Lecture 18. Localisation

6.3 Localisation

We next discuss an important construction that is the algebraic analogue of the geometric notion of concentrating attention near a point.

Let A be an integral domain and $S \subset A$ a subset not containing 0. We can then consider the subring

$$S^{-1}A = A[\{1/s : s \in S\}] \subset \mathrm{Frac}(A).$$

There is the inclusion homomorphism $A \to S^{-1}A$, and every element of S becomes invertible in $S^{-1}A$. Recall that in the construction of $\mathrm{Frac}(A)$ we use the equivalence relation which declares $\frac{a}{s} = \frac{b}{s'}$ if $as' - bs = 0$. More generally, if A is not an integral domain but S is a multiplicative set that does not contain any zero divisors, then the same construction works.

Now let A be any ring and $S \subset A$ a multiplicative set, that is, $1 \in S$ and S is closed under multiplication, which possibly contains zero divisors. We would like to construct a ring B and a homomorphism $\iota : A \to B$ such that $\iota(s)$ is invertible for all $s \in S$. If $as = 0$ for some $a \in A$ and $s \in S$, then $\iota(a)\iota(s) = 0$ forces $\iota(a) = 0$, so ι cannot be injective: such elements a have to be killed first. It turns out that this can be done directly, by imitating the construction of the field of fractions with a small modification of the equivalence relation.

Definition 6.13. *Let A be a ring and $S \subset A$ a multiplicative set. Consider the set $A \times S$ with the equivalence relation*

$$(a, s) \sim (b, s') \text{ if and only if there exists } t \in S \text{ such that } t(as' - bs) = 0.$$

Writing $\frac{a}{s}$ for the equivalence class of (a, s), we define the **ring of fractions** $S^{-1}A \coloneqq \{\frac{a}{s} : a \in A, s \in S\}$ *with the following rules for addition and multiplication:*

$$\frac{a}{s} + \frac{b}{s'} \coloneqq \frac{as' + bs}{ss'}, \qquad \frac{a}{s} \cdot \frac{b}{s'} \coloneqq \frac{ab}{ss'}.$$

The ring $S^{-1}A$ is also called the **localisation** of A with respect to S.

◁ Verify that $\sim$ is an equivalence relation and that addition and multiplication are well defined, so that $S^{-1}A$ is a ring. The transitivity of $\sim$ is instructive: it is where the element t in the definition is needed.

▶ There is a homomorphism $\iota : A \to S^{-1}A$ given by $a \mapsto \frac{a}{1}$, and it satisfies $\iota(s)\frac{1}{s} = 1$ for all $s \in S$. One often omits ι from the notation and writes $a \in S^{-1}A$ instead of $\iota(a) \in S^{-1}A$ for $a \in A$. The kernel of ι is $\ker(\iota) = \{a \in A : \frac{a}{1} = \frac{0}{1}\} = \{a \in A : \exists\, t \in S,\ at = 0\}$. In particular, $S^{-1}A = \{0\}$ if and only if $0 \in S$, and ι is injective if A is an integral domain and $0 \notin S$.

▶ If $I \subset A$ is an ideal, we write $S^{-1}I$ for the ideal of $S^{-1}A$ generated by $\iota(I)$. Explicitly, $S^{-1}I = \{\frac{a}{s} : a \in I,\ s \in S\}$, since the right-hand side is an ideal containing $\iota(I)$, and $\frac{a}{s} = \frac{1}{s}\,\iota(a)$.

▶ Localisation satisfies the following universal property: suppose B is a ring and $g : A \to B$ is a ring homomorphism such that $g(s)$ is invertible for all $s \in S$. Then there exists a unique homomorphism $h : S^{-1}A \to B$ such that $g = h \circ \iota$. Indeed, $h(\frac{a}{s})\,g(s) = h(\frac{a}{s})\,h(\frac{s}{1}) = h(\frac{a}{1}) = g(a)$ forces $h(\frac{a}{s}) = g(a)g(s)^{-1}$, and one verifies that this formula gives a well-defined homomorphism of rings.

◁ Let $\mathfrak{n} = \ker(\iota) = \{a \in A : \exists\, t \in S,\ at = 0\}$, and let $\overline{S}$ be the image of S in $A/\mathfrak{n}$. Show that the multiplicative set $\overline{S}$ does not contain any zero divisors of $A/\mathfrak{n}$, and that $\overline{S}^{-1}(A/\mathfrak{n})$ is isomorphic to $S^{-1}A$.

The following two are the main examples of localisation.

(a) Let $f \in A$ and $S = \{1, f, f^2, \ldots\}$. We write $A_f \coloneqq S^{-1}A$. We have $\ker(\iota) = \{a \in A : \exists\, n \geq 0,\ af^n = 0\}$, and $A_f = \{0\}$ if and only if f is nilpotent.

◁ Prove that A_f is isomorphic to the ring $A[X]/(Xf - 1)$.

▶ For $0 \neq b \in \mathbb{Z}$ we have $\mathbb{Z}_b = \{\frac{a}{b^s} \in \mathbb{Q} : a \in \mathbb{Z},\ s \geq 0\} = \mathbb{Z}[1/b]$.

◁ Let $A = k[X, Y]/(XY)$, the coordinate ring of the union of the two axes in k^2. Show that A_X is isomorphic to the ring $k[X, X^{-1}]$, see Figure 12.

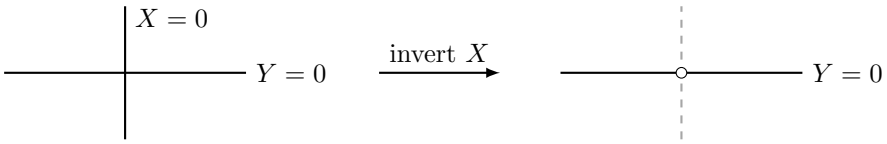


Figure 12: The ring $A = k[X, Y]/(XY)$ is the coordinate ring of the union of the two axes. Inverting X removes the line $X = 0$; the function Y vanishes on what remains, so it becomes zero in A_X.

(b) Let $\mathfrak{p} \subset A$ be a prime ideal. Then $S \coloneqq A \setminus \mathfrak{p}$ is a multiplicative set, by the definition of a prime ideal, and we write $A_\mathfrak{p} \coloneqq S^{-1}A$. This is a local ring with unique maximal ideal $S^{-1}\mathfrak{p}$: by Lemma 6.14(ii) below, its prime ideals are the ideals $S^{-1}\mathfrak{q}$ with $\mathfrak{q} \subset \mathfrak{p}$. Beware of the notation: in A_f the element f is inverted, while in $A_\mathfrak{p}$ the elements outside $\mathfrak{p}$ are inverted.

▶ If A is an integral domain, then (0) is a prime ideal, and $A_{(0)}$ is the field of fractions of A.

▶ For the prime ideal $(p) \subset \mathbb{Z}$ we get $\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} : p \nmid b\}$.

▶ For the maximal ideal $(X) \subset k[X]$ we get $k[X]_{(X)} = \{\frac{f}{g} \in k(X) : X \nmid g\}$. Since $X \nmid g$ means $g(0) \neq 0$, these are the rational functions that are defined at the point 0.

Here are some basic properties of localisation.

Lemma 6.14. *(i) Let $J \subset S^{-1}A$ be an ideal and $I = \iota^{-1}(J) \subset A$. Then $J = S^{-1}I$. In particular, $S^{-1}A$ is Noetherian if A is Noetherian.*

(ii) The map $\iota^* : \mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$, $\mathfrak{q} \mapsto \iota^{-1}(\mathfrak{q})$, is an inclusion-preserving bijection onto the subset $\{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \cap S = \emptyset\}$, with inverse given by $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$.

(iii) (Localisation commutes with quotients) Let I be an ideal in A, and write $\overline{S}$ for the image of S under the canonical ring homomorphism $A \to A/I$. Then $\overline{S}^{-1}(A/I) \simeq S^{-1}A/S^{-1}I$.

▶ For $A = \mathbb{Z}$, part (ii) of the lemma says that $\mathrm{Spec}(\mathbb{Z}[1/6])$ consists of the points of $\mathrm{Spec}(\mathbb{Z})$ other than (2) and (3), while $\mathrm{Spec}(\mathbb{Z}_{(5)})$ consists only of (5) and (0), see Figure 13.

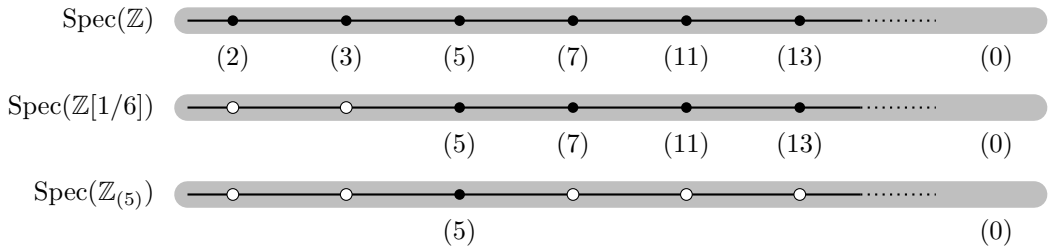


Figure 13: The spectra of $\mathbb{Z}[1/6]$ and $\mathbb{Z}_{(5)}$ as subsets of $\mathrm{Spec}(\mathbb{Z})$; open circles are the points that are removed. The point (0), drawn as a grey band, belongs to all three.

Proof. (i) Since J is an ideal containing $\iota(I)$, we have $S^{-1}I \subset J$. Conversely, let $x = \frac{a}{s} \in J$. Then $\frac{a}{1} = \frac{s}{1} \cdot \frac{a}{s} \in J$, hence $a \in \iota^{-1}(J) = I$ and $x \in S^{-1}I$. If A is Noetherian, then $I = (a_1, \ldots, a_n)$ for some $a_i \in A$, and $J = S^{-1}I$ is generated by $\frac{a_1}{1}, \ldots, \frac{a_n}{1}$; so every ideal of $S^{-1}A$ is finitely generated.

(ii) The key point is the following: if $\mathfrak{p} \subset A$ is a prime ideal with $\mathfrak{p} \cap S = \emptyset$, then for $a \in A$ and $s \in S$,

$$\frac{a}{s} \in S^{-1}\mathfrak{p} \iff a \in \mathfrak{p}.$$

Indeed, if $\frac{a}{s} = \frac{p}{r}$ with $p \in \mathfrak{p}$ and $r \in S$, then there exists $u \in S$ such that $u(ar - ps) = 0$, so $a \cdot ru = p \cdot us \in \mathfrak{p}$. Since $ru \in S$ does not lie in $\mathfrak{p}$, we get $a \in \mathfrak{p}$. It follows that $S^{-1}\mathfrak{p}$ is a prime ideal of $S^{-1}A$: it is proper, since $1 \notin \mathfrak{p}$, and if $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st} \in S^{-1}\mathfrak{p}$, then $ab \in \mathfrak{p}$, so $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. It also follows that $\iota^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$.

Conversely, let $\mathfrak{q} \subset S^{-1}A$ be a prime ideal and $\mathfrak{p} = \iota^{-1}(\mathfrak{q})$, a prime ideal of A. Then $\mathfrak{p} \cap S = \emptyset$, since otherwise $\mathfrak{q}$ would contain a unit of $S^{-1}A$, and $S^{-1}\mathfrak{p} = \mathfrak{q}$ by part (i). So the two maps are inverse to each other, and both clearly preserve inclusions.

(iii) Both sides have the universal property for ring homomorphisms $g : A \to C$ which map the elements of S to units of C and the elements of I to 0, so they are isomorphic. More concretely, one checks directly that $\frac{a}{s} + S^{-1}I \mapsto \frac{a+I}{s+I}$ is a well-defined ring isomorphism from $S^{-1}A/S^{-1}I$ to $\overline{S}^{-1}(A/I)$. □