Lecture 19. Local rings: Nakayama's lemma and Krull's height theorem

Part (ii) of Lemma 6.14 describes the spectra of the two localisations A_f and $A_\mathfrak{p}$ of Lecture 18.

Corollary 6.15. *(i) For $f \in A$, the map ι^* identifies* $\mathrm{Spec}(A_f)$ *with the basic open set $D(f) = \{\mathfrak{q} \in \mathrm{Spec}(A) : f \notin \mathfrak{q}\}$.*

(ii) For a prime ideal $\mathfrak{p} \subset A$, the map ι^* identifies $\mathrm{Spec}(A_\mathfrak{p})$ *with $\{\mathfrak{q} \in \mathrm{Spec}(A) : \mathfrak{q} \subset \mathfrak{p}\}$, preserving inclusions. In particular,* $\dim A_\mathfrak{p} = h(\mathfrak{p})$.

Proof. A prime ideal $\mathfrak{q}$ is disjoint from $\{1, f, f^2, \ldots\}$ if and only if $f \notin \mathfrak{q}$, and disjoint from $A \setminus \mathfrak{p}$ if and only if $\mathfrak{q} \subset \mathfrak{p}$. For the last statement, every chain of prime ideals contained in $\mathfrak{p}$ can be extended to end at $\mathfrak{p}$, so the supremum of their lengths is $h(\mathfrak{p})$. □

Thus the localisation A_f plays the role of the ring of "regular functions" on the open set $D(f)$.

▶ The passage from A to $A_\mathfrak{p}$ cuts out all the prime ideals except those contained in $\mathfrak{p}$. This is in contrast to the passage from A to $A/\mathfrak{p}$, which cuts out all the prime ideals except those containing $\mathfrak{p}$. In Figure 10, for $\mathfrak{p} = (Y - X^2)$ the first keeps the generic points of the plane and of the parabola, the second keeps the parabola with all its points. Given prime ideals $\mathfrak{q} \subset \mathfrak{p}$, we can also localise with respect to $\mathfrak{p}$ and quotient with respect to $\mathfrak{q}$: by Lemma 6.14(iii), the prime ideals of $A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p} \simeq (A/\mathfrak{q})_{\mathfrak{p}/\mathfrak{q}}$ are those lying between $\mathfrak{q}$ and $\mathfrak{p}$.

◁ Let I be an ideal in A. Show that $\iota^{-1}(S^{-1}I) = \{a \in A : as \in I \text{ for some } s \in S\}$.

The ring $A_\mathfrak{p}$ is the algebraic analogue of the germs of functions defined near the point $\mathfrak{p}$ of $\mathrm{Spec}(A)$. The dimension of A can be computed from these local rings: by definition $\dim A = \sup_\mathfrak{p} h(\mathfrak{p})$, and $h(\mathfrak{p}) \leq h(\mathfrak{m})$ when $\mathfrak{p} \subset \mathfrak{m}$, so

$$\dim A = \sup_{\mathfrak{p} \in \mathrm{Spec}(A)} \dim A_\mathfrak{p} = \sup_{\mathfrak{m} \in \mathrm{mSpec}(A)} \dim A_\mathfrak{m}.$$

6.4 Dimension for local rings

Next, we come to a very useful lemma in the study of local rings, which will later on help us understand how to compute $\dim A_\mathfrak{p}$.

Lemma 6.16. *(Nakayama's Lemma) Let A be a local ring with maximal ideal $\mathfrak{m}$. Let M be a finitely generated A-module and $N \subset M$ be a submodule with the property that $M = N + \mathfrak{m}M$. Then $M = N$.*

Proof. The A-module $\overline{M} = M/N$ is finitely generated and satisfies $\overline{M} = \mathfrak{m}\overline{M}$. We want to show that $\overline{M} = \{0\}$. If not, let $x_1, \ldots, x_n \in \overline{M}$ be generators, $\overline{M} = Ax_1 + Ax_2 + \ldots + Ax_n$, with $n \geq 1$ minimal. Since $\overline{M} = \mathfrak{m}\overline{M}$, we have $x_n = \sum_{i=1}^n a_i x_i$ with $a_i \in \mathfrak{m}$. Then $(1 - a_n)x_n = \sum_{i=1}^{n-1} a_i x_i$, and $1 - a_n \notin \mathfrak{m}$ is a unit, because A is local and every non-unit lies in a maximal ideal. It follows that $x_n \in Ax_1 + Ax_2 + \ldots + Ax_{n-1}$, contradicting the minimality of n. □

This is a simple but important result that allows us to make deductions about properties of M as an A-module from the properties of $M/\mathfrak{m}M$ as an $A/\mathfrak{m}$-vector space. Thus, many properties of the finitely generated modules over a local ring A can be studied via linear algebra over the field $A/\mathfrak{m}$.

Here is a first application of Nakayama's lemma. For a finitely generated module M, write

$$\mu(M) \coloneqq \min\{n \geq 0 : M = Ax_1 + \ldots + Ax_n \text{ for some } x_1, \ldots, x_n \in M\}$$

for the minimal number of generators of M.

Proposition 6.17. *Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$, and let M be a finitely generated A-module. Elements $x_1, \ldots, x_n \in M$ generate M if and only if their images $\overline{x}_1, \ldots, \overline{x}_n$ span the k-vector space $M/\mathfrak{m}M$. In particular, $\mu(M) = \dim_k M/\mathfrak{m}M$.*

Proof. If $M = Ax_1 + \ldots + Ax_n$, then $M/\mathfrak{m}M$ is spanned by $\overline{x}_1, \ldots, \overline{x}_n$. Conversely, suppose that $\overline{x}_1, \ldots, \overline{x}_n$ span $M/\mathfrak{m}M$, and let $N = Ax_1 + \ldots + Ax_n \subset M$. By Nakayama's lemma, it suffices to show that $M = N + \mathfrak{m}M$. Let $x \in M$. Then $\overline{x} = \sum_{i=1}^n \overline{a}_i \overline{x}_i$ for some $a_i \in A$, so $x - \sum_{i=1}^n a_i x_i \in \mathfrak{m}M$, and $x \in N + \mathfrak{m}M$. Finally, a k-vector space spanned by n elements has dimension at most n, and a basis of $M/\mathfrak{m}M$ lifts to a generating set of M; hence $\mu(M) = \dim_k M/\mathfrak{m}M$. □

Now let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Then $\mathfrak{m}$ is a finitely generated A-module, and we write

$$\mu(A) \coloneqq \mu(\mathfrak{m}) = \dim_k \mathfrak{m}/\mathfrak{m}^2$$

for the minimal number of generators of $\mathfrak{m}$.

◁ Let $\mathfrak{m}$ be a maximal ideal of a ring A. Show that $A_\mathfrak{m}/\mathfrak{m}^2 A_\mathfrak{m} \simeq A/\mathfrak{m}^2$, and deduce that $\mathfrak{m}A_\mathfrak{m}/(\mathfrak{m}A_\mathfrak{m})^2 \simeq \mathfrak{m}/\mathfrak{m}^2$ (use Lemma 6.14(iii)).

▶ Here is the geometric meaning of $\mu(A)$. Let $V \subset k^n$ be an affine variety with $\mathcal{I}(V) = (f_1, \ldots, f_m)$, let $\mathbf{v} \in V$, and let $A = k[V]_{\mathfrak{m}_\mathbf{v}}$ be the local ring of V at $\mathbf{v}$, so that $\mathfrak{m}/\mathfrak{m}^2 \simeq \mathfrak{m}_\mathbf{v}/\mathfrak{m}_\mathbf{v}^2$ by the exercise. Every $g \in k[X_1, \ldots, X_n]$ can be written as $g = g(\mathbf{v}) + d_\mathbf{v} g + r$, where $d_\mathbf{v} g = \sum_i \frac{\partial g}{\partial X_i}(\mathbf{v})(X_i - v_i)$ is the linear part of g at $\mathbf{v}$, which we view as a linear form on k^n, and $r \in (X_1 - v_1, \ldots, X_n - v_n)^2$. Then $g \mapsto d_\mathbf{v} g$ induces an isomorphism $\mathfrak{m}_\mathbf{v}/\mathfrak{m}_\mathbf{v}^2 \simeq (k^n)^*/\langle d_\mathbf{v} f_1, \ldots, d_\mathbf{v} f_m\rangle$, and the right hand side is the dual of the **tangent space**

$$T_\mathbf{v} V = \{\mathbf{w} \in k^n : d_\mathbf{v} f_j(\mathbf{w}) = 0 \text{ for } j = 1, \ldots, m\}.$$

So $\mu(A) = \dim_k T_{\mathbf{v}}V$. For the parabola $V = \mathcal{V}(Y - X^2)$, the tangent space at the origin is the line $Y = 0$, so $\mu(A) = 1 = \dim A$. For the cusp $V = \mathcal{V}(Y^2 - X^3)$ of Figure 8, we have $d_{\mathbf{0}}(Y^2 - X^3) = 0$, so the tangent space at the origin is all of k^2 and $\mu(A) = 2$, while $\dim A = 1$. At every other point of the cusp the tangent space is a line, see Figure 14.

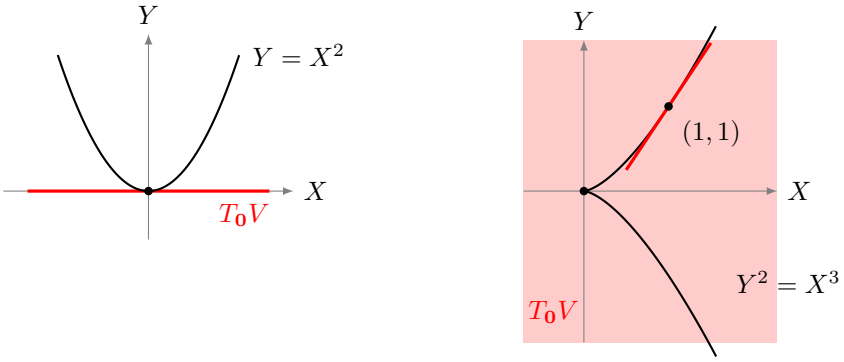


Figure 14: Tangent spaces (red). At the origin of the parabola, $T_{\mathbf{0}}V$ is the line $Y = 0$ and $\mu = 1 = \dim$. At the origin of the cusp, $T_{\mathbf{0}}V$ is the whole plane and $\mu = 2 > 1 = \dim$; at the point $(1, 1)$ the tangent space is again a line.

Question: What is the relation between $\mu(A)$ and $\dim A$?

The answer will be given by the following important theorem of Krull, which we prove in Lectures 20 and 21. It provides a lower estimate for the number of generators of an ideal in a Noetherian ring (and for the number of equations needed to describe an affine variety).

Theorem 6.18. *(Krull's height theorem) Let A be a Noetherian ring, and let $I = (a_1, \ldots, a_r)$ be an ideal generated by r elements. Let $\mathfrak{p} \subset A$ be a prime ideal which is minimal (with respect to inclusion) among the prime ideals containing I. Then*

$$h(\mathfrak{p}) \leq r.$$

Corollary 6.19. *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Then $\dim A \leq \mu(A)$. In particular, $\dim A$ is finite.*

Proof. Every chain of prime ideals of A can be extended to end at $\mathfrak{m}$, so $\dim A = h(\mathfrak{m})$, and $\mathfrak{m}$ is the only prime ideal containing $\mathfrak{m}$. So Theorem 6.18 applied to $I = \mathfrak{m} = (a_1, \ldots, a_r)$ with $r = \mu(A)$ gives $\dim A = h(\mathfrak{m}) \leq \mu(A)$. □

Theorem 6.18 makes precise the intuition that each additional equation can decrease the dimension of the space of solutions by at most one. Note that one can give a lower bound for $\dim A$ by exhibiting a chain of prime ideals in A, but an upper bound is not straightforward to obtain from the definition. This is another reason why Theorem 6.18 is important.

Definition 6.20. *A Noetherian local ring A is said to be* **regular** *if $\dim A = \mu(A)$.*

▶ For a point $\mathbf{v} \in k^n$, the local ring $A = k[X_1, \ldots, X_n]_{\mathfrak{m}_\mathbf{v}}$ is regular of dimension n. Indeed, $\mu(A) = n$, since $\mathfrak{m}_\mathbf{v}/\mathfrak{m}_\mathbf{v}^2$ has basis $X_1 - v_1, \ldots, X_n - v_n$, and the chain of prime ideals $(0) \subsetneq (X_1 - v_1) \subsetneq (X_1 - v_1, X_2 - v_2) \subsetneq \ldots \subsetneq \mathfrak{m}_\mathbf{v}$, whose quotients are polynomial rings, gives $h(\mathfrak{m}_\mathbf{v}) \geq n$, so $\dim A = n$ by Corollary 6.19. More generally, a point $\mathbf{v}$ of an affine variety V is called a **smooth point** if $\dim_k T_\mathbf{v} V = \dim k[V]_{\mathfrak{m}_\mathbf{v}}$, that is, if the local ring of V at $\mathbf{v}$ is regular. By Corollary 6.19, $\dim_k T_\mathbf{v} V \geq \dim k[V]_{\mathfrak{m}_\mathbf{v}}$ always, and the origin of the cusp is not a smooth point.

An important theorem due to Auslander and Buchsbaum states that regular local rings are UFDs. Unfortunately, we do not have time to prove this result in this course; see [4], where you can also find a proof that the localisation $k[X_1, \ldots, X_n]_\mathfrak{p}$ at any prime ideal $\mathfrak{p}$ is regular. (These are applications of the homological methods in algebra, which we do not get into in this course.)