Algebraic Curves – Exam 2018

1) (i) (5/25) Let f be a holomorphic map $S_1 \to S_2$ between compact Riemann surfaces. State the definition of the branching index b_f. State the Riemann-Hurwitz formula in terms of Euler characteristics of S_i and the branching index b_f of f.

(ii) (10/25) Find the compact connected Riemann surface S associated to the affine curve $C \subset \mathbb{C}^2$ defined by the irreducible polynomial $f(x,y) = x^3 + y^3 - 1$ (Hint: Consider the projectivization and check that its smooth). How many points are there in $S \setminus C$? Show that projection x induces a map $S \to \mathbb{P}^1$ of degree 3. Using Riemann-Hurwitz formula for this projection show that the genus of S is 1.

(iii) (10/25) Find the compact connected Riemann surface S associated to the affine curve $C \subset \mathbb{C}^2$ defined by the irreducible polynomial $f(x,y) = y^2 - x^6 + 1$. (Hint: Projectivization in $\mathbb{P}^2$ gives a singular curve. Find S by studying the map $C \to \mathbb{C}$ given by projection to x.) How many points are there in $S \setminus C$? Show that the projection to x induces a map $S \to \mathbb{P}^1$ of degree 2. Using the Riemann-Hurwitz formula for this projection show that the genus of S is 2.

2) (i) (5/25) Consider the family of projective curves C_λ given by

$$C_\lambda = \{[X,Y,Z] \in \mathbb{P}^2 : Z^2Y^2 = X^4 + \lambda(Y^4 + Z^4)\}, \quad \lambda \in \mathbb{C}.$$

Determine the values of λ such that C_λ is smooth.

(ii) (5/25) For those λ for which C_λ is singular, determine all the singular points.

(iii) (15/25) Consider the curve C_1 where $\lambda = 1$ in the above given equation. Consider the holomorphic map $f : C_1 \to \mathbb{P}^1$ given by projection given by $[X : Y : Z] \to [X : Y]$. Check that f gives a well-defined holomorphic map when restricted to C_1. What is the degree of f? What are the branch points of f? Compute the branching index b_f and use this together with the Riemann-Hurwitz formula to compute the genus of C_1.

3) (i) (15/25) Recall that every elliptic curve over $\mathbb{C}$ can be defined by an equation

$$Y^2Z = 4X^3 + g_2XZ^2 + g_3Z^3$$

for some g_2 and g_3. This is called the Weierstrass form. The j-invariant of an elliptic curve given in the Weierstrass form can be computed as

$$\frac{g_2^3}{g_2^3 - 27g_3^2}.$$

By projectively transforming the following elliptic curves to the Weierstrass form, compute their j-invariants

a) (5/25) $Y^2Z = X(X-Z)(X-2Z)$.
b) (10/25) $X^3 + Y^3 + Z^3 = 0$.

(ii) (10/25) Consider the singular projective curve C defined by $Y^2Z = X^3$ in $\mathbb{P}^2$. Show that the map $\phi : \mathbb{P}^1 \to C$ given by $[U,V] \to [UV^2, V^3, U^3]$ induces an isomorphism from $\mathbb{C}$ to $C \setminus \{(0,0,1)\}$. Moreover, show that there cannot be an isomorphism between C and $\mathbb{P}^1$.

4) Let $C = \{[X,Y,Z] \in \mathbb{P}^2 : F(X,Y,Z) = 0\}$ defined by a homogeneous polynomial F with no repeated factors. Assume that C is smooth. Consider the map

$$[X,Y,Z] \to \left[\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right]$$

from $\mathbb{P}^2$ to $\mathbb{P}^2$. This map is called the *polar mapping* of C.

(i) (5/25) Show that the polar mapping is well-defined as a map from $\mathbb{P}^2$ to $\mathbb{P}^2$.

(ii) (5/25) Recall that a conic is a curve in $\mathbb{P}^2$ defined by an equation:

$$F(X,Y,Z) = aX^2 + 2bXY + 2cXZ + dY^2 + 2eYZ + fZ^2$$

for some complex constants, a, b, c, d, e, f, not all zero. Show that the conic defined by $F(X,Y,Z)$ is smooth if and only if the matrix

$$M = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

is invertible.

(iii) (5/25) If C is a smooth conic, then show that the image of C under the polar mapping of C is also a smooth conic.

(iv) (10/25) Find the equations of the dual curves for the following conics.

a) (5/10) $XY + YZ + ZX = 0$,
b) (5/10) $XY + Z^2 = 0$.

5) (i) (5/100) State Belyi's theorem.

(ii) (5/100) Give a Belyi function for the Riemann surface associated to

$$z^2 = w(w-1)(w-2).$$

(iii) (5/100) Give the definition of a dessin d'enfant and explain how to get two permutations associated to a dessin d'enfant.

(iv) (10/100) Let $\sigma_0 = (1,5,4)(2,6,3)$ and $\sigma_1 = (1,2)(3,4)(5,6)$ be permutations in $\mathfrak{S}_6$. Compute the genus of the surface determined by the dessin d'enfant associated to this pair of permutations.

Solutions:

1) i) Branching index is defined by

$$b_f = \sum_{s\in S_2} (\deg(f) - |f^{-1}(s)|)$$

or it can also be defined by

$$b_f = \sum_{s\in S_2} \sum_{t\in f^{-1}(s)} (v_f(t) - 1)$$

where $v_f(t)$ is the ramification index of f at t (in local coordinates around t, f is given by $z \to z^{v_f(t)}$).

Riemann-Hurwitz states that

$$2g(S_1) - 2 = \deg(f)(2g(S_2) - 2) + b_f$$

ii) The projectivization gives the Fermat cubic $X^3 + Y^3 - Z^3 = 0$, which is non-singular (the derivatives vanish only at $X = Y = Z = 0$, which is not a point in $\mathbb{P}^2$. The points at ∞ are given by setting $Z = 0$, and $X = 1$, thus we get $Y^3 = -1$, which gives 3 points $[1, Y_1, 0], [1, Y_2, 0], [1, Y_3, 0]$ where Y_i are the 3 distinct solutions to $Y^3 + 1 = 0$. Thus, the projection to $x = X/Z$ has no branching at infinity. Moreover, in the affine part we see that projection x of the curve $x^3 + y^3 - 1$ has branched points when $x^3 = 1$, and along these points there is a single geometric preimage. Hence, we get

$$2g(S) - 2 = 3.(-2) + 3.2 = 0$$

Hence $g(S) = 1$.

iii) First, we check that $y^2 = x^6 - 1$ is smooth as an affine curve. Indeed, derivatives vanish for $x = y = 0$, and this point is not on the curve. So, the surface S will be obtained by adding points at infinity to C.

Consider the projection to x. For $|x| > N$ where N is sufficiently large, using the coordinates $z = 1/x$, $w = y$, C is given by $w^2 = \frac{1-z^6}{z^6}$. Choosing a branch of the square root in a neighborhood of $z = 1$, we can parametrize points of C by $(z, \frac{\sqrt{1-z^6}}{z^3})$ and $(z, \frac{-\sqrt{1-z^6}}{z^3})$. As these two are disjoint for $|z|$ sufficiently small, we see that the preimage of the punctured neighborhood of the infinity $D^\times = \{x \in \mathbb{C} : |x| > N\}$ has two components, and we can compactify C by adding those 2 points at infinity.

The projection to x has no branching at infinity since as $|x| \to \infty$, the six roots of the equation $x^6 = 1$ gives all the branching, and along these points there is a single geometric preimage. Hence, we get

$$2g(S) - 2 = 2.(-2) + 6.1 = 2$$

Hence $g(S) = 2$.

(A slightly tricky way of arguing that there is no branching at infinity is to observe that, if there were branching than the branching index b_f would be 7 which is an odd number, but this cannot be.)

2) i) + ii) Let $F_\lambda(X, Y, Z) = X^4 + \lambda(Y^4 + Z^4) - Y^2Z^2$. We compute

$$\begin{aligned} \partial F_\lambda/\partial X &= 4X^3 \\ \partial F_\lambda/\partial Y &= 4\lambda Y^3 - 2YZ^2 \\ \partial F_\lambda/\partial Z &= 4\lambda Z^3 - 2Y^2Z \end{aligned}$$

To find singular points, we set these to zero, which gives $X = 0$, $YZ^2 = 2\lambda Y^3$, $Y^2Z = 2\lambda Z^3$. To analyze the solutions, suppose first $\lambda = 0$, then we get $YZ^2 = Y^2Z = 0$, and this leads to solutions. $\{(0, 0, 1), (0, 1, 0)\}$. Next, suppose $\lambda \neq 0$. This implies that $Y \neq 0$, since otherwise we get $2\lambda Z^3 = 0$, which gives $Z = 0$ but as we are using projective co-ordinates, not all of X, Y, Z can be 0 at the same time. Thus, we may set $Y = 1$. As a result, we get simplified equations: $Z^2 = 2\lambda$ and $Z = 2\lambda Z^3$. It follows that $Z^4 = 1$, and the possible solutions are given as follows. For $\lambda = 1/2$, we have $\{(0, 1, 1), (0, 1, -1)\}$ and for $\lambda = -1/2$, we have $\{(0, 1, i), (0, 1, -i)\}$.

iii) If $X = Y = 0$, then it follows that $Z = 0$, hence the projection $[X : Y]$ is well-defined. Observe that for $X = 1, Y = 0$, the points on C_1 satisfy, $Z^4+1 = 0$, hence there are 4 distinct solutions in this case, hence $\deg(f) = 4$. To find the branch points, we may restrict to the chart $Y = 1$, then the equation of the curve becomes $Z^4 - Z^2 + 1 + X^4 = 0$. Hence,

$$Z^2 = \frac{1 \pm \sqrt{1 - 4(1 + X^4)}}{2}$$

This can fail to have 4 distinct solutions in two ways: 1) $1 = 4(1 + X^4)$, 2) $0 = 4(1 + X^4)$. In the first case, which happens for 4 distinct X values $\{X_1, X_2, X_3, X_4\}$, and the preimage consists of 2 distinct solutions $\{(1/\sqrt{2}, X_i), (-1/\sqrt{2}, X_i)\}$ (each with multiplicity 2). In the second case, which also happens for 4 distinct X values $\{X_1', X_2', X_3', X_4'\}$, the preimage consists of 3 distinct solutions $\{(0, X_i'), (1, X_i'), (-1, X_i')\}$. Now, the Riemann-Hurwitz formula gives

$$2g(C_1) - 2 = (\deg f)(2g(\mathbb{P}^1) - 2) + \sum_b (\deg f) - |f^{-1}(b)|) = -4.2 + 4.2 + 4.1 = 4$$

from which we conclude that $g(C_1) = 3$.

3) i) a) $Y^2Z = X(X - Z)(X - 2Z) = X^3 - 3ZX^2 + 2Z^2X$. Send $X \to X + Z$ to get

$$Y^2Z = (X + Z)^3 - 3Z(X + Z)^2 + 2Z^2(X + Z) = X^3 - XZ^2.$$

Hence, $g_3 = 0$, and the j-invariant is 1.

b) $X^3 + Y^3 + Z^3 = 0$. Send $Z \to Z - Y$ to get

$X^3 + Y^3 + (Z - Y)^3 = X^3 + Z^3 - 3Z^2Y + 3ZY^2 = 0$.

Next, send $Y \to (Y + \frac{Z}{2})$ to get

$X^3 + \frac{Z^3}{4} + 3ZY^2 = 0$.

Hence $g_2 = 0$, and the j-invariant is 0.

ii) Clearly, ϕ is well-defined as $UV^2 = V^3 = U^3 = 0$ implies $U = V = 0$. Furthermore, and inverse of ϕ on $C \backslash \{(0,0,1)\}$ is given by $\psi : [X,Y,Z] \to [X,Y]$, as one checks that $\psi \circ \phi : [U,V] \to [UV^2, V^3]$ is the identity map away from $[U,V] = [1,0]$.

There cannot be an isomorphism between C and $\mathbb{P}^1$ because such an isomorphism would show that C is smooth, however C is singular at $[0,0,1]$, which we can see by observing that, for $F = X^3 - Y^2Z$, we have

$$\begin{aligned}\partial F/\partial X &= 3X^2\\ \partial F/\partial Y &= -2YZ\\ \partial F/\partial Z &= Y^2\end{aligned}$$

vanish at $[0,0,1]$.

4) i) The polar mapping is well defined because by the Euler's identity not all of $\partial F_\lambda/\partial X$, $\partial F_\lambda/\partial Y$, $\partial F_\lambda/\partial Z$ can vanish except at a singular point of C, but we assume that C is non-singular.

ii) We observe that

$$\begin{aligned}\partial F/\partial X &= 2aX + 2bY + 2cZ\\ \partial F/\partial Y &= 2bX + 2dY + 2eZ\\ \partial F/\partial Z &= 2cX + 2eY + 2fZ\end{aligned}$$

In other words,

$$\begin{pmatrix}\partial F/\partial X\\ \partial F/\partial Y\\ \partial F/\partial Z\end{pmatrix} = 2M \begin{pmatrix}X\\ Y\\ Z\end{pmatrix}$$

Hence, there is a singular point (X,Y,Z) of C if and only if (X,Y,Z) is in the kernel of the multiplication by the given matrix. In other words, this is the case if and only if the matrix is not invertible.

iii) We have

$$F(X,Y,Z) = \begin{pmatrix}X & Y & Z\end{pmatrix} \begin{pmatrix}a & b & c\\ b & d & e\\ c & e & f\end{pmatrix} \begin{pmatrix}X\\ Y\\ Z\end{pmatrix} = \begin{pmatrix}X & Y & Z\end{pmatrix} M \begin{pmatrix}X\\ Y\\ Z\end{pmatrix}$$

If C is smooth, M is invertible, hence we can write

$$\begin{pmatrix}\partial F/\partial X & \partial F/\partial Y & \partial F/\partial Z\end{pmatrix} M^{-1} = \begin{pmatrix}2X & 2Y & 2Z\end{pmatrix}$$

Thus, equation of C turns into the equation (up to scalar multiple):

$$\begin{pmatrix}\partial F/\partial X & \partial F/\partial Y & \partial F/\partial Z\end{pmatrix} M^{-1} \begin{pmatrix}\partial F/\partial X \\ \partial F/\partial Y \\ \partial F/\partial Z\end{pmatrix} = 0$$

for the dual curve. In particular, dual of a smooth conic defined by M is a smooth conic defined by M^{-1}.

iv) a) M is given by

$$M = \begin{pmatrix}0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0\end{pmatrix}$$

Its inverse is

$$M^{-1} = \begin{pmatrix}-1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1\end{pmatrix}$$

Thus, the dual conic is defined by the equation:

$$-(X^2 + Y^2 + Z^2) + 2(XY + XZ + YZ) = 0$$

b) M is given by

$$M = \begin{pmatrix}0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1\end{pmatrix}$$

Its inverse is

$$M^{-1} = \begin{pmatrix}0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix}$$

Thus, the dual conic is defined by the equation:

$$Z^2 + 4XY = 0$$

5) i) Let S be a compact Riemann surface. The following statements are equivalent: 1) S can be defined over $\overline{\mathbb{Q}}$. 2) S admits a morphism $f : S \to \mathbb{P}^1$ with at most 3 branching points.

ii) Let C be the surface associated to the given polynomial. The projection to w is a degree 2 map which has ramification over $\{0, 1, 2, \infty\}$. Composing this with $w \to 1/w$, we get a function with ramification over $\{0, 1/2, 1, \infty\}$. Finally, we can compose this with the Belyi function $p(w) = 4w(1 - w)$ which is ramified only at $1/2$ and ∞, and has values $p(0) = p(1) = 0$, $p(1/2) = 1$, $p(\infty) = \infty$.

iii) A dessin denfant, or simply a dessin, is a pair $(X, \mathcal{D})$ where X is an oriented compact topological surface, and $\mathcal{D} \subset X$ is a finite graph such that: (i) $\mathcal{D}$ is connected (ii) $\mathcal{D}$ is bicoloured, i.e. the vertices are coloured either black or white, and the vertices connected by an edge have different colours. (iii) $X \setminus \mathcal{D}$ is the union of finitely many topological disks, which we call facets of $\mathcal{D}$.

We label the edges of $\mathcal{D}$ with numbers. Around each vertex of the dessin, we can draw a little disk which has an induced orientation from the orientation of the surface, hence it makes sense to talk about counter-clockwise rotation in this disk. The edge labelled i is incident to a white vertex, at such a vertex, we set $\sigma_0(i) = j$, where j is the edge that follows i in the counter-clockwise direction (among edges incident to that particular white vertex). We define σ_1 using black vertices in a similar fashion.

iv) From the given decomposition of σ_0 and σ_1, we deduce that there are 2 white vertices and 3 black vertices. $\sigma_1\sigma_0 = (1, 6, 4, 2, 5, 3)$, hence there is a unique face. Finally, the number of edges is 6. Hence, we can compute the Euler characteristic by:

$$2 - 2g = 5 - 6 + 1 = 0$$

which gives $g = 1$.