Algebraic Curves - Homework 1 Solutions

Matthew Habermann, based on notes by Dogancan Karabas, Autumn 2019

1. **(i)** Show that $\mathbb{H}$ and $\mathbb{D}$ are isomorphic Riemann surfaces via the map:

$$\begin{aligned} \mathbb{H} &\to \mathbb{D} \\ z &\mapsto \frac{z-i}{z+i} \end{aligned}$$

Solution: We need to show that the map $f : \mathbb{H} \to \mathbb{D}$ with $f(z) = (z-i)/(z+i)$ is bijective and holomorphic.

Injectivity: $f(z) = f(w)$ implies $(z-i)/(z+i) = (w-i)/(w+i)$ which gives $z = w$, hence f is injective.

Surjectivity: Pick $w \in \mathbb{D}$. By definition of $\mathbb{D}$, $|w| < 1$. Examine the equation $w = f(z) = (z-i)/(z+i)$, by modifying it we get $z = i(1+w)/(1-w) = i(1+w)(1-\bar{w})/|1-w|^2$. We want to show $z \in \mathbb{H}$, i.e. $\text{Im}(z) > 0$. By the above relation,

$$\text{Im}(z) = \text{Re}((1+w)(1-\bar{w})) = \text{Re}(1-|w|^2+w-\bar{w}) = \text{Re}(1-|w|^2+2\text{Im}(w)i) = 1-|w|^2 > 0$$

since $|w| < 1$, hence $z \in \mathbb{H}$. This shows f is surjective.

Note: Bijectivity is a lot easier to show if you recognise f as Mobius transformation, and use its properties. Hint: it is automatically injective and it maps lines to circles which can be used to show the surjectivity.

Holomorphicity: We need to check the holomorphicity for each chart. But both $\mathbb{H}$ and $\mathbb{D}$ are subsets of $\mathbb{C}$ with the single charts $\phi = \text{id}\colon \mathbb{H} \to \mathbb{H} \subset \mathbb{C}$ and $\psi = \text{id}\colon \mathbb{D} \to \mathbb{D} \subset \mathbb{C}$. So, we only need to check $\psi \circ f \circ \phi^{-1} = \text{id} \circ f \circ \text{id}^{-1} = f$ is holomorphic. This is easy to see since f is quotient of two polynomials (polynomials are holomorpic and quotient of two holomorphic functions is holomorphic on the points which don't make the denominator equal to zero). f is defined on U, and its denominator $(z+i)$ is not zero for $z \in U$. Hence f is holomorphic.

$f\colon \mathbb{H} \to \mathbb{D}$ is bijective and holomorphic function, therefore it is an isomorphism of Riemann surfaces $\mathbb{H}$ and $\mathbb{D}$, which shows $\mathbb{H}$ and $\mathbb{D}$ are isomorphic.

(ii) Show that $\mathbb{C}$ and $\mathbb{D}$ are not isomorphic Riemann surfaces, by using Liouvilles theorem (a bounded entire function is necessarily constant).

Solution: Assume they are isomorphic. Then there exists a bijective holomorphic function $f\colon \mathbb{C} \to \mathbb{D}$ (f holomorphic means it is holomorphic on charts, but since $\mathbb{C}$ and $\mathbb{D}$ are subsets of $\mathbb{C}$ and they have single charts which are identities as in the previous part, f on charts equals to f itself). Since f is holomorphic and defined on $\mathbb{C}$, it is entire (this is the definition of being entire). Then by Liouville's theorem, it is constant, which shows that is not bijective. This contradicts with the assumption. Hence $\mathbb{C}$ and $\mathbb{D}$ are not isomorphic.

2. Show that the projective line

$$\mathbb{P}^1 = (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times \ ,$$

the space of lines through origin in $\mathbb{C}^2$ is a Riemann surface which is isomorphic to the Riemann sphere.

Solution: First, note that $\mathbb{P}^1$ is endowed with quotient topology. We need to show it is Hausdorff. For that, you can refer to Problem 9, or show directly by finding disjoint open neighbourhoods around any distinct pair of points in $\mathbb{P}^1$, or see the end of the solution.

Next, define $U_0 = \{[1 : z] : z \in \mathbb{C}\}$ and $U_1 = \{[z : 1] : z \in \mathbb{C}\}$. Obviously, they build an open cover of $\mathbb{P}^1$. Consider the maps $\phi_0 : U_0 \to \mathbb{C}$ and $\phi_1 : U_1 \to \mathbb{C}$ where $\phi_0([1 : z]) = z$ and $\phi_1([z : 1]) = z$ for all $z \in \mathbb{C}$. ϕ_0 and ϕ_1 are obviously homeomorphisms (if we go into detail: $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ is the projection and it is a quotient map, and when restricted to $V_0 = \{(1, z) : z \in \mathbb{C}\}$, it is injective, hence it is a homeomorphism. So $\pi|_{V_0} : V_0 \to U_0$ is invertible. Then ϕ_0 can be seen as the composition $\phi_0 : U_0 \xrightarrow{(\pi|_{V_0})^{-1}} V_0 \xrightarrow{\pi_2} \mathbb{C}$ where $\pi_2 : \mathbb{C}^2 \to \mathbb{C}$ is the projection to the second coordinate and $\pi_2|_{V_0}$ is clearly a homeomorphism, hence ϕ_0 is a homeomorphism. Similarly, ϕ_1 is a homeomorphism).

Finally, consider the map $\phi_1 \circ \phi_0^{-1} : \mathbb{C}^\times \to \mathbb{C}^\times$. $\phi_1 \circ \phi_0^{-1}(z) = \phi_1([1 : z]) = \phi_1([1/z : 1]) = 1/z$. Hence it is holomorphic. Note that we don't need to check $\phi_0 \circ \phi_1^{-1}$ is holomorphic since it is the inverse of the holomorphic map $\phi_1 \circ \phi_0^{-1}$, hence it is automatically holomorphic. In conclusion, $\mathbb{P}^1$ is a Riemann surface.

To see that $\mathbb{P}^1$ is isomorphic to the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$, consider the map $f : S^2 \to \mathbb{P}^1$ such that $f(z) = [z : 1]$ for $z \in \mathbb{C}$ and $f(\infty) = [1 : 0]$. This map is clearly a bijection. To see that it is holomorphic, we need to consider charts. The charts of S^2 are $\psi_0 : S^2 \setminus \{\infty\} \to \mathbb{C}$ and $\psi_1 : S^2 \setminus \{0\} \to \mathbb{C}$ such that $\psi_0(z) = z$ and $\psi_1(z) = 1/z$ and $\psi_1(\infty) = 0$. The charts of $\mathbb{P}^1$ are ϕ_0 and ϕ_1. Then we have

$$\begin{aligned}
\phi_0 \circ f \circ \psi_0^{-1}(z) &= 1/z \\
\phi_1 \circ f \circ \psi_0^{-1}(z) &= z \\
\phi_0 \circ f \circ \psi_1^{-1}(z) &= z \\
\phi_1 \circ f \circ \psi_1^{-1}(z) &= 1/z
\end{aligned}$$

which are all holomorphic. Hence f is holomorphic and it is isomorphism. Therefore, S^2 and $\mathbb{P}^1$ are isomorphic.

Note: Another way to show $\mathbb{P}^1$ is Hausdorff is as follows: S^2 and $\mathbb{P}^1$ are isomorphic, in particular they are homeomorphic. S^2 is Hausdorff, hence $\mathbb{P}^1$ is Hausdorff.

3. **(i)** Show that $\mathbb{C}/\Lambda$ is a Riemann surface for any lattice Λ.

   Solution: A lattice is $\Lambda := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 = \{s\omega_1 + t\omega_2 : (s, t) \in \mathbb{Z}^2\} \subset \mathbb{C}$ where $\omega_1, \omega_2 \in \mathbb{C}^\times$ such that $\omega_1/\omega_2 \notin \mathbb{R}$. Note that $\mathbb{C}/\Lambda$ has quotient topology and it is topologically a torus. Hence it is Hausdorff (to show Hausdorffness explicitly with a different method, see the end).

   Let $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ be the projection map. Define

   $$U'_{i,j} := (i, i + 1)\omega_1 \times (j, j + 1)\omega_2 = \{s\omega_1 + t\omega_2 : (s, t) \in (i, i + 1) \times (j, j + 1)\} \subset \mathbb{C}$$

   for $i, j \in \{0, 1/2\}$ and define $U_{i,j} = \pi(U'_{i,j})$. Then it is obvious that $\{U_{i,j} : i, j \in \{0, 1/2\}\}$ is an open cover for $\mathbb{C}/\Lambda$. Note that $\pi|_{U'_{i,j}}$ is homeomorphism (since π is a quotient map and it is injective on $U'_{i,j}$), so we can define the charts $\phi_{i,j} = (\pi|_{U'_{i,j}})^{-1} : U_{i,j} \to U'_{i,j}$ for $i, j \in \{0, 1/2\}$ which are homeomorphisms.

   Finally, we need to show $\phi_{i',j'} \circ \phi_{i,j}^{-1}$ is holomorphic for $i, j, i', j' \in \{0, 1/2\}$. We have

   $$\phi_{i',j'} \circ \phi_{i,j}^{-1} = (\pi|_{U'_{i',j'}})^{-1} \circ \pi : \phi_{i,j}(U_{i,j} \cap U_{i',j'}) \to \phi_{i',j'}(U_{i,j} \cap U_{i',j'}) \ .$$

   From this expression one can see that $\phi_{i',j'} \circ \phi_{i,j}^{-1}$ is translation on the connected components of the domain. Hence it is holomorphic and consequently $\mathbb{C}/\Lambda$ is a Riemann surface.

   To show this explicitly for one transition patch, we can show $\phi_{1/2,0} \circ \phi_{0,0}^{-1}$ is holomorphic. The other patches work in the same way. We have

   $$\begin{aligned}
   \phi_{0,0}(U_{0,0} \cap U_{1/2,0}) &= (0, 1/2)\omega_1 \times (0, 1)\omega_2 \sqcup (1/2, 1)\omega_1 \times (0, 1)\omega_2 \\
   \phi_{1/2,0}(U_{0,0} \cap U_{1/2,0}) &= (1/2, 1)\omega_1 \times (0, 1)\omega_2 \sqcup (1, 3/2)\omega_1 \times (0, 1)\omega_2 \ .
   \end{aligned}$$

Hence we have

$$\phi_{1/2,0} \circ \phi_{0,0}^{-1}(z) = z + \omega_1 \qquad \text{if } z \in (0, 1/2)\omega_1 \times (0,1)\omega_2$$
$$\phi_{1/2,0} \circ \phi_{0,0}^{-1}(z) = z \qquad \text{if } z \in (1/2, 1)\omega_1 \times (0,1)\omega_2$$

which shows $\phi_{1/2,0} \circ \phi_{0,0}^{-1}$ is translation on the connected components of the domain. Hence it is holomorphic.

Note: Using the open sets $U_{i,j}$, we can show $\mathbb{C}/\Lambda$ is Hausdorff as follows: For any two points $z_1 \neq z_2$ in $\mathbb{C}/\Lambda$, it is easy to see that there exists $i, j \in \mathbb{R}$ (not necessarily 0 or 1/2) such that $z_1, z_2 \in U_{i,j} \subset \mathbb{C}/\Lambda$. But $U_{i,j}$ is homeomophic to $U'_{i,j} \subset \mathbb{C}$ which is Hausdorff, hence $U_{i,j}$ is Hausdorff. So, there are disjoint open neighbourhoods of z_1, z_2 in $U_{i,j}$, and since $U_{i,j} \subset \mathbb{C}/\Lambda$, same holds true in $\mathbb{C}/\Lambda$. Hence $\mathbb{C}/\Lambda$ is Hausdorff.

(ii) Show that any such Riemann surface $\mathbb{C}/\Lambda$ is isomorphic to $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ for some $\tau \in \mathbb{H}$.

Solution: Let $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ for some $\omega_1, \omega_2 \in \mathbb{C}^\times$ such that $\omega_1/\omega_2 \notin \mathbb{R}$. Without loss of generality, assume $\omega_2/\omega_1 \in \mathbb{H}$ (if not, we have $\omega_1/\omega_2 \in \mathbb{H}$, so work with that). Then let $\tau = \omega_2/\omega_1$, which is in $\mathbb{H}$, and define $f\colon \mathbb{C}/\Lambda \to \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ with $f([z]) = [z/\omega_1]$. We will show that this is an isomorphism.

First, we need to show that it is well-defined: Let $[z_1] = [z_2] \in \mathbb{C}/\Lambda$, then $z_1 - z_2 \in \Lambda$, i.e. $z_1 - z_2 = s\omega_1 + t\omega_2$ for some $s, t \in \mathbb{Z}$. We have

$$\frac{z_1}{\omega_1} - \frac{z_2}{\omega_1} = \frac{s\omega_1 + t\omega_2}{\omega_1} = s + t\frac{\omega_2}{\omega_1} = s + t\tau \in \mathbb{Z} \oplus \mathbb{Z}\tau$$

hence $f([z_1]) = [z_1/\omega_1] = [z_2/\omega_1] = f([z_2])$ which shows f is well-defined.

Next, to show f is bijective, consider the map $g\colon \mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z}) \to \mathbb{C}/\Lambda$ with $g([z]) = [z\omega_1]$. We claim it is the inverse of f. We need to show g is well-defined: Let $[z_1] = [z_2] \in \mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$, then $z_1 - z_2 \in \mathbb{Z}\oplus\tau\mathbb{Z}$, i.e. $z_1 - z_2 = s + t\tau$ for some $s, t \in \mathbb{Z}$. We have

$$z_1\omega_1 - z_2\omega_1 = (s + t\tau)\omega_1 = s\omega_1 + t\omega_2 \in \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$$

hence $g([z_1]) = [z_1\omega_1] = [z_2\omega_1] = f([z_2])$ which shows g is well-defined. Also, $g \circ f([z]) = [z]$ and $f \circ g([z]) = [z]$, so g is the inverse of f, which shows f is bijective.

Finally, to show f is holomorphic, we need to consider charts. Charts of $\mathbb{C}/\Lambda$ are as in part (i). Charts of $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ are $\psi_{i,j} : V_{i,j} \to V'_{i,j}$ for $i, j \in \{0, 1/2\}$ where $V'_{i,j} = (i, i+1) \times (j, j+1)\tau$, $\pi'\colon \mathbb{C} \to \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ is the projection, $V_{i,j} = \pi'(V'_{i,j})$ and $\psi_{i,j} = (\pi'|_{V'_{i,j}})^{-1}$. Then

$$\psi_{i',j'} \circ f \circ \phi_{i,j}^{-1}(z) = (\pi'|_{V'_{i',j'}})^{-1} \circ f \circ \pi(z) = (\pi'|_{V'_{i',j'}})^{-1} \circ f([z]) = (\pi'|_{V'_{i',j'}})^{-1}([z/\omega_1]) = (z/\omega_1) + v$$

for some $v \in \mathbb{Z} \oplus \tau\mathbb{Z}$ on a connected component of the domain, hence it is holomorphic, which shows f is holomorphic.

In the end, f is isomorphism and therefore $\mathbb{C}/\Lambda$ is isomorphic to $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$.

4. Show that the subset of $\mathbb{C}^2$ consisting of the points of the form

$$(t^2, t^3 + 1), \qquad t \in \mathbb{C}$$

is an algebraic curve.

Solution: To show that it is an algebraic curve, we must show that it is the zero set of a polynomial equation. The above points are the zero set of the equation $f(z, w) = w^2 - z^3 - 2w + 1$, since

$$(t^3 + 1)^2 - t^6 - 2(t^3 + 1) + 1 = t^6 + 2t^3 + 1 - t^6 - 2t^3 - 2 + 1 = 0$$

5. **(i)** The algebraic curve defined by $f(z,w) = w^2 - p(z)$ where $p \in \mathbb{C}[z]$ a polynomial, is smooth if and only if $p(z)$ has no multiple roots.

   Solution: Note that equivalently, we can show f is singular if and only if $p(z)$ has multiple roots.

   Assume f is singular. Then there is a singular point $(z_0, w_0) \in \mathbb{C}^2$ of f, which means

$$\begin{aligned} f(z_0, w_0) &= w_0^2 - p(z_0) = 0 \\ f_z(z_0, w_0) &= -p'(z_0) = 0 \\ f_w(z_0, w_0) &= 2w_0 = 0 \end{aligned}$$

   which gives $p(z_0) = p'(z_0) = 0$. But if you take the Taylor expansion of p around z_0 we get

$$\begin{aligned} p(z) &= p(z_0) + p'(z_0)(z - z_0) + \frac{p''(z_0)}{2!}(z - z_0)^2 + \ldots + \frac{p^{(d)}(z_0)}{d!}(z - z_0)^d \\ &= \frac{p''(z_0)}{2!}(z - z_0)^2 + \ldots + \frac{p^{(d)}(z_0)}{d!}(z - z_0)^d \\ &= (z - z_0)^2 \left(\frac{p''(z_0)}{2!} + \ldots + \frac{p^{(d)}(z_0)}{d!}(z - z_0)^{d-2} \right) \end{aligned}$$

   hence $p(z)$ has a multiple root z_0.

   Conversely, assume $p(z)$ has a multiple root z_0. Then $p(z_0) = 0$ and $p(z)$ can be written as $p(z) = (z - z_0)^2 q(z)$ for some polynomial $q(z)$. Taking derivative we get $p'(z) = 2(z - z_0)q(z) + (z - z_0)^2 q'(z)$, hence $p'(z_0) = 0$ also. Then $(z_0, 0)$ is a singular point of f, since

$$\begin{aligned} f(z_0, 0) &= -p(z_0) = 0 \\ f_z(z_0, 0) &= -p'(z_0) = 0 \\ f_w(z_0, 0) &= 0 \end{aligned}$$

   and therefore f is singular.

6. Show that a smooth connected curve is irreducible (Hint. Show that the points at which two or more components of a curve intersect are singular.)

   Solution: Let $C = \{f = 0\}$ be a smooth connected curve for some smooth polynomial f. Assume it is reducible. Then $f = gh$ for some nonconstant polynomials g and h. Note that $C = \{g = 0\} \cup \{h = 0\}$ and $\{g = 0\}$ and $\{h = 0\}$ cannot be disjoint, because $\{g = 0\}$ and $\{h = 0\}$ are closed sets in C and if they are disjoint, $\{g = 0\} = C \setminus \{h = 0\}$ is open in C, hence $\{g = 0\}$ is either empty or whole C by the connectivity of C, which are not true since $\{g = 0\}$ and $\{h = 0\}$ are nonempty. So, $\{g = 0\}$ and $\{h = 0\}$ intersect, which means there exists (z_0, w_0) such that $g(z_0, w_0) = h(z_0, w_0) = 0$. But then we have

$$\begin{aligned} f(z_0, w_0) &= g(z_0, w_0)h(z_0, w_0) = 0 \\ f_z(z_0, w_0) &= g_z(z_0, w_0)h(z_0, w_0) + g(z_0, w_0)h_z(z_0, w_0) = 0 \\ f_w(z_0, w_0) &= g_w(z_0, w_0)h(z_0, w_0) + g(z_0, w_0)h_w(z_0, w_0) = 0 \end{aligned}$$

   which shows (z_0, w_0) is a singular point of f which is a contradiction since f is smooth. Hence, f should be irreducible.

7. **i)** Show that the polynomial $f(z,w) = z^p w + w^q z - 1$ is irreducible for all $p, q \geq 1$.

Solution: By the previous question, we must only show that the curve is smooth. A singular point, (z_0, w_0), must satisfy

$$\begin{aligned} f(z_0, w_0) &= 0 \\ f_z(z_0, w_0) &= pz_0^{p-1} w_0 + w_0^q = 0 \\ f_w(z_0, w_0) &= z_0^p + q w_0^{q-1} z_0 = 0. \end{aligned}$$

Note that we must have $z_0, w_0 \neq 0$, and so we can devide through by z_0 and w_0 where necessary to get

$$pz_0^{p-1} + w_0^{q-1} = 0 \tag{1}$$
$$z_0^{p-1} + q w_0^{q-1} = 0 \tag{2}$$
$$z_0^p w_0 + w_0^q z_0 - 1 = 0. \tag{3}$$

By multiplying the second equation by p, we have $w_0^{q-1} = pq w_0^{q-1} = 0$, which gives $(pq-1)w_0^{q-1}$. So either $w_0 = 0$, which is a contradiction, or $p = q = 1$ Therefore the curve is smooth if one of p or q is greater than 1. For $p = q = 1$, we have that the only solution is that $z_0 = w_0 = 0$, and so this case is smooth too. Therefore the curve defined by $f(z, w)$ is smooth $\implies$ irreducible.

ii) Show that the polynomial defined by $f(z, w) = z^p + w^q$ is irreducible if and only if $\gcd(p, q) = 1$.

Solution: We prove the forwards direction by contrapositive, i.e., by showing that if $gcd(p, q) > 1$, then the polynomial is reducible. Suppose that $\gcd(p, q) = r > 1$. Then $p = r\alpha$, $q = r\beta$, and we have

$$f(z, w) = (z^\alpha)^r + (w^\beta)^r.$$

If r is even, then we have

$$f(z, w) = (z^{\frac{r\alpha}{2}} + i w^{\frac{r\alpha}{2}})(z^{\frac{r\alpha}{2}} - i w^{\frac{r\alpha}{2}}).$$

If r is odd, then we have

$$f(z, w) = (z^\alpha + w^\beta)\left(\sum_{m=0}^{r-1} (-1)^m (z^\alpha)^{r-m-1} (w^\beta)^m \right).$$

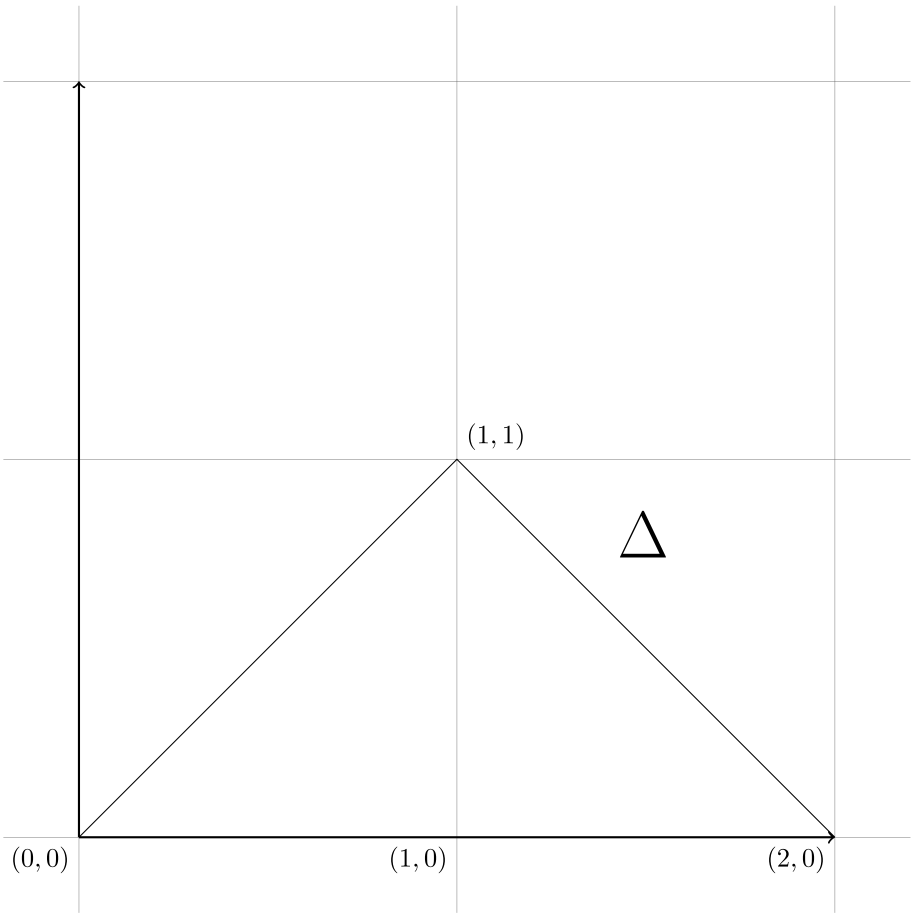
To show the backwards direction, suppose that $\gcd(p, q) = 1$. Then, assume that the Newton polytope is integrally decomposable and $\Delta = \Delta_1 + \Delta_2$. As in Proposition 1.16 of the notes, we have that if $v \in \Delta = \Delta_1 + \Delta_2$ is a *vertex*, then $v = v_1 + v_2$ is a unique decomposition of v such that $v_1 \in \Delta_1$ and $v_2 \in \Delta_2$. Since Δ is a line, we have that both Δ_i must be lines. As such, may assume that S_i, where $\Delta_i = \text{Conv}(S_i)$, has exactly two points, since any more points would be internal integral points, and would not affect the convex hull. Denote $(p, 0) = v$, $(0, q) = w$, and let $S_i = \{v_i, w_i\}$ be such that the unique decomposition of v and w is given by

$$\begin{aligned} v &= v_1 + w_1, \\ w &= v_2 + w_2. \end{aligned}$$

We must also have $v_1 + w_2 \in \Delta$ and $v_2 + w_1 \in \Delta$, and since there are only two points in Δ, we must have $v_1 + w_2 \in \{v, w\}$ and $v_2 + w_1 \in \{v, w\}$. If $v_1 + w_2 = v$, then by the uniqueness of the decomposition, we get that $w_1 = w_2$, which is a contradiction. In the case $v_1 + w_2 = w$, we have that $v_1 = v_2$, which is also a contradiction. Therefore, we have that if $\gcd(p, q) = 1$, then Δ is integrally indecomposable.

iii) Show that all polynomials of the form $f(z, w) = a + bz + cz^2 + dzw$ are irreducible.

Solution: The Newton polytope of $f(z, w)$ is given by



Similary to part (i), assume that it is integrally decomposable $\Delta = \Delta_1 + \Delta_2$. As in part (i), we have that each vertex is uniquely decomposable (not necessarily each point though!). Note that now we cannot make an assumption about the size of the set S_i, where $\text{Conv}(S_i) = \Delta_i$, although we must have at least two distinct elements in each S_i. Denote $r = (0, 0),\ s = (1, 0),\ t = (2, 0)$, and $u = (1, 1)$, and let $v_i, w_i \in S_i$ be the unique points such that

$$\begin{aligned} r &= v_1 + w_1, \\ t &= v_2 + w_1. \end{aligned}$$

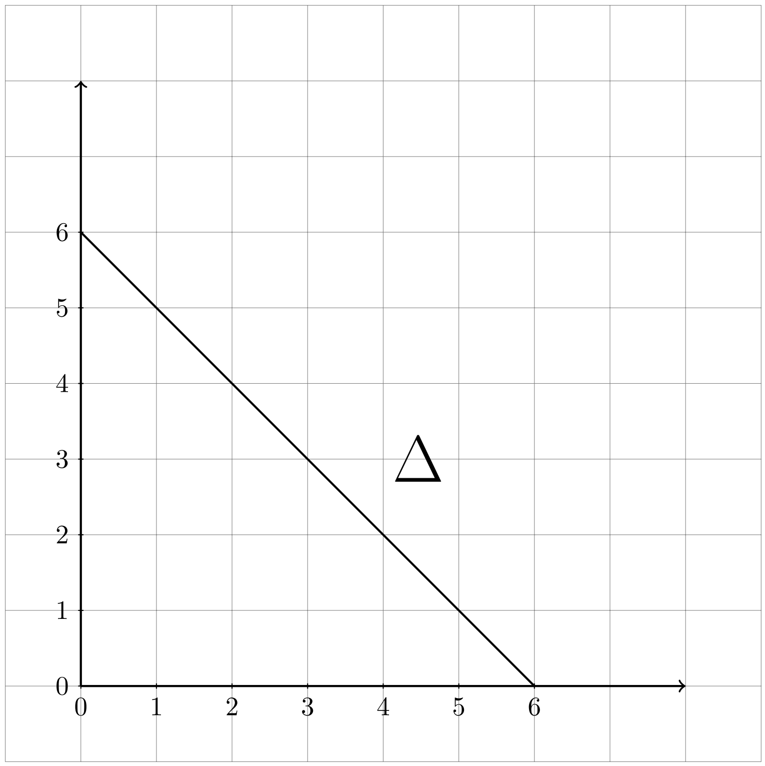
is the unique decomposition of r and t. Note that there are at least two other points in the sum, namely, $v_2 + w_2$ and $v_1 + w_2$. these points cannot be r or t, since the vertex decomposition is unique, and we are assuming that v_i and w_i are distinct. They cannot both be the point u for the same reason, and if both were s, then we would have $v_1 = v_2$, which is a contradiction. Therefore, we know that one must be u and one must be s. Suppose that $u = v_2 + w_2$, then, since $r = (0, 0)$, we have

$$\begin{aligned} (1, 1) = u &= u + r \\ &= v_1 + v_2 + w_1 + w_2 \\ &= s + t = (3, 0), \end{aligned}$$

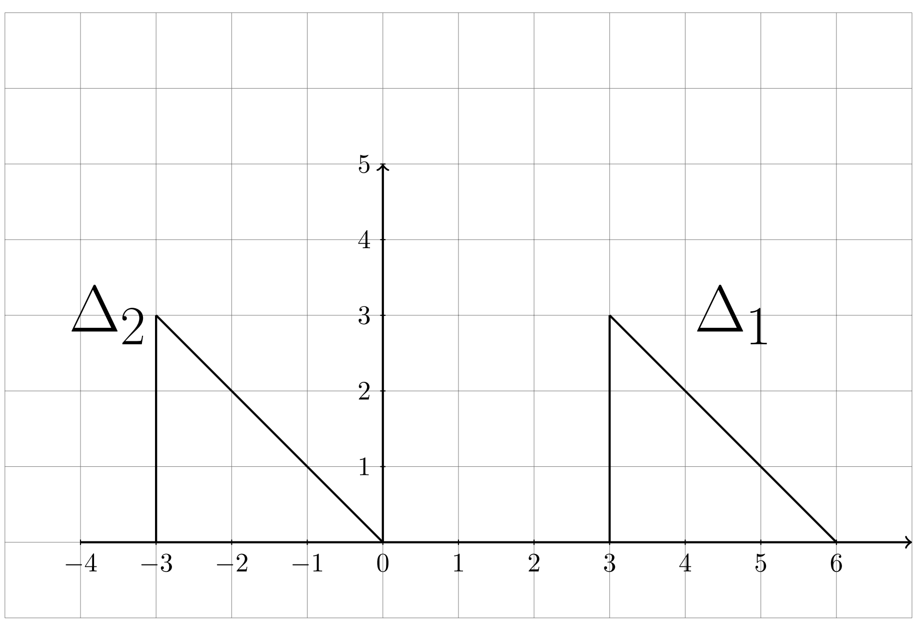
which is a contradiction. If, on the other hand, $u = v_1 + w_2$, then we draw a contradiction in a similar way. Therefore Δ is integrally indecomposable, and so $f(z, w)$ is irreducible.

iv) Show that although $f(z,w) = z^6 + w^6 + 1$ is irreducible, its Newton polytope is integrally decomposable. Find a field over which $f(z,w)$ is reducible.

Solution: Since $f(z,w)$ is smooth, it is irreducible. Its Newton polytope is given by



and this can decompose as



Note that this is not a unique decomposition. Over $\mathbb{Z}/3$, we have that $f(z,w) = (z^2+w^2+1)^3 = z^6+w^6+1$. We also have that $f(z,w) = (x^3+y^3+1)^2$ over $\mathbb{Z}/2$.

8. Examine the tangent directions at $(0,0)$ of the curves:

   (i) $f(x,y) = y^2 - x^3 - x^2$

   (ii) $g(x,y) = (x^4+y^4)^2 - x^2y^2$

   (iii) $h(x,y) = (x^4+y^4-x^2-y^2)^2 - 9x^2y^2$

   What are the multiplicites at $(0,0)$, and which singularities are ordinary?

   Solution:
   (i) We can see that the multiplicity of the equation in part (i) is two, and so

$$\begin{aligned} TC(x,y) &= \sum_{i+j=2} \frac{\partial^2 f}{\partial x^i \partial y^j}(0,0)\frac{x^i y^j}{i!j!} \\ &= x^2 - y^2 \\ &= (x+y)(x-y). \end{aligned}$$

   So this is an ordinary singularity, and the two tangent directions are $x \pm y = 0$

   (ii) From the definition, we see that the multiplicity of the singularity at $(0,0)$ is four. We then have that

$$\begin{aligned} TC(x,y) &= \sum_{i+j=4} \frac{\partial^4 g}{\partial x^i \partial y^j}(0,0)\frac{x^i y^j}{i!j!} \\ &= x^2y^2. \end{aligned}$$

   Therefore the tangent directions are the x and y axes, and the singularity is not ordinary.

   (iii) The multiplicity of the singularity at $(0,0)$ is four, and so

$$\begin{aligned} TC(x,y) &= \sum_{i+j=4} \frac{\partial^4 h}{\partial x^i \partial y^j}(0,0)\frac{x^i y^j}{i!j!} \\ &= x^4 - 7x^2y^2 + y^4 \\ &= (x^2-3xy+y^2)(x^2+3xy+y^2) \\ &= (x - \frac{3+\sqrt{5}}{2}y)(x - \frac{3-\sqrt{5}}{2}y)(x + \frac{3+\sqrt{5}}{2}y)(x + \frac{3-\sqrt{5}}{2}y), \end{aligned}$$

   and so this is an ordinary singularity whose tangent diretions are given by $(x \pm \frac{3\pm\sqrt{5}}{2}y)$, $(x \pm \frac{3\mp\sqrt{5}}{2}y)$.

9. Using a modification of the map $t \mapsto (\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1})$, show directly that the curve C defined by $f(z,w) = z^2+w^2-1$ is isomorphic to $\mathbb{C}^\times$.

   Solution: First, note that the map $f(t) = (\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1})$ is not defined on $i, -i$, but it is defined on the other points of $\mathbb{C}$. Also, by considering limits, we can even define it at ∞ by $f(\infty) = (0,1)$. So, our strategy is to show $f : S^2 \setminus \{i,-i\} = (\mathbb{C} \setminus \{i,-i\}) \cup \{\infty\} \to C$ is an isomorphism and $S^2 \setminus \{i,-i\}$ is isomorphic to $S^2 \setminus \{0,\infty\} = \mathbb{C}^\times$, which will show C is isomorphic to $\mathbb{C}^\times$.

$f\colon S^2 \setminus \{i, -i\} \to C$ is well-defined, since $f(t) = (\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1})$ and

$$\left(\frac{2t}{t^2+1}\right)^2 + \left(\frac{t^2-1}{t^2+1}\right)^2 - 1 = \frac{4t^2 + t^4 - 2t^2 + 1 - t^4 - 2t^2 - 1}{(t^2+1)^2} = 0$$

which shows $f(t) \in C$ if $t \neq \infty$. Also, $f(\infty) = (0, 1) \in C$ obviously.

To show f is injective, note that $f^{-1}((0,1)) = \{\infty\}$ only, so it is enough to show that f on $\mathbb{C} \setminus \{i, -i\}$ is injective. Pick $t, s \in \mathbb{C} \setminus \{i, -i\}$ and assume $f(t) = f(s)$. Then $(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}) = (\frac{2s}{s^2+1}, \frac{s^2-1}{s^2+1})$ which gives $\frac{t^2-1}{t^2+1} = \frac{s^2-1}{s^2+1}$ and that implies $t^2 = s^2$. Using this in the first component of the equality, we get $t = s$ and hence f is injective.

To show f is surjective, note that $f(\infty) = (0, 1) \in C$, so pick $(z_0, w_0) \neq (0, 1) \in C$ and show that there exists t such that $f(t) = (z_0, w_0)$ as follows: Consider the equation $f(t) = (\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}) = (z_0, w_0)$ that gives $w_0 = \frac{t^2-1}{t^2+1}$ which has two solutions by the Fundemental Theorem of Algebra (since $w_0 \neq 1$), which are t and $-t$. Note that $t \neq 0$ unless $w_0 = -1$. We know

$$f(t) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) = \left(\frac{2t}{t^2+1}, w_0\right) \in C$$
$$f(-t) = \left(\frac{-2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) = \left(\frac{-2t}{t^2+1}, w_0\right) \in C$$

which shows $z = \frac{2t}{t^2+1}$ and $z = \frac{-2t}{t^2+1}$ are solutions of the equations $f(z, w_0) = z^2 + w_0^2 - 1$. Assume $w_0 \neq -1$, then $t \neq 0$ and by the Fundamental Theorem of Algebra, these are all solutions. Since z_0 is also a solution for this equation, it need to be equal to $\frac{2t}{t^2+1}$ or $\frac{-2t}{t^2+1}$. Hence we have $f(t) = (z_0, w_0)$ or $f(-t) = (z_0, w_0)$. If we assume $w_0 = -1$, then $z_0 = 0$ and $f(0) = (0, -1) = (z_0, w_0)$. Therefore f is surjective.

To see f is holomorphic, we need to consider the charts of $S^2 \setminus \{i, -i\}$ (we don't need to consider the charts of the affine curve C explicitly, since by checking the holomorphicity of each component, we are implicitly using the charts of C). $S^2 \setminus \{i, -i\}$ has the charts $\phi_1 : S^2 \setminus \{i, -i, \infty\} = \mathbb{C} \setminus \{i, -i\} \to \mathbb{C}$ with $\phi_1(t) = t$ and $\phi_2 : S^2 \setminus \{i, -i, 0\} = (\mathbb{C} \setminus \{i, -i, 0\}) \cup \{\infty\} \to \mathbb{C}$ with $\phi_2(t) = 1/t$ for $t \neq \infty$ and $\phi_2(\infty) = 0$. Then

$$f \circ \phi_1^{-1}(t) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$$
$$f \circ \phi_2^{-1}(t) = \left(\frac{2/t}{(1/t)^2+1}, \frac{(1/t)^2-1}{(1/t)^2+1}\right) = \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$$

whose components are all holomorphic. Hence f is holomorphic, and consequently $f\colon S^2 \setminus \{i, -i\} \to C$ is an isomorphism.

Next, to show $\mathbb{C}^\times = S^2 \setminus \{0, \infty\}$ is isomorphic to $S^2 \setminus \{i, -i\}$, we can refer to Mobius transformations. It is a known fact that a Mobius transformation is an isomorphism between $S^2 \to S^2$. Also, by fixing the values of any three points, we can find a unique Mobius transformation corresponding to that. So let's define a Mobius transformation $m\colon S^2 \to S^2$ by $m(0) = i$, $m(\infty) = -i$ and $m(1) = 1$ (the last one is just to make m unique). By the argument before, there is a unique m satisfying these (one can give the formula of m explicitly, see any elementary complex analysis book). Then $m\colon S^2 \to S^2$ is an isomorphism and in this case it is obvious to see that if we restrict m to $S^2 \setminus \{0, \infty\}$, then $m\colon S^2 \setminus \{0, \infty\} \to S^2 \setminus \{i, -i\}$ is still an isomorphism (since it is still a bijection, and holomorphicity is preserved under restriction).

In the end, we get

$$\mathbb{C}^\times = S^2 \setminus \{0, \infty\} \xrightarrow{m} S^2 \setminus \{i, -i\} \xrightarrow{f} C$$

and since f and m are isomorphisms, the composition $f \circ m \colon \mathbb{C}^\times \to C$ is an isomorphism, which shows $\mathbb{C}^\times$ and C are isomorphic.

10. Given a non-constant polynomial $f(z, w)$ that is not independent of w, show that for all but finitely many values of z, there exists $w \in \mathbb{C}$ such that $f(z, w) = 0$. Deduce that algebraic curves in $\mathbb{C}^2$ are non-compact.

Solution: Let $f(z, w)$ be not independent of w. We can express it as

$$f(z, w) = p_0(z) + p_1(z)w + \ldots + p_d(z)w^d$$

where $p_i(z)$ are polynomials in z and $p_d(z)$ is not identically zero. Since $f(z, w)$ is not independent of w, we have $d \geq 1$. Note that $p_d(z) = 0$ only for finitely many z by Fundamental Theorem of Algebra, since $p_d(z)$ is not identically zero. Then if we take z_0 such that $p_d(z_0) \neq 0$, we get

$$f(z_0, w) = p_0(z_0) + p_1(z_0)w + \ldots + p_d(z_0)w^d$$

which is a degree d polynomial in the variable w. Since $d \geq 1$, it has a solution w_0, i.e. $f(z_0, w_0) = 0$. So we have shown that for all but finitely many values of z, there exists $w \in \mathbb{C}$ such that $f(z, w) = 0$.

Next, we want to show that an algebraic curve C in $\mathbb{C}^2$ is non-compact. Assume C is given by a polynomial $f(z, w)$ which is not independent of w. We will construct a sequence in C as follows: Pick $z = n \in \mathbb{N}$, then by the previous result there exists w_n (for all but finitely many n) such that $f(n, w_n) = 0$, i.e. $(n, w_n) \in C$. Then we have the sequence $\{(n, w_n) : n \in \mathbb{N} \setminus \{\text{finitely many points}\}\}$ in C. It is obvious to see that any subsequence of this sequence is not Cauchy, hence it is not convergent. This means C is not sequentially compact, hence C is not compact.

If we assume C is given by a polynomial $f(z, w)$ which is independent of w, i.e. $f(z, w) = g(z)$, then there exists z_0 such that $g(z_0) = 0$ (since by definition of an algebraic curve, $f(z, w) = g(z)$ is non-constant), so for any w, $f(z_0, w) = g(z_0) = 0$, i.e. $(z_0, w) \in C$. So we can construct a sequence $\{(z_0, n) : n \in \mathbb{N}\}$ in C. Any subsequence of this sequence is not Cauchy, hence it is not convergent. This means C is not sequentially compact, hence C is not compact.

Hence, any algebraic curve in $\mathbb{C}^2$ is non-compact.