Lecture 17. Krull dimension; Artinian rings

6.2 Dimension of a commutative ring

Definition 6.5. *Let X be a topological space. We define*

$$\dim X \coloneqq \sup\{n \geq 0 : \exists\ X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n \subset X \text{ with } X_i \text{ closed and irreducible}\}.$$

For a ring A, the **Krull dimension** *is defined to be*

$$\dim A \coloneqq \dim \operatorname{Spec}(A) = \sup\{n \geq 0 : \exists\ \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n \text{ with } \mathfrak{p}_i \subset A \text{ prime}\}.$$

For a prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$, we define its **height** *as*

$$h(\mathfrak{p}) \coloneqq \sup\{n \geq 0 : \exists\ \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n = \mathfrak{p} \text{ with } \mathfrak{p}_i \text{ prime}\}.$$

The two descriptions of $\dim A$ agree because the irreducible closed subsets of $\operatorname{Spec}(A)$ are the sets $\mathscr{V}(\mathfrak{p})$, and $\mathscr{V}(\mathfrak{p}) \subsetneq \mathscr{V}(\mathfrak{q})$ if and only if $\mathfrak{q} \subsetneq \mathfrak{p}$. In Figure 10, the chain $(0) \subsetneq (Y - X^2) \subsetneq (X - 1, Y - 1)$ is the chain point $\subset$ parabola $\subset$ plane.

▶ An integral domain A has $\dim A = 0$ if and only if it is a field: (0) is a prime ideal, and it is the only one exactly when it is maximal.

▶ Let $A = k[X_1, \ldots, X_n]$. The chain of prime ideals $(0) \subsetneq (X_1) \subsetneq (X_1, X_2) \subsetneq \ldots \subsetneq (X_1, X_2, \ldots, X_n)$ shows that $\dim A \geq n$. We will see later that in fact $\dim A = n$.

▶ Suppose A is a UFD and $\mathfrak{p} \subset A$ is a prime ideal. Then $h(\mathfrak{p}) = 1$ if and only if $\mathfrak{p} = (a)$ with $a \in A$ an irreducible element. In a PID every non-zero prime ideal is of this form, so $\dim \mathbb{Z} = 1$ and $\dim k[X] = 1$.

You can also use Macaulay2 to compute the Krull dimension of a ring:

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R = QQ[X,Y,Z]
dim R
R = QQ[X,Y]/(X^2,Y^3,X*Y^2)
dim R
R = ZZ[X]
dim R
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The answers are 3, 0 and 2; for the last one, note the chain $(0) \subsetneq (2) \subsetneq (2, X)$ in $\mathbb{Z}[X]$.

We next want to understand rings which have $\dim A = 0$ that are not fields. It turns out there is a nice characterisation of such rings in terms of chain conditions on their ideals.

Definition 6.6. *Let A be a ring and M an A-module.*

(a) We say that M is **Noetherian** *if every ascending chain of A-submodules*

$$M_1 \subset M_2 \subset \ldots \subset M$$

becomes stationary.

(b) We say that M is **Artinian** *if every descending chain of A-submodules*

$$M \supset M_1 \supset M_2 \supset \ldots$$

becomes stationary.

(c) A ring A is Noetherian or Artinian if A as an A-module is Noetherian or Artinian.

The A-submodules of A are just the ideals of A, so for rings this is the notion of Noetherian ring that we saw before.

$\lhd$ Show that if A is Noetherian and M is a finitely generated A-module, then M is Noetherian.

Despite the similarity of the definitions, being an Artinian ring is much more restrictive. For example, $\mathbb{Z}$ is Noetherian but not Artinian, since $(2) \supsetneq (4) \supsetneq (8) \supsetneq \ldots$, whereas a k-algebra which is finite-dimensional as a k-vector space is both, since its ideals are subspaces. Here is a first indication of how restrictive the condition is.

Proposition 6.7. *In an Artinian ring every prime ideal is maximal.*

Proof. Let A be an Artinian ring and $\mathfrak{p} \subset A$ a prime ideal. Then $A/\mathfrak{p}$ is an integral domain, and it is Artinian, since its ideals correspond to the ideals of A containing $\mathfrak{p}$. We show that it is a field. Let $a \in A/\mathfrak{p}$ be non-zero. The descending chain of ideals

$$(a) \supset (a^2) \supset \ldots$$

has to stabilise, so $(a^n) = (a^{n+1})$ for some $n > 0$. Hence $a^n = ba^{n+1}$ for some b, that is, $a^n(1 - ba) = 0$. Since $a^n \neq 0$ and $A/\mathfrak{p}$ is an integral domain, we deduce that $ba = 1$, so a is invertible. $\square$

We can now state the characterisation of 0-dimensional Noetherian rings.

Theorem 6.8. *Let A be a Noetherian ring. Then* $\dim A = 0$ *if and only if A is also Artinian.*

To prove that $\dim A = 0$ implies that A is Artinian, we need a few preliminary results.

Definition 6.9. *We say that an A-module M is* **simple** *if $M \neq \{0\}$ and $\{0\}$ and M are the only A-submodules of M.*

▶ The simple A-modules are, up to isomorphism, the modules $A/\mathfrak{m}$ with $\mathfrak{m} \subset A$ a maximal ideal. Indeed, the submodules of A/I correspond to the ideals of A containing I, so A/I is simple if and only if I is maximal. Conversely, if M is simple and $v \in M$ is non-zero, then $A \cdot v$ is a non-zero submodule, so $A \cdot v = M$. Thus M is cyclic, hence isomorphic to A/I with $I = \operatorname{ann}(M)$, and I is maximal by the above.

A chain of A-submodules $M = M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_n = \{0\}$ is called a **Jordan–Hölder series** or **composition series** of length n if each quotient M_i/M_{i+1} is a simple A-module, in other words, if no submodule can be inserted into the chain. If M admits a composition series, the **length** $\ell(M)$ of M is the smallest length of a composition series of M; we will see in a moment that all composition series of M have the same length.

Proposition 6.10. *(Jordan–Hölder) An A-module M admits a composition series if and only if M is Noetherian and Artinian. In this case, every chain of submodules $M = K_0 \supsetneq K_1 \supsetneq \ldots \supsetneq K_m$ has length $m \leq \ell(M)$, and every composition series of M has length $\ell(M)$.*

Proof. Suppose that M is Noetherian and Artinian. If $N \subset M$ is a non-zero submodule, then by the Noetherian property the collection of proper submodules of N has a maximal element N', and N/N' is simple. Starting from $M_0 = M$, this produces a chain $M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \ldots$ with simple quotients, which continues as long as $M_i \neq \{0\}$. By the Artinian property it stops, so $M_n = \{0\}$ for some n.

Conversely, suppose that M admits a composition series, and let $M = M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_n = \{0\}$ be one of smallest length $n = \ell(M)$. We claim that every proper submodule $N \subsetneq M$ admits a composition series, and $\ell(N) < \ell(M)$. Put $N_i = N \cap M_i$. Since $N_{i+1} = N_i \cap M_{i+1}$, the inclusion $N_i \subset M_i$ induces an injective homomorphism of A-modules $N_i/N_{i+1} \to M_i/M_{i+1}$, $n + N_{i+1} \mapsto n + M_{i+1}$. Its image is a submodule of the simple module M_i/M_{i+1}, hence is either $\{0\}$ or everything: either $N_i = N_{i+1}$ or $N_i/N_{i+1} \simeq M_i/M_{i+1}$ is simple. Deleting the repetitions from $N = N_0 \supset N_1 \supset \ldots \supset N_n = \{0\}$ therefore gives a composition series of N, so $\ell(N) \leq n$. If $\ell(N) = n$, there are no repetitions, so $N_i/N_{i+1} \to M_i/M_{i+1}$ is an isomorphism for all i. Then $N_{n-1} = M_{n-1}$, and $N_{i+1} = M_{i+1}$ implies $N_i = M_i$, so by descending induction $N = N_0 = M_0 = M$, a contradiction. This proves the claim.

Now let $M = K_0 \supsetneq K_1 \supsetneq \ldots \supsetneq K_m$ be a chain of submodules. Applying the claim repeatedly gives $\ell(M) > \ell(K_1) > \ldots > \ell(K_m) \geq 0$, so $m \leq \ell(M)$. In particular, every ascending or descending chain of submodules of M becomes stationary, so M is Noetherian and Artinian. Finally, a composition series of length m is such a chain, so $m \leq \ell(M)$, and $m \geq \ell(M)$ by the definition of $\ell(M)$. □

◁ Prove that the simple quotients of a composition series are also unique: if $M = M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_n = \{0\}$ and $M = N_0 \supsetneq N_1 \supsetneq \ldots \supsetneq N_n = \{0\}$ are two composition series of M, then there is a permutation σ of $\{1, \ldots, n\}$ such that $M_{i-1}/M_i \simeq N_{\sigma(i)-1}/N_{\sigma(i)}$ for all i. The proof is the same as in the case of finite groups.

▶ A vector space V over a field k has $\ell(V) = \dim_k V$. For the second ring in the Macaulay2 example above, $A = k[X, Y]/(X^2, XY^2, Y^3)$, the monomials $1, X, Y, XY, Y^2$ form a basis over

k, and Figure 11 shows a composition series of A: each quotient is one-dimensional over k. So $\ell(A) = 5$.

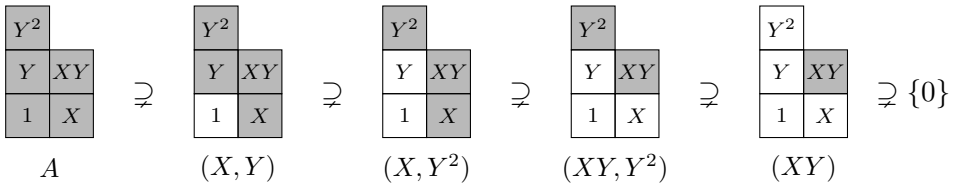


Figure 11: A composition series of $A = k[X, Y]/(X^2, XY^2, Y^3)$. The boxes are the monomials $1, X, Y, XY, Y^2$, which form a basis of A over k; the shaded ones span the ideal written below. Each step removes one box, so $\ell(A) = 5$.

Proof of Theorem 6.8. If A is Artinian, then every prime ideal of A is maximal by Proposition 6.7, so $\dim A = 0$.

Conversely, suppose that $\dim A = 0$. Then every prime ideal of A is both minimal and maximal, and since A is Noetherian, there are only finitely many of them by Proposition 6.4, say $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$. Let $I = \sqrt{(0)} = \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \ldots \cap \mathfrak{m}_r$ (Proposition 2.7). Distinct maximal ideals are coprime, that is, $\mathfrak{m}_i + \mathfrak{m}_j = A$ for $i \neq j$, so by the Sunzi remainder theorem (Theorem 2.3)

$$\mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_r = \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \ldots \cap \mathfrak{m}_r = I.$$

Next, $I^s = \{0\}$ for some s: by the Noetherian property $I = (a_1, \ldots, a_n)$, and there exists N such that $a_i^N = 0$ for all i; a product of Nn of the generators contains some a_i at least N times, so $I^{Nn} = \{0\}$. Consider the chain of ideals

$$A \supset \mathfrak{m}_1 \supset \mathfrak{m}_1 \mathfrak{m}_2 \supset \ldots \supset \mathfrak{m}_1 \cdots \mathfrak{m}_r = I \supset I\mathfrak{m}_1 \supset I\mathfrak{m}_1\mathfrak{m}_2 \supset \ldots \supset I^2 \supset \ldots \supset I^s = \{0\}.$$

Two consecutive terms are of the form $L \supset L\mathfrak{m}_i$ for some i and some ideal L of A. The quotient $L/L\mathfrak{m}_i$ is annihilated by $\mathfrak{m}_i$, so it is a vector space over the field $A/\mathfrak{m}_i$, and its A-submodules are its subspaces. It is finite-dimensional, since L is finitely generated. Hence a chain of subspaces with one-dimensional quotients is a composition series of the A-module $L/L\mathfrak{m}_i$. Inserting the preimages of these subspaces between $L\mathfrak{m}_i$ and L, for all consecutive terms, we obtain a composition series of A. By Proposition 6.10, A is Artinian. □

In fact, an Artinian ring is automatically Noetherian.

Theorem 6.11. *(Akizuki) A ring A is Artinian if and only if A is Noetherian and $\dim A = 0$.*

Proof. By Theorem 6.8, it remains to show that an Artinian ring A is Noetherian. By Proposition 6.7, every prime ideal of A is maximal.

Step 1: A has only finitely many maximal ideals. Suppose that $\mathfrak{m}_1, \mathfrak{m}_2, \ldots$ are infinitely many distinct maximal ideals. The chain $\mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset \ldots$ becomes stationary, so $\mathfrak{m}_1 \cdots \mathfrak{m}_k = \mathfrak{m}_1 \cdots \mathfrak{m}_{k+1} \subset \mathfrak{m}_{k+1}$ for some k. Since $\mathfrak{m}_{k+1}$ is prime, it contains $\mathfrak{m}_i$ for some $i \leq k$, hence is

equal to it, a contradiction. So the maximal ideals of A are $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$, and as in the proof of Theorem 6.8 we get $I = \sqrt{(0)} = \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_r$.

Step 2: $I^s = \{0\}$ for some s. The chain $I \supset I^2 \supset \ldots$ becomes stationary, so there is s with $J := I^s = I^{s+i}$ for all $i \geq 0$; in particular $J^2 = J$. Suppose that $J \neq \{0\}$. Then the collection of ideals K with $JK \neq \{0\}$ is non-empty, as it contains J, so by the Artinian property it has a minimal element K. Let $x \in K$ with $Jx \neq \{0\}$. Then $K = (x)$ by minimality. Moreover, $J(Jx) = J^2x = Jx \neq \{0\}$ and $Jx \subset (x)$, hence $Jx = (x)$ by minimality again. So $x = xy$ for some $y \in J$, and therefore $x = xy = xy^2 = \ldots$. But $y \in J \subset \sqrt{(0)}$ is nilpotent, so $x = 0$, a contradiction.

Step 3: A is Noetherian. Consider the chain of ideals from the proof of Theorem 6.8. For consecutive terms $L \supset L\mathfrak{m}_i$, the quotient $L/L\mathfrak{m}_i$ is a vector space over $A/\mathfrak{m}_i$ whose subspaces satisfy the descending chain condition, since A is Artinian; so it is finite-dimensional. As before, A admits a composition series, and it is Noetherian by Proposition 6.10. □

A local Artinian ring is called a **primary ring**. By Theorems 6.8 and 6.11, these are precisely the Noetherian rings with exactly one prime ideal.

▶ $\mathbb{Z}/(p^s)$, $k[X]/(X^s)$ and the ring $k[X,Y]/(X^2, XY^2, Y^3)$ of Figure 11 are primary rings.

Theorem 6.12. *Let A be a Noetherian ring with $\dim A = 0$. Then A is isomorphic to a finite product of primary rings (in a unique way).*

Proof. We saw in the proof of Theorem 6.8 that A has finitely many distinct maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$, that $\sqrt{(0)} = \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_r$, and that $\sqrt{(0)}^s = \{0\}$ for some s. Since $\mathfrak{m}_i$ and $\mathfrak{m}_j$ are coprime for $i \neq j$, so are $\mathfrak{m}_i^s$ and $\mathfrak{m}_j^s$: a maximal ideal containing $\mathfrak{m}_i^s + \mathfrak{m}_j^s$ would contain both $\mathfrak{m}_i$ and $\mathfrak{m}_j$. Hence, by the Sunzi remainder theorem (Theorem 2.3),

$$\{0\} = \mathfrak{m}_1^s\mathfrak{m}_2^s\cdots\mathfrak{m}_r^s = \mathfrak{m}_1^s \cap \mathfrak{m}_2^s \cap \ldots \cap \mathfrak{m}_r^s \quad \text{and} \quad A \simeq A/\mathfrak{m}_1^s \times A/\mathfrak{m}_2^s \times \ldots \times A/\mathfrak{m}_r^s.$$

Each factor $A/\mathfrak{m}_i^s$ is Noetherian, and its only prime ideal is $\mathfrak{m}_i/\mathfrak{m}_i^s$, because a prime ideal containing $\mathfrak{m}_i^s$ contains $\mathfrak{m}_i$. So it is a primary ring. We leave the uniqueness statement as an exercise. □

You might have the impression that due to this theorem, we can classify all the Artinian rings. This is far from the truth. In fact, even in the case of finite-dimensional k-algebras over an algebraically closed field k, there are infinitely many isomorphism types and a pattern of classification is not known. Up to dimension 6, there are finitely many isomorphism types and a classification is known. From dimension 7 onwards, there are infinitely many isomorphism types.[11]

To give you an impression of these classification results, here is a complete list (up to isomorphism) of 4-dimensional (over k) local Artinian algebras over an algebraically closed field of characteristic not equal to 2.

$$k[X]/(X^4),\ k[X,Y]/(X^2, XY, Y^3),\ k[X,Y]/(X^2, Y^2),\ k[X,Y,Z]/(X,Y,Z)^2$$

[11] B. Poonen, Isomorphism types of commutative algebras of finite rank over an algebraically closed field, in: Computational Arithmetic Geometry, Contemp. Math. 463, Amer. Math. Soc., 2008, 111–120.

In characteristic 2, you have to also add $k[X,Y]/(X^2+Y^2, XY)$ to the list.