King's College London

UNIVERSITY OF LONDON

PLACE THIS PAPER AND ANY ANSWER BOOKLETS **in the EXAM ENVELOPE provided**

Candidate No: .... **Desk No:** ....

MSc AND MSci EXAMINATION

7CCMMS16T ALGEBRAIC CURVES (MSc)

SUMMER 2018

TIME ALLOWED: TWO HOURS

ALL QUESTIONS CARRY EQUAL MARKS.
FULL MARKS WILL BE AWARDED FOR COMPLETE ANSWERS TO FOUR QUESTIONS.
IF MORE THAN FOUR QUESTIONS ARE ATTEMPTED, THEN ONLY THE BEST FOUR WILL COUNT.

NO CALCULATORS ARE PERMITTED.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

TURN OVER WHEN INSTRUCTED

1. (i) (5/25) Let f be a holomorphic map $S_1 \to S_2$ between compact Riemann surfaces. State the definition of the branching index b_f. State the Riemann-Hurwitz formula in terms of Euler characteristics of S_i and the branching index b_f of f.

(ii) (10/25) Find the compact connected Riemann surface S associated to the affine curve $C \subset \mathbb{C}^2$ defined by the irreducible polynomial

$$f(x, y) = x^3 + y^3 - 1.$$

(Hint: Consider the projectivization and check that its smooth). How many points are there in $S \setminus C$? Show that projection to x induces a map $S \to \mathbb{P}^1$ of degree 3. Using the Riemann-Hurwitz formula for this projection show that the genus of S is 1.

(iii) (10/25) Find the compact connected Riemann surface S associated to the affine curve $C \subset \mathbb{C}^2$ defined by the irreducible polynomial

$$f(x, y) = y^2 - x^6 + 1.$$

(Hint: Projectivization in $\mathbb{P}^2$ gives a singular curve. Find S by studying the map $C \to \mathbb{C}$ given by projection to x.) How many points are there in $S \setminus C$? Show that the projection to x induces a map $S \to \mathbb{P}^1$ of degree 2. Using the Riemann-Hurwitz formula for this projection show that the genus of S is 2.

2. (i) (5/25) Consider the family of projective curves C_λ given by

$$C_\lambda = \{[X, Y, Z] \in \mathbb{P}^2 : Z^2Y^2 = X^4 + \lambda(Y^4 + Z^4)\}, \quad \lambda \in \mathbb{C}.$$

Determine the values of λ such that C_λ is smooth.

(ii) (5/25) For those λ for which C_λ is singular, determine all the singular points.

(iii) (15/25) Consider the curve C_1 where $\lambda = 1$ in the above given equation. Consider the holomorphic map $f : C_1 \to \mathbb{P}^1$ given by projection given by

$$[X : Y : Z] \to [X : Y].$$

Check that f gives a well-defined holomorphic map when restricted to C_1. What is the degree of f? What are the branch points of f? Compute the branching index b_f and use this together with the Riemann-Hurwitz formula to compute the genus of C_1.

3. (i) (15/25) Recall that every elliptic curve over $\mathbb{C}$ can be defined by an equation

$$Y^2Z = 4X^3 + g_2XZ^2 + g_3Z^3$$

for some g_2 and g_3. This is called the Weierstrass form. The j-invariant of an elliptic curve given in the Weierstrass form can be computed as

$$\frac{g_2^3}{g_2^3 - 27g_3^2}.$$

By projectively transforming the following elliptic curves to the Weierstrass form, compute their j-invariants

a) (5/25) $Y^2Z = X(X - Z)(X - 2Z)$.

b) (10/25) $X^3 + Y^3 + Z^3 = 0$.

(ii) (10/25) Consider the singular projective curve C defined by $Y^2Z = X^3$ in $\mathbb{P}^2$. Show that the map $\phi : \mathbb{P}^1 \to C$ given by

$$[U, V] \to [UV^2, V^3, U^3]$$

induces an isomorphism from $\mathbb{C}$ to $C \setminus \{(0, 0, 1)\}$. In addition, show that there cannot be an isomorphism between C and $\mathbb{P}^1$.

4. Let $C = \{[X, Y, Z] \in \mathbb{P}^2 : F(X, Y, Z) = 0\}$ defined by a homogeneous polynomial F with no repeated factors. Assume that C is smooth. Consider the map

$$[X, Y, Z] \to \left[\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right]$$

from $\mathbb{P}^2$ to $\mathbb{P}^2$. This map is called the *polar mapping* of C.

(i) (5/25) Show that the polar mapping is well-defined as a map from $\mathbb{P}^2$ to $\mathbb{P}^2$.

(ii) (5/25) Recall that a conic is a curve in $\mathbb{P}^2$ defined by an equation:

$$F(X, Y, Z) = aX^2 + 2bXY + 2cXZ + dY^2 + 2eYZ + fZ^2$$

for some complex constants, a, b, c, d, e, f, not all zero. Show that the conic defined by $F(X, Y, Z)$ is smooth if and only if the matrix

$$M = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

is invertible.

(iii) (5/25) If C is a smooth conic, then show that the image of C under the polar mapping of C is also a smooth conic.

(iv) (10/25) Find the equations of the dual curves for the following conics.

a) (5/10) $XY + YZ + ZX = 0$,

b) (5/10) $XY + Z^2 = 0$.

5. (i) (5/25) State Belyi's theorem.

(ii) (5/25) Give a Belyi function for the Riemann surface associated to

$$z^2 = w(w-1)(w-2).$$

(iii) (5/25) Give the definition of a dessin d'enfant and explain how to get two permutations associated to a dessin d'enfant.

(iv) (10/25) Let $\sigma_0 = (1,5,4)(2,6,3)$ and $\sigma_1 = (1,2)(3,4)(5,6)$ be permutations in $\mathfrak{S}_6$. Compute the genus of the surface determined by the dessin d'enfant associated to this pair of permutations.