module code

candidate number

7CCMMS16T

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I affirm that I did not give or receive any unauthorised help on this exam and all working out is my own.

1-i) Let C be the curve defined by F. We have a regular map $\phi : \mathbb{P}^1 \to C$ defined by $[X:Y] \to [X^2 : XY : Y^2]$. Since $X^2Y^2 - (XY)^2 = 0$, we see that the image of F is in C.

Suppose $\phi([X_0 : Y_0]) = \phi([X_1 : Y_1])$. Then, we have

$$X_0^2 = \lambda X_1^2 \quad , \quad X_0Y_0 = \lambda X_1 Y_1 \quad , \quad Y_0^2 = \lambda Y_1^2 \quad \text{for } \lambda \in \mathbb{C}^\times$$

Suppose $X_0 \neq 0$ then $X_1 \neq 0$ from first equation. Dividing the second equation by the first one, we get

$$Y_0 / X_0 = Y_1 / X_1$$

Hence, $[X_0 : Y_0] = [X_1 : Y_1]$. If $X_0 = 0$, then we have $X_1 = 0$ from the first equation so again $[X_0 : Y_0] = [X_1 : Y_1]$

Thus ϕ is injective.

To prove surjectivity, given $[X_0 : Y_0 : Z_0]$ such that $X_0 Z_0 = Y_0^2$, let $X, Y \in \mathbb{C}$ such that $X^2 = X_0$, $Y^2 = Z_0$ then $(XY)^2 = Y_0^2$. If $XY = Y_0$ we have $\phi[X;Y] = [X_0, Y_0, Z_0]$

otherwise $\phi([X; -Y]) = [X_0; Y_0; Z_0]$

Draw a line

1-ii) Taking derivatives and setting them to zero we get

$$3(X^2 - aYZ) = 0$$
$$3(Y^2 - aXZ) = 0$$
$$3(Z^2 - aXY) = 0$$

Hence $a \neq 0$ as otherwise we get $X = Y = Z = 0$ which is not a point on $\mathbb{CP}^2$.

1-ii) continues.

We also note that if $X=0$, the second and third equations give $Y=Z=0$. By symmetry we conclude that $X \neq Y \neq Z \neq 0$.

We have
$$X^2 = aYZ$$
$$Y^2 = aXZ$$
$$Z^2 = aXY$$
multiplying both sides we get

$$X^2Y^2Z^2 = a^3X^2Y^2Z^2 \Rightarrow \boxed{a^3 = 1}$$

By multiplying the first equation by X, second by Y third by Z we get

$$X^3 = Y^3 = Z^3 = aXYZ$$

Without loss of generality we can assume $Z=1$

then $\boxed{X^3 = Y^3 = 1}$

Now let $\xi = e^{\frac{2\pi i}{3}}$ so that $1, \xi, \xi^2$ are 3rd roots of unity.

We can verify that the only singularities are given by.

$X:Y:Z$: $[1:1:1], [\xi;\xi^2;1], [\xi^2:\xi;1]$ for $a=1$

$[\xi,\xi:1], [1,\xi^2:1], [\xi^2,1,1]$ for $a=\xi$

$[\xi:1:1], [1;\xi;1], [\xi^2,\xi^2,1]$ for $a=\xi^2$

1 - iii) Following the hint we consider lines joining the singular points of C_a for $a = 1, \zeta, \zeta^2$

Let $a = 1$. The singular points are
$$s_1 = [1;1;1]$$
$$s_2 = [\zeta;\zeta^2;1]$$
$$s_3 = [\zeta^2;\zeta;1]$$

The lines through pairs of these are

$$l_{s_1,s_2} : \quad \zeta^2 X + Y + \zeta Z = 0$$
$$l_{s_2,s_3} : \quad X + Y + Z = 0$$
$$l_{s_3,s_1} \quad \zeta X + Y + \zeta^2 Z = 0$$

where we note that $1 + \zeta + \zeta^2 = 0$. It remains to observe that

$$(X+Y+Z)(\zeta X + Y + \zeta^2 Z)(\zeta^2 X + Y + \zeta Z)$$
$$= (\zeta X^2 + Y^2 + \zeta^2 Z + (1+\zeta)XY + (\zeta+\zeta^2)XZ + (1+\zeta^2)YZ)(\zeta^2 X + Y + \zeta Z)$$
$$= X^3 + Y^3 + Z^3 - 3XYZ.$$

The other cases are similar.

Start with a new page

2-i) Branching index is defined by

$$b_f = \sum_{s \in S_2} \left(\deg(f) - |f^{-1}(s)| \right)$$

or it can be defined by

$$b_f = \sum_{s \in S_2} \sum_{t \in f^{-1}(s)} \left(\nu_f(t) - 1 \right)$$

where $\nu_f(t)$ is the ramification index of f at t.
(in local co-ordinates around t, f is given by $z \to z^{\nu_f(t)}$)

Riemann-Hurwitz states that

$$2h-2 = \deg(f)(2g-2) + b_f.$$

draw a line

2-ii) For a non-constant holomorphic map Riemann-Hurwitz gives $2h-2 = d(2g-2)+b$ where $d = \deg(f) \geq 1$ and $b = b_f \geq 0$

If $g \geq h+1$, we have that

$$d(2g-2) + b \geq d(2h) + b \geq 2h$$

Hence $d(2g-2)+b$ cannot be equal to $2h-2$.

Contradiction.

2-iii) To check smoothness we observe that the only solutions to the system of equations

$$dX^{d-1} = dY^{d-1} = dZ^{d-1} = 0 \quad \text{is given by}$$

$$X = Y = Z = 0 \quad \text{which does not represent a point on } \mathbb{CP}^2.$$

π is well-defined because $[\lambda X : \lambda Y : \lambda Z] \to [\lambda X ; \lambda Y] = [X ; Y]$

For fixed $[X : Y]$ there are d solutions to $X^d + Y^d + Z^d = 0$ hence π has degree d. The only branching is when $X^d + Y^d = 0$. We may assume that $X = 1$ so there are d distinct branching points and the preimage above those points is geometrically unique., so the ramification index is $d - 1$. Thus applying Riemann-Hurwitz we have

$$2g(X_F) - 2 = -2d + d(d-1)$$

$$g(X_F) = \frac{(d-1)(d-2)}{2} \quad \text{as required.}$$

Start with a new page.

3-i) Let $\tau = u + iv$ with $v > 0$ and $z = x + iy$

we have

$$\left| e^{\pi i n^2 \tau + 2\pi i n z} \right| = \left| e^{\pi i u n^2 - \pi v n^2} \right| \left| e^{2\pi i n x - 2\pi n y} \right|$$

$$= e^{-\pi v n^2 - 2\pi n y}$$

For n large enough $|n| < \pi n (v n + 2y)$ hence

$$\left| e^{\pi i n^2 \tau + 2\pi i n z} \right| = e^{-\pi v n^2 - 2\pi n y} < e^{-|n|}$$

As a result, the series converges absolutely, and

draw a line.

very rapidly so.

3-ii) a) Clear since $e^{2\pi i n (z+1)} = e^{2\pi i n z}$

1) $\vartheta(z+\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n (z+\tau)}$

Re-indexing $n \to n-1$ we get

$$\sum_{n \in \mathbb{Z}} e^{\pi i (n-1)^2 \tau + 2\pi i (n-1)(z+\tau)} = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z} e^{\pi i \tau} e^{-2\pi i (z+\tau)}$$

$$= \left(\sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z} \right) e^{-\pi i n \tau - 2\pi i z}$$

3-iii) continues

c) The independence of a follows from part a)

Suppose $b>0$, we apply part b) to deduce

$$\vartheta(z+b\tau) = \vartheta(z+(b-1)\tau+\tau) = e^{-\pi i\tau - 2\pi i(z+(b-1)\tau)}\vartheta((z+b-1)\tau)$$

$$= e^{-(2b-1)\pi i\tau - 2\pi iz}\vartheta(z+(b-1)\tau)$$

By induction, this is equal to

$$e^{-(2b-1)\pi i\tau - 2\pi iz - \pi(b-1)^2\tau - 2\pi i(b-1)z}\vartheta(z)$$

$$= e^{-\pi ib^2\tau - 2\pi ibz}\vartheta(z) \quad \text{as required.}$$

If $b<0$ by what we proved we have

$$\vartheta(z-b\tau) = e^{-\pi ib^2\tau + 2\pi ibz}\vartheta(z)$$

Now let $z \to z+b\tau$ to conclude that

$$\vartheta(z+b\tau) = e^{\pi ib^2 z - 2\pi ib(z+b\tau)}\vartheta(z) = e^{-\pi ib^2\tau - 2\pi ibz}\vartheta(z)$$

as required.

Start with a new page.

4-i) Let $f(z,w), g(z,w) \in \mathbb{C}[z,w]$ be two polynomials

(Weak form of Bezout's theorem) If f and g are relatively prime, then the curves $f(z,w)=0$ and $g(z,w)=0$ intersect only at finitely many points.

(Weak form of Hilbert's Nullstellensatz) If f is irreducible and g vanishes at all points of the curve $f(z,w)=0$ then f divides g.

draw a line.

4-ii) A singularity of the affine curve defined by a polynomial $f(z,w)$ is given by common zeroes of $f(z,w)$, $\partial_z f(z,w)$, $\partial_w f(z,w)$.

Suppose $f(z,w)$ and $\partial_z f(z,w)$ have infinitely many common zeroes. then by weak form of Bezout's theorem $f(z,w)$ and $\partial_z f(z,w)$ must have a common divisor. But $f(z,w)$ is irreducible hence $f(z,w)$ must divide $\partial_z f(z,w)$ but this is impossible for degree reasons unless $\partial_z f(z,w)$ is identically zero in which case we can argue similarly using $\partial_w f(z,w)$.

4-iii) We compute the derivatives

$$\partial_X F = 4(X^2 - Z^2)X$$
$$\partial_Y F = -6YZ(Y+Z)$$
$$\partial_Z F = -4(X^2 - Z^2)Z - 6Y^2Z - 2Y^3$$

From which we conclude $\xi_1 = [-1:0;1]$, $\xi_2 = [1:0:1]$, $\xi_3 = [0;-1:1]$ are the singular points.

We compute the multiplicities in the affine chart $\{Z = 1\}$ which includes all the singular points. So let

$$f(x,y) = F(X;Y;1) = (x^2-1)^2 - y^2(2y+3).$$

$$\partial_{xx} f = 12x^2 - 4$$
$$\partial_{xy} f = 0$$
$$\partial_{yy} f = -12y$$

Thus we see that ∂_{xx} does not vanish for any of the singular points. Hence, the multiplicity of each of the singular points is $\underline{\underline{2}}$.

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5-i) Let S be a compact Riemann surface. The following statements are equivalent.

1) S can be defined over $\bar{\mathbb{Q}}$

2) S admits a morphism $f: S \to \mathbb{P}^1$ with at most 3 branching points.

draw a line.

5-ii) Recall that $\text{Branch}(g\circ f) = \text{Branch}(g) \cup g(\text{Branch}(f))$

We have $g(z) = z^d$ so $\text{Branch}(g) = \{0, \infty\}$

We have $\text{Branch}(f) = \{0, 1, \infty\}$

Thus $g(\text{Branch}(f)) = \{0, 1, \infty\}$. Hence, we see that $\text{Branch}(g\circ f) = \{0, 1, \infty\}$, so $g\circ f$ is a Belyi function.

draw a line

5-iii) The branch points of f are b such that $f^{-1}(b)$ consists of less than 4 distinct solutions (including possibly $b = \infty$) We immediately note that $f^{-1}(\infty) = \{1/2, \infty\}$ which shows that $b = \infty$ is a branch point. For $z \neq \infty$ we compute

$$f'(z) = -4\frac{(z-1)z(4z^2-4z+2)}{(2z-1)^3}$$

So ramification points are $\{0, 1/2, 1, \frac{1+i}{2}, \frac{1-i}{2}\}$ The branch points are obtained by calculating the value of f at these points.

$$f(0) = f(1) = 0 \quad f(1/2) = \infty \quad f\left(\frac{1+i}{2}\right) = f\left(\frac{1-i}{2}\right) = 1$$

5-iii) continues

Thus, the white vertices of the dessin are 0 and 1, and the black vertices are at $\frac{1+i}{2}$, $\frac{1-i}{2}$

Each vertex has degree 2, so the dessin looks like