Lecture 20. Krull's principal ideal theorem

The general case of Krull's height theorem is obtained by induction on the number r of generators. The case $r = 1$ is the hardest part of the proof, and we state it separately.

Theorem 6.21. *(Krull's principal ideal theorem / Hauptidealsatz) Let A be a Noetherian ring, let $a \in A$ be a non-unit, and let $\mathfrak{p}$ be a prime ideal which is minimal among the prime ideals containing a. Then $h(\mathfrak{p}) \leq 1$.*

Proof. We may suppose that A is a local ring with maximal ideal $\mathfrak{p}$. Indeed, by Corollary 6.15(ii), the prime ideals of $A_\mathfrak{p}$ containing $\frac{a}{1}$ correspond to the prime ideals of A contained in $\mathfrak{p}$ and containing a, so $\mathfrak{p}A_\mathfrak{p}$ is minimal among the prime ideals of $A_\mathfrak{p}$ containing $\frac{a}{1}$, and $h(\mathfrak{p}) = \dim A_\mathfrak{p} = h(\mathfrak{p}A_\mathfrak{p})$.

So let A be local with maximal ideal $\mathfrak{p}$, and let $\mathfrak{q} \subsetneq \mathfrak{p}$ be a prime ideal. We want to show that $h(\mathfrak{q}) = 0$, that is, that $\mathfrak{q}A_\mathfrak{q}$ is the only prime ideal of the local ring $A_\mathfrak{q}$. Since $\sqrt{(0)}$ is the intersection of all prime ideals, it suffices to show that the maximal ideal $\mathfrak{q}A_\mathfrak{q}$ is nilpotent. Note that $a \notin \mathfrak{q}$, as $\mathfrak{p}$ is minimal among the prime ideals containing a, so $\frac{a}{1}$ is a unit in $A_\mathfrak{q}$.

Let $\iota : A \to A_\mathfrak{q}$ be the localisation map, and let

$$\mathfrak{q}^{(n)} := \iota^{-1}(\mathfrak{q}^n A_\mathfrak{q}) = \{x \in A : xs \in \mathfrak{q}^n \text{ for some } s \in A \setminus \mathfrak{q}\}$$

be the n-th **symbolic power** of $\mathfrak{q}$, see the exercise in Lecture 19. These form a chain of ideals $\mathfrak{q} = \mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \dots$ in A, and by Lemma 6.14(i) we have $\mathfrak{q}^{(n)}A_\mathfrak{q} = \mathfrak{q}^n A_\mathfrak{q}$.

Step 1: $(a) + \mathfrak{q}^{(n)} = (a) + \mathfrak{q}^{(n+1)}$ for some n. The prime ideals of $A/(a)$ correspond to the prime ideals of A containing a, and $\mathfrak{p}$ is the only one of these, as A is local and $\mathfrak{p}$ is minimal over a. So $\dim A/(a) = 0$, and $A/(a)$ is Artinian by Theorem 6.8. Hence the chain of images of $(a) + \mathfrak{q}^{(1)} \supset (a) + \mathfrak{q}^{(2)} \supset \dots$ in $A/(a)$ becomes stationary, which gives the claim.

Step 2: $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$. Let $x \in \mathfrak{q}^{(n)}$. By Step 1, $x = ra + y$ with $r \in A$ and $y \in \mathfrak{q}^{(n+1)}$. Then $ra = x - y \in \mathfrak{q}^{(n)}$, so $\frac{r}{1}\frac{a}{1} \in \mathfrak{q}^n A_\mathfrak{q}$, and since $\frac{a}{1}$ is a unit, $\frac{r}{1} \in \mathfrak{q}^n A_\mathfrak{q}$, that is, $r \in \mathfrak{q}^{(n)}$. This shows that

$$\mathfrak{q}^{(n)} = a\mathfrak{q}^{(n)} + \mathfrak{q}^{(n+1)} \subset \mathfrak{p}\mathfrak{q}^{(n)} + \mathfrak{q}^{(n+1)}.$$

As A is Noetherian, $\mathfrak{q}^{(n)}$ is a finitely generated A-module, and Nakayama's lemma for the submodule $\mathfrak{q}^{(n+1)} \subset \mathfrak{q}^{(n)}$ gives $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$.

Step 3: $\mathfrak{q}A_\mathfrak{q}$ is nilpotent. By Step 2, $\mathfrak{q}^n A_\mathfrak{q} = \mathfrak{q}^{(n)}A_\mathfrak{q} = \mathfrak{q}^{(n+1)}A_\mathfrak{q} = \mathfrak{q}^{n+1}A_\mathfrak{q} = (\mathfrak{q}A_\mathfrak{q})(\mathfrak{q}^n A_\mathfrak{q})$. The ring $A_\mathfrak{q}$ is Noetherian by Lemma 6.14(i), so Nakayama's lemma for the $A_\mathfrak{q}$-module $\mathfrak{q}^n A_\mathfrak{q}$ gives $\mathfrak{q}^n A_\mathfrak{q} = \{0\}$. □

▶ Geometrically, let $f \in k[X_1, \ldots, X_n]$ be non-constant, with k algebraically closed. By the dictionary of Lecture 14, the irreducible components of the hypersurface $\mathcal{V}(f)$ are the sets $\mathcal{V}(\mathfrak{p})$, where $\mathfrak{p}$ runs over the minimal prime ideals containing f. The theorem says that each of these has $h(\mathfrak{p}) \leq 1$, and since $(0) \subsetneq \mathfrak{p}$, in fact $h(\mathfrak{p}) = 1$. Thus every irreducible component of a hypersurface has codimension one.

◁ Carefully examine the proof to see why we needed to use the symbolic powers $\mathfrak{q}^{(n)}$ rather than the ordinary powers $\mathfrak{q}^n$ of the ideal $\mathfrak{q}$.

◁ Let $\mathfrak{p} = (X^4 - YZ, Y^2 - XZ, X^3Y - Z^2) \subset k[X, Y, Z]$. This is the kernel of the homomorphism $k[X, Y, Z] \to k[T]$ sending X, Y, Z to T^3, T^5, T^7, hence a prime ideal. Show that $\mathfrak{p}^{(2)} \neq \mathfrak{p}^2$.

Before we go on to the proof of Krull's height theorem, let us see an application of the principal ideal theorem.

Theorem 6.22. *Let $\mathfrak{p} \subset \mathfrak{q}$ be prime ideals in a Noetherian ring A. If there exists a prime ideal strictly between them, then there are infinitely many.*

Proof. The prime ideals between $\mathfrak{p}$ and $\mathfrak{q}$ correspond to the prime ideals between (0) and $\mathfrak{q}/\mathfrak{p}$ in the integral domain $A/\mathfrak{p}$, so we may suppose that A is an integral domain and $\mathfrak{p} = (0)$. Suppose for a contradiction that $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, with $n \geq 1$, are the only prime ideals strictly between (0) and $\mathfrak{q}$. By prime avoidance (Proposition 2.9), $\mathfrak{q} \not\subset \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_n$, so we can pick $a \in \mathfrak{q}$ with $a \notin \mathfrak{p}_i$ for all i. Then $\mathfrak{q}$ is minimal among the prime ideals containing a: such a prime ideal contained in $\mathfrak{q}$ is (0), one of the $\mathfrak{p}_i$, or $\mathfrak{q}$, and the first two do not contain a. By Theorem 6.21, $h(\mathfrak{q}) \leq 1$, contradicting the chain $(0) \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{q}$. □

▶ In $k[X, Y]$, the prime ideals strictly between (0) and (X, Y) are the ideals (f) with f irreducible and $f(0, 0) = 0$, that is, the irreducible curves through the origin, see Figure 15.

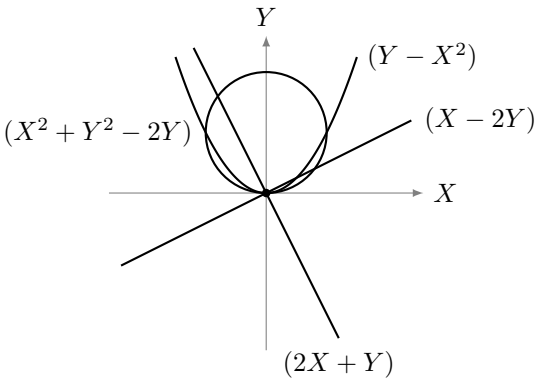


Figure 15: Some of the infinitely many prime ideals strictly between (0) and (X, Y) in $k[X, Y]$: the irreducible curves through the origin.

◁ Let A be a Noetherian ring with minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$, so that $\sqrt{(0)} = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_s$ by Proposition 6.4. Show that every element of $\bigcup_{i=1}^{s} \mathfrak{p}_i$ is a zero-divisor. Deduce that, with the assumptions of Theorem 6.21, $h(\mathfrak{p}) = 1$ if a is not a zero-divisor.