.

Question: How can A be viewed as functions on Spec$A$?

Let $f \in A$ and $\mathfrak{p} \in \text{Spec}A$, we define the "value" of f at $\mathfrak{p}$ as follows. Since $\mathfrak{p}$ is a prime ideal, $A/\mathfrak{p}$ is an integral domain, so we let $k(\mathfrak{p}) = Frac(A/\mathfrak{p})$ be the fraction field. This field is called the **residue field** at $\mathfrak{p}$. Then, we define $f(\mathfrak{p})$ as the image of f under the composition of the ring homomorphisms:

$$A \to A/\mathfrak{p} \to k(\mathfrak{p})$$

Thus, an element $f \in A$ can be "evaluated" at a point $\mathfrak{p}$ as $f(\mathfrak{p}) \in k(\mathfrak{p})$. Moreover, $f(\mathfrak{p}) = 0$ if and only $(f) \subset \mathfrak{p}$, or equivalently $\mathfrak{p} \in \mathscr{V}((f))$.

Conversely, one can characterize prime ideals in A as the kernel of homomorphisms $A \to k$ where k can be arbitrary field. Thus, we add a point $\mathfrak{p}$ for every possible evaluation homomorphism from A to some field $k(\mathfrak{p})$.

In the geometric case, when $A = k[X_1, \ldots, X_n]/I$ is a finitely generated k-algebra and k is algebraically closed, a maximal ideal is given by $\mathfrak{m} = (X_1 - a_1, X_2 - a_2, \ldots, X_n - a_n)$ and one can canonically identify $k(\mathfrak{m}) = k[X_1, \ldots, X_n]/\mathfrak{m}$ with k via the homomorphism $A \to k$ that corresponds to the evaluation homomorphism

$$A \to k \quad , \quad f \to f(a_1, a_2, \ldots, a_n)$$

So the above assignment gives back the usual notion of functions on $\mathcal{V}(I)$ under the bijection between maximal ideals in A and the points of $\mathcal{V}(I)$.

In general, the construction views A as a generalized function on Spec(A) since it possibly takes values in different fields $k(\mathfrak{p})$ depending on the point $\mathfrak{p}$. This is the starting point of Grothendieck's scheme theory. You should take a course on Algebraic Geometry to see how this idea develops further.

▶ Let's consider the case $A = \mathbb{Z}$. Then, Spec$(\mathbb{Z})$ consist of maximal ideal (p) for prime $p \in \mathbb{Z}$ and the non-maximal, prime ideal (0). An element $n \in \mathbb{Z}$ is viewed as a function which sends

$$(0) \to n \in \mathbb{Q} \quad , \quad (p) \to n \pmod{p} \in \mathbb{F}_p$$

◁ Let A be an integral domain. Prove that $\{(0)\}$ is a dense point in Spec(A) in the Zariski topology.

◁ Prove that Spec$\mathbb{Z}$ and Spec$\overline{\mathbb{Q}}[X]$ are homeomorphic.

Definition 6.3. *Suppose X is a topological space.*

(a) We say that X is Noetherian if all the chains $Y_1 \supset Y_2 \supset \dots$ with $Y_i \subset X$ closed subspaces stabilise: There exists N such that $Y_N = Y_{N+i}$ for all $i > 0$.

(b) We say that X is reducible if $X = X_1 \cup X_2$ with X_1, X_2 proper closed subspaces of X. Otherwise, we say X is irreducible.

The following statements are proven exactly the same way as in Corollary 4.37 and Lemma 4.39.

▶ Let A be a Noetherian ring, then $X = \mathrm{Spec}A$ is a Noetherian topological space.

▶ Let $Y \subset \mathrm{Spec}(A)$ for a ring A be a closed subset. Then, Y is irreducible if and only if $\mathscr{I}(Y)$ is prime.

▶ If X is a Noetherian topological space, then $X = Y_1 \cup Y_2 \cup \dots \cup Y_r$ with Y_i closed and irreducible subspaces.

▶ If A is a Noetherian ring and I is a proper ideal. Then there exist a finite number of prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r$ with $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for $i \neq j$ such that $\sqrt{I} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \dots \cap \mathfrak{p}_r$. Moreover, if $\mathfrak{q} \in \mathrm{Spec}(A)$ with $I \subset \mathfrak{q}$, then there exists i such that $\mathfrak{p}_i \subset \mathfrak{q}$. In particular, A contains finitely many minimal primes.