Algebraic Curves - Homework 2 Solutions

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1. Show that the projective curve $X^2+Y^2 = Z^2$ is non-singular and in fact isomorphic to $\mathbb{P}^1$ as a Riemann surface.

   Solution For the curve $\overline{C} = \{X^2 + Y^2 - Z^2 = 0\}$ to be non-singular, we require that the there are no points $[X_0 : Y_0 : Z_0] \in \mathbb{P}^2$ such that $F(X_0, Y_0, Z_0) = F_X(X_0, Y_0, Z_0) = F_Y(X_0, Y_0, Z_0) = F_Z(X_0, Y_0, Z_0) = 0$. We have

$$F_X(X, Y, Z) = 2X,$$
$$F_Y(X, Y, Z) = 2Y,$$
$$F_Z(X, Y, Z) = -2Z.$$

   It is clear that the only point that the only solution to these equations is $X_0 = Y_0 = Z_0 = 0$, but this is not a point in projective space, so the curve is smooth.

   We showed in Question 2 of HW1 that $\mathbb{P}^1$ is isomorphic to the Riemann Sphere $S^2 = \mathbb{C} \cup \{\infty\}$. Therefore, it suffices to show that S^2 is isomorphic to $\overline{C}$ as Riemann surfaces.

   Consider the map

$$\begin{aligned} g : \mathbb{C} \cup \{\infty\} &\to \overline{C}, \\ t &\mapsto [2t : t^2 - 1 : t^2 : 1] \qquad \text{for } t \neq \infty \\ \infty &\mapsto [0 : 1 : 1]. \end{aligned}$$

   Note that $g(t) \in \overline{C}$, and where ∞ is mapped to comes from analysing the limit - away from $t = \pm i$, $[2t : t^2 - 1 : t^2 + 1] = [\frac{2t}{t^2+1} : \frac{t^2-1}{t^2+1} : 1]$, and this converges to $[0 : 1 : 1]$ as $t \to \infty$.

   Since Z is not a factor of F, we have that $[X : Y : 0] \in \overline{C}$ is a finite collection of points in $\mathbb{P}^2$ (in fact, there are exactly two, since F is a degree 2 polynomial, and the line $Z = 0$ is degree one). These two points are given by $\{[-i : 1 : 0], [i : 1 : 0]\}$. Note that

$$\overline{C} = C \cup \{[-i : 1 : 0], [i : 1 : 0]\},$$

   where $C = F(X, Y, 1) = 0$ is an affine curve in $\mathbb{C}^2$, and that $g(\pm i) = [\mp i : 1 : 0]$. To show that g is injective, we need to show that $g|_{S^2 \setminus \{\pm i\}} : S^2 \setminus \{\pm i\} \to C$ is bijective. Define

$$\begin{aligned} f = g|_{S^2 \setminus \{\pm i\}} : S^2 \setminus \{\pm i\} &\to C \subset \mathbb{C}^2 \\ t &\mapsto \left(\frac{2t}{t^2 + 1}, \frac{t^2 - 1}{t^2 + 1} \right), \qquad t \neq \infty \\ \infty &\mapsto (0, 1). \end{aligned}$$

   To check injectivity, it is enough to check on $\mathbb{C} \setminus \{\pm i\}$, since $f^{-1}(0, 1) = \infty$ only. Let $s, t \in \mathbb{C} \setminus \{\pm i\}$ such that $f(s) = f(t)$. Then $\left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) = \left(\frac{2s}{s^2+1}, \frac{s^2-1}{s^2+1}\right)$, which implies $s^2 = t^2$. Since $\frac{2t}{t^2+1} = \frac{2s}{s^2+1}$, we have that $s = t$.

To show f is surjective, note that $f(\infty) = (0,1) \in \mathbb{C}$, so pick $(z_0, w_0) \neq (0,1) \in \mathbb{C}$ and show that there exists t such that $f(t) = (z_0, w_0)$ as follows: Consider the equation $f(t) = (\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}) = (z_0, w_0)$ that gives $w_0 = \frac{t^2-1}{t^2+1}$ which has two solutions by the Fundemental Theorem of Algebra (since $w_0 \neq 1$), which are t and $-t$. Note that $t \neq 0$ unless $w_0 = -1$. We know

$$f(t) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) = \left(\frac{2t}{t^2+1}, w_0\right) \in \mathbb{C}$$
$$f(-t) = \left(\frac{-2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) = \left(\frac{-2t}{t^2+1}, w_0\right) \in \mathbb{C}$$

which shows $z = \frac{2t}{t^2+1}$ and $z = \frac{-2t}{t^2+1}$ are solutions of the equations $f(z, w_0) = z^2 + w_0^2 - 1$. Assume $w_0 \neq -1$, then $t \neq 0$ and by the Fundamental Theorem of Algebra, these are all solutions. Since z_0 is also a solution for this equation, it needs to be equal to $\frac{2t}{t^2+1}$ or $\frac{-2t}{t^2+1}$. Hence we have $f(t) = (z_0, w_0)$ or $f(-t) = (z_0, w_0)$. If we assume $w_0 = -1$, then $z_0 = 0$ and $f(0) = (0, -1) = (z_0, w_0)$. Therefore f is surjective.

Since f is injective and we have already seen that $g(\pm i) = [\mp i : 1 : 0]$, we have that g is bijective.

To show g is holomorphic, we need to consider charts of S^2 (we don't need to consider the charts of $\overline{C}$ explicitly, because when we check the holomorphicity of the function for each component, we are implicitly considering the charts of $\overline{C}$). S^2 has two charts $\phi_1 \colon S^2 \setminus \{\infty\} \to \mathbb{C}$ and $\phi_2 \colon S^2 \setminus \{0\} \to \mathbb{C}$ where $\phi_1 = \text{id}$ and $\phi_2(t) = 1/t$. Then $g \circ \phi_1^{-1}(t) = g(t) = [2t, t^2-1, t^2+1]$, and each component is holomorphic, which makes $g \circ \phi_1^{-1}$ holomorphic. Also, $g \circ \phi_2^{-1}(t) = g(1/t) = [2/t, 1/t^2 - 1, 1/t^2 + 1] = [2t, 1-t^2, 1+t^2]$ (note that $t = 0$ corresponds to ∞ in this chart and it goes indeed to $[0, 1, 1]$). Again each component is holomorphic, hence $g \circ \phi_2^{-1}$ is holomorphic. Therefore g is holomorphic.

We showed g is bijective and holomorphic, hence it is an isomorphism and S^2 (or equivalenly $\mathbb{P}^1$) is isomorphic to $\overline{C} = \{X^2 + Y^2 = Z^2\}$.

2. For which values of $\lambda \in \mathbb{C}$, are the curves defined by

$$X^3 + Y^3 + Z^3 + \lambda XYZ = 0$$

non-singular?

Solution: The projective curve above is defined by the homogeneous polynomial $F = X^3 + Y^3 + Z^3 + \lambda XYZ$. Let's find the values of λ which makes the curve singular: If $[X, Y, Z]$ is a singular point, then we have

$$F_X = 3X^2 + \lambda YZ = 0$$
$$F_Y = 3Y^2 + \lambda XZ = 0$$
$$F_Z = 3Z^2 + \lambda XY = 0\ .$$

Observe also that all X, Y and Z are non-zero (otherwise all would be zero which is not a point in $\mathbb{P}^2$). Then by multiplying the equations by X, Y, Z, respectively, and then leaving λ alone, the above equations can be equivalently expressed as

$$\lambda = \frac{-3X^3}{XYZ} = \frac{-3Y^3}{XYZ} = \frac{-3Z^3}{XYZ}$$

and by multiplying all terms we get $\lambda^3 = -3^3$ which has three roots $\lambda_i = -3s^i$ for $i = 0, 1, 2$ and $s = e^{2\pi i/3}$. To see that all these λ's give nonzero solutions X, Y, Z, note that above equations are equivalent to

$$X^3 = Y^3 = Z^3$$
$$\lambda = \frac{-3X^2}{YZ}$$

so by choosing $Y = Z = 1$ and $X = 1$, we get $\lambda = \lambda_0 = -3$; choosing $Y = Z = 1$ and $X = s$, we get $\lambda = \lambda_2 = -3s^2$; and choosing $Y = Z = 1$ and $X = s^2$, we get $\lambda = \lambda_1 = -3s$. Hence the curve is singular if and only if $\lambda^3 = -3$ or equivalently, the curve is nonsingular if and only if $\lambda^3 \neq -3$.

3. Consider the affine curve C defined by

$$y^2 = (x - a_1)(x - a_2) \dots (x - a_{2g+1})$$

where $a_1, \dots, a_{2g+1}$ are distinct points. Show that the affine curve C is non-singular. Show that the projectivization $\overline{C}$ intersects the line at infinity at a unique point. Show that for $g > 1$, this point is singular.

Solution: The curve is defined by the polynomial $f = \prod_{i=1}^{2g+1}(x - a_i) - y^2$. Assume it is singular, then there exists a point (x, y) such that

$$f = \prod_{i=1}^{2g+1} (x - a_i) - y^2 = 0$$
$$f_x = \sum_{j=1}^{2g+1} \prod_{\substack{i=1 \\ i \neq j}}^{2g+1} (x - a_i) = 0$$
$$f_y = -2y = 0$$

where the last equation gives $y = 0$, the first equation gives $x = a_i$ for $i = 1, \dots, 2g + 1$, but none of these x's solves the second equation. Hence the affine curve C is nonsingular.

Next, the projectivisation of C can be found by homogenising f, which is

$$F = \prod_{i=1}^{2g+1} (X - a_i Z) - Y^2 Z^{2g-1} \ .$$

Then $\overline{C} = \{F = 0\}$. The line at infinity in this case is the set of the points where $Z = 0$, so by setting $Z = 0$ in $\overline{C}$ we get $X^{2g+1} = 0$, which gives $X = 0$. But Y is free, so we have $[0, Y, 0]$. But Y can't be zero, then it can be scaled down to 1. Hence $\overline{C}$ and the line at infinity intersects at a unique point $[0, 1, 0]$.

To show that $[0, 1, 0]$ is a singular point of $\overline{C}$ for $g > 1$, consider

$$F_X = \sum_{j=1}^{2g+1} \prod_{\substack{i=1 \\ i \neq j}}^{2g+1} (X - a_i Z)$$
$$F_Y = 2YZ^{2g-1}$$
$$F_Z = \sum_{j=1}^{2g+1} (-a_j \prod_{\substack{i=1 \\ i \neq j}}^{2g+1} (X - a_i Z)) - (2g - 1)Y^2 Z^{2g-2}$$

If we set $X = 0$, $Y = 1$, and $Z = 0$, then $F_X = F_Y = F_Z = 0$ if $g > 1$, so $[0, 1, 0]$ is a singular point of $\overline{C}$ if $g > 1$. Note that if $g = 1$, $F_Z = -(2g - 1) \neq 0$.

4. Let A be a matrix in $\mathrm{GL}(3,\mathbb{C})$. Show that A induces a map from $\mathbb{P}^2$ to itself. Let F be a homogeneous polynomial defining a projective curve $C \subset \mathbb{P}^2$. Show that A sends C to another projective curve $A(C) \subset \mathbb{P}^2$ defined by another polynomial F_A. Show that if C is non-singular, so is $A(C)$ and the map $A\colon C \to A(C)$ gives an isomorphism of Riemann surfaces.

Solution: $\mathrm{GL}(3,\mathbb{C})$ is the set of invertible linear maps from $\mathbb{C}^3$ to $\mathbb{C}^3$ (or equivalently the set of 3×3 complex invertible matrices). $A \in \mathrm{GL}(3,\mathbb{C})$ gives the following map:

$$\begin{aligned} A\colon \mathbb{P}^2 &\to \mathbb{P}^2 \\ [X,Y,Z] &\mapsto [A(X,Y,Z)] \ . \end{aligned}$$

In short, if we denote $w = (X,Y,Z)$, we can equivalently describe it as $A([w]) = [A(w)]$ for $w \in \mathbb{C}^3 \setminus \{0\}$. The question is if this map is well-defined or not. To check, remember that $[w] = [\lambda w]$ for any $\lambda \in \mathbb{C}^\times$. Then they should be sent to the same points by A. Indeed, $A([\lambda w]) = [A(\lambda w)] = [\lambda A(w)] = [A(w)] = A([w])$. Also, we need to show that the image of each point is indeed in $\mathbb{P}^2$, i.e. $A(w) \neq 0$ for any $w \in \mathbb{C}^3 \setminus \{0\}$. But since A is invertible linear map, only point that goes to zero is zero, and since $w \neq 0$, $A(w)$ is never zero. This show that A is well-defined and that it induces a map from $\mathbb{P}^2$ to itself.

Next, let $C = \{F = 0\} \subset \mathbb{P}^2$ be a projective curve defined by a homogeneous polynomial F of degree d for some d. Let $F_A = F \circ A^{-1}$. It is easy to see that F_A is a polynomial of degree d. To see that it is homogeneous, we need to show $F_A(\lambda w) = \lambda^d F_A(w)$ for all $\lambda \in \mathbb{C}$. Indeed, $F_A(\lambda w) = F \circ A^{-1}(\lambda w) = F(\lambda A^{-1}(w)) = \lambda^d F(A^{-1}(w)) = \lambda^d F_A(w)$. Therefore it defines the projective curve $C' = \{F_A = 0\} \subset \mathbb{P}^2$.

We claim that A sends C to C', i.e. $A(C) = C'$. Pick $w \in C$, then $F(w) = 0$. We claim $A(w) \in C'$, i.e. $F_A(A(w)) = 0$. Indeed, $F_A(A(w)) = F \circ A^{-1} \circ A(w) = F(w) = 0$. Hence $A(C) \subset C'$. Now pick a point $v \in C'$, then $F_A(v) = F(A^{-1}(v)) = 0$. Let $w = A^{-1}(v)$. Then $F(w) = F(A^{-1}(v)) = 0$, hence $w \in C$ and $A(w) = A(A^{-1}(v)) = v$. This shows $C' \subset A(C)$. Therefore, $A(C) = C'$, i.e. A sends the projective curve defined by F to the projective curve defined by $F_A = F \circ A^{-1}$.

Next, assume $A(C) = C' = \{F_A = 0\}$ is singular. Then there exists $v \neq 0$ such that $(F_A)_X(v) = (F_A)_Y(v) = (F_A)_Z(v) = 0$. Since $F_A = F \circ A^{-1}$, by the chain rule we have

$$D(F_A)|_v = DF|_{A^{-1}(v)} \circ D(A^{-1})|_v$$

where $DG|_v$ denotes the derivative matrix of the map G evaluted at the point v. Since A is a linear map, its linearisation is itself, and so we have $D(A^{-1})|_v = A^{-1}$. Also,

$$D(F_A)|_v = [(F_A)_X(v)\ (F_A)_Y(v)\ (F_A)_Z(v)] = [0\ 0\ 0] = 0 \ .$$

So we have

$$0 = DF|_{A^{-1}(v)} \circ A^{-1}$$

which gives $DF|_{A^{-1}(v)} = 0$ since A^{-1} is invertible. Explicitly, we have

$$DF|_{A^{-1}(v)} = [F_X(A^{-1}(v))\ F_Y(A^{-1}(v))\ F_Z(A^{-1}(v))] = [0\ 0\ 0]$$

i.e. $F_X(A^{-1}(v)) = F_Y(A^{-1}(v)) = F_Z(A^{-1}(v)) = 0$. We have $A^{-1}(v) \neq 0$ since $v \neq 0$ and A is invertible linear map. Hence $A^{-1}(v)$ is a singular point of $C = \{F = 0\}$ and C is singular. Or to state differently, if C is a nonsingular curve, $A(C) = C'$ also needs to be a nonsingular curve.

Lastly, if C is nonsingular, $A(C) = C'$ is nonsingular and hence both are Riemann surfaces. Consider the map $A : C \to C'$. We want to show that A is bijective and holomorphic. Since $C' = A(C)$, A is already surjective. To show A is injective, assume $A([w]) = A([v])$ (w and v are nonzero). Then $[A(w)] = [A(v)]$ and $A(w) = \lambda A(v)$ for some $\lambda \in \mathbb{C}^\times$, which implies $A(w) = A(\lambda v)$. By applying A^{-1}, we get $w = \lambda v$ and hence $[w] = [v]$. Therefore A is injective.

A is holomorphic, because it is holomorphic as a map from $\mathbb{C}^3$ to $\mathbb{C}^3$, since it has only linear terms of X, Y, Z in each of its components. Therefore, A is an isomorphism between C and $A(C) = C'$.

5. **i)** If a line of slope t intersects the circle $x^2+y^2=1$ in the points $(-1,0)$ and (x,y), show x and y are both rational functions of t. By taking $t=p/q$, construct the general of the equation $x^2+y^2=z^2$ for x,y,z coprime integers.

Solution: The line of slope t is given by $y=tx+c$. Since the line intersects the $x-$axis at $(-1,0)$, we have that $c=t$. Therefore $y=t(x+1)$. Substituting this into the equation $x^2+y^2=1$, we get $x^2(t^2+1)+2t^2x+t^2-1=0$. This has two solutions, one at $x=-1$, and one at $x=\frac{1-t^2}{t^2+1}$. Substituting the second solution in to $x^2+y^2=1$, we get that $y=\frac{2t}{t^2+1}$. Therefore x and y are both rational functions of t.

Substituting $t=p/q$, we get

$$x=\frac{p^2-q^2}{p^2+q^2}, \qquad y=\frac{2pq}{p^2+q^2}.$$

Substituting this into the equation $x^2+y^2=1$ and clearing the denominator, we get

$$(p^2-q^2)^2+(2pq)^2=(p^2+q^2)^2.$$

We then have that

$$\begin{aligned} x &= p^2-q^2, \\ y &= 2pq, \\ z &= p^2+q^2 \end{aligned}$$

is the general solution to $x^2+y^2=z^2$. To see that x,y,z are coprime, note that if any of the two integers are relatively prime, then they are all pairwise relatively prime. To see this, note that if two of x,y, or z are relatively prime, then x,y,z have no common factor. To see that this implies that x,y,z are pairwise relatively prime, suppose that x,y, and z are not pairwise relatively prime, and suppose WLOG that $p|x$, $p|y$ and $p>1$. Then since $x^2+y^2=z^2$, we have that $p|z^2$, and so $p|z$. Therefore, if there is no common factor, then they are pairwise relatively prime.

Now, all we need to show is that to of x,y, and z are relatively prime. Consider $d=\gcd(x,z)=\gcd(p^2-q^2,p^2+q^2)$. Therefore, $d|2p^2$ and $d|2q^2$ (their sum and difference of the factors). Since p^2-q^2 and p^2+q^2 are odd, d must be odd, so $d|p^2$ and $d|q^2$. Since $\gcd(p,q)=1$, we have that $\gcd(p^2,q^2)=1$, and so $d=1$. Therefore x,y, and z are coprime.

ii) Given any smooth conic in $\mathbb{P}^2$ defined with coefficients in the rational numbers, find a change of co-ordinates over $\mathbb{Q}$ such that

$$\alpha x^2+\beta y^2=z^2, \qquad \alpha,\beta\in\mathbb{Q}.$$

Solution: We want to find an $A\in \mathrm{GL}(3,\mathbb{Q})$ such that $F_A(x,y,z)=\alpha x^2+\beta y^2-z^2=F\circ A^{-1}(x,y,z)$. Let $\boldsymbol{x}=(x,y,z)^T$, and $M=\begin{pmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{pmatrix}$. Then we have that

$$F(x,y,z)=\boldsymbol{x}^TM\boldsymbol{x},$$

and so F is a quadratic form. We are therefore looking for a matrix $B=A^{-1}$ such that

$$F(B(\boldsymbol{x}))=\boldsymbol{x}^TB^TMB\boldsymbol{x}=\boldsymbol{x}^T\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -1 \end{pmatrix}\boldsymbol{x}=\alpha x^2+\beta y^2-z^2.$$

Since the curve is smooth we have that

$$\begin{pmatrix} 2a & b & d \\ b & 2c & e \\ d & e & 2f \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

for all x, y, z. This means that the matrix is full rank, and so $\det M \neq 0$. Our goal is to find an invertible matrix with rational co-efficients, P, and a diagonal matrix $D = \begin{pmatrix} \delta & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \eta \end{pmatrix}$, where $\delta, \nu, \eta \in \mathbb{Q}$, such that $P^T MP = D$. In other words, we want to diagonalise M by a similarity transform. Since $\det M \neq 0$ and P is invertible, we have that $\det D \neq 0$, and so all of the $\delta, \nu, \eta \neq 0$. Finding such a P is possible, and there is a canonical way to construct it. By Gauss-Jordan elimination, we have that

$$\left(\begin{array}{ccc|ccc} a & 0 & 0 & 1 & \frac{-b}{2a} & \frac{-d}{2a} \\ \frac{b}{2} & c - \frac{b^2}{4a} & \frac{e}{2} - \frac{bd}{4a} & 0 & 1 & 0 \\ \frac{d}{2} & \frac{e}{2} - \frac{bd}{4a} & f - \frac{d^2}{4a} & 0 & 0 & 1, \end{array}\right)$$

and so if we define $P_1 = \begin{pmatrix} 1 & \frac{-b}{2a} & \frac{-d}{2a} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we have that

$$P_1^T MP_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & c - \frac{b^2}{4a} & \frac{e}{2} - \frac{bd}{4a} \\ 0 & \frac{e}{2} - \frac{bd}{4a} & f - \frac{d^2}{4a}. \end{pmatrix}.$$

Continuing this process, we find

$$\left(\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 0 & c - \frac{b^2}{4a} & 0 & 0 & 1 & -\left(\frac{\frac{e}{2} - \frac{bd}{4a}}{c - \frac{b^2}{4a}}\right) \\ 0 & \frac{e}{2} - \frac{bd}{4a} & \frac{-4acf+ae^2+b^2f-bde+cd^2}{b^2-4ac} & 0 & 0 & 1, \end{array}\right),$$

and so we have that $P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \left(\frac{2ae-bd}{b^2-4ac}\right) \\ 0 & 0 & 1 \end{pmatrix}$. Now, by defining

$$P = P_1 P_2 = \begin{pmatrix} 1 & -\frac{b}{2a} & \frac{2cd-be}{b^2-4ac} \\ 0 & 1 & \frac{2ae-bd}{b^2-4ac} \\ 0 & 0 & 1 \end{pmatrix},$$

we have that

$$P^T MP = P_2^T P_1^T MP_1 P_2 = \begin{pmatrix} a & 0 & 0 \\ 0 & c - \frac{b^2}{4a} & \\ 0 & 0 & \frac{cd^2-bde+ae^2+b^2f-4acf}{b^2-4ac}. \end{pmatrix}$$

By the rationality of all of the entries of M we have that all of the entries in both P and D are rational. We have therefore shown that the original conic is projectively equivalent over $\mathbb{Q}$ to $\delta x^2 + \nu y^2 + \eta z^2 = 0$. We can then divide through by $\frac{-1}{\eta}$ to put the curve into the desired form. Therefore the correct transformation is $A = P^{-1} \in \mathrm{GL}(3, \mathbb{Q})$.

6. **i)** Show that if $f : S \to T$ is a proper non-constant holomorphic map between connected Riemann surfaces then $f^{-1}(t)$ is a finite set for all $t \in T$.

Solution: Fix a point $t \in T$. f is a proper map and $\{t\}$ is a compact set, hence $f^{-1}(\{t\})$ is compact. Our aim is to show that $f^{-1}(t)$ also a discrete subset of S, which will show that $f^{-1}(t)$ is finite.

Pick a point $s \in f^{-1}(t)$. We want to show that there is an open neighbourhood $W \subset S$ of s such that $f^{-1}(t) \cap W = \{s\}$. Consider the charts $\phi\colon U \to U' \subset \mathbb{C}$ where $s \in U$ and $\psi\colon V \to V' \subset \mathbb{C}$ where $t \in V$ and let $s' = \phi(s)$ and $t' = \psi(t)$. Then consider the representation $\hat{f} = \psi \circ f \circ \phi^{-1}$ of f on these charts. Since f is nonconstant holomorphic map and S connected, f is not constant even locally anywhere, so $\hat{f}$ is nonconstant. $\hat{f}$ is also holomorphic since f is holomorphic. Then by the identity theorem for the holomorphic functions on connected open subsets of complex plane, we know $(\hat{f})^{-1}(t')$ needs to be a discrete subset of U' (otherwise $\hat{f}$ would be constant), in particular, since $s' \in (\hat{f})^{-1}(t')$, there is an open neighborhood $W' \subset U'$ of s' such that $(\hat{f})^{-1}(t') \cap W' = \{s'\}$. Let $W = \phi^{-1}(W')$. Then W is an open neighbourhood of s in S and it can be easily seen that $f^{-1}(t) \cap W = \{s\}$. Hence $f^{-1}(t)$ is a discrete subset of S. It is also compact, hence $f^{-1}(t)$ is finite. Note that we don't need T to be connected to show this.

ii) Show that if S is connected, a non-constant holomorphic function $f : S \to T$ between Riemann surfaces is an open mapping i.e. it sends open sets to open sets.

Solution: Pick an open subset W of S. We will show that $f(W)$ is open in T. For that, pick $t \in f(W)$. We need to show that there is an open neighbourhood $Y \subset f(W)$ of t.

Pick $s \in f^{-1}(t)$ and consider the charts $\phi\colon U \to U' \subset \mathbb{C}$ where $s \in U$ and $\psi\colon V \to V' \subset \mathbb{C}$ where $t \in V$. Then consider the representation $\hat{f} = \psi \circ f \circ \phi^{-1}$ of f on these charts. Define $W' = \phi(W \cap U) \subset U'$. Since W and U are open and ϕ is a homeomorphism, W' is open in U'. Since f is nonconstant holomorphic map and S connected, f is not constant even locally anywhere, so $\hat{f}$ is nonconstant. $\hat{f}$ is also holomorphic since f is holomorphic. Then by the open mapping theorem for nonconstant holomorphic functions on connected open subsets of complex plane, $\hat{f}$ is an open mapping and $\hat{f}(W')$ is open. Let $Y = \psi^{-1}(\hat{f}(W')) \subset T$. Since ψ is a homeomorphism, Y is also open. It is easy to see that $Y \subset f(W)$ and $t \in Y$, hence Y is an open neighbourhood of t in $f(W)$. Hence $f(W)$ is open and f is an open mapping.

Better solution: We can prove this without using the open mapping theorem for complex plane as follows: Pick an open subset W of S and $s \in W$. f is not constant near s, hence by Theorem 23 in lecture notes, f can be represented on a neighbourhood $U_s \subset W$ of s as $z \mapsto z^n$ (on charts), which is an open mapping. Hence $f|_{U_s}$ is an open mapping, in particular $f(U_s)$ is open in T. Then

$$f(W) = f\left(\bigcup_{s \in W} U_s\right) = \bigcup_{s \in W} f(U_s)$$

and since $f(U_s)$ is open for any $s \in W$, $f(W)$ is open in T. This shows f is an open mapping.

Bonus Consequence: If S is connected and compact, and T is connected, then any non-constant holomorphic function $f : S \to T$ between Riemann surfaces is surjective and T is compact.

Proof. S is compact, hence $f(S)$ is also compact by continuity of f. Since T is a Riemann surface, it is Hausdorff, and $f(S)$ is a compact subset of T, hence $f(S)$ is closed in T. S is connected and f is nonconstant holomorphic function, by Problem 6(ii) f is an open mapping, which shows $f(S)$ is open in T since S is open in S. So, $f(S)$ is open and closed in T and T is connected, hence $f(S) = T$, i.e. f is surjective and $T = f(S)$ is compact. □

Note: It is easy to see this also proves Theorem 20 in the lecture notes: Every holomorphic function $f\colon S \to \mathbb{C}$ defined on a compact connected Riemann surface S is constant.

7. View a non-constant polynomial $f(z) \in \mathbb{C}[z]$ as a holomorphic function from $f: \mathbb{P}^1 \to \mathbb{P}^1$ by sending ∞ to ∞. Show directly that $v_f(\infty)$ is the same as the degree of the polynomial.

   Solution: Let f be a polynomial of degree d, then it can be written as $f(z) = a_0 + a_1 z + \ldots + a_d z^d$ for some $a_0, a_1, \ldots, a_d \in \mathbb{C}$ where $a_d \neq 0$. We have $f(\infty) = \infty$, so to find $v_f(\infty)$, express f in local coordinates (centered at ∞ (in domain) and ∞ (in range)): Pick the chart $\phi\colon \mathbb{P}^1 \setminus \{0\} \to \mathbb{C}$ for both the domain and range where $\phi(z) = 1/z$ and $\phi(\infty) = 0$. Then the local presentation will be $\hat{f} = \phi \circ f \circ \phi^{-1}$ and we get

$$\hat{f}(z) = \frac{1}{f(1/z)} = \frac{1}{a_0 + a_1 \frac{1}{z} + \ldots + a_d \frac{1}{z^d}} = \frac{1}{a_d} z^d \left(\frac{1}{1 + \ldots \frac{a_0}{a_d} z^d} \right).$$

   Domain of $\hat{f}$ is around $z = 0$, so by shrinking the domain, we can make sure that z is very close to 0, which makes the denominator above close to 1, hence it is nonzero. So $1/(1 + \ldots \frac{a_0}{a_d} z^d)$ is holomorphic in the domain, hence it has the Taylor expansion $1 + b_1 z + b_2 z^2 + \ldots$ for some $b_1, b_2, \ldots \in \mathbb{C}$. Therefore we can write

$$\hat{f}(z) = \frac{1}{a_d} z^d (1 + b_1 z + b_2 z^2 + \ldots)$$

   which is the Taylor expansion of $\hat{f}$ around 0. The lowest power whose coefficient is nonzero is d, hence $v_f(\infty) = d$, which is the degree of the polynomial f.

   Note: Using this, one can prove the Fundamental Theorem of Algebra! Hint: Observe that $v_f(\infty) = d_f(\infty) = \deg f$ ($\deg f$ is the degree defined in Definition 26 in lecture notes) and we showed $v_f(\infty) = d$, so the classical degree d of a polynomial is the same as $\deg f$. Then consider also $\deg f = d_f(0)$, and calculate $d_f(0)$ explicitly.

8. Another proof of the fact $K(\mathbb{P}^1)$ is isomorphic to $\mathbb{C}(z)$ can be given as follows. Given $f \in K(\mathbb{P}^1)$, let z_i for $i = 1, \ldots, n$ be the zeros of f and p_j for $j = 1, \ldots, m$ be the poles of f (repeated as necessary). Consider the function:

$$g(z) = \frac{\prod_{i=1}^n (z - z_i)}{\prod_{j=1}^m (z - p_j)}$$

   Show that $f(z)/g(z)$ is a meromorphic function with no zeroes and poles. Hence, it is constant.

   Solution: First, note that zeroes and poles of $f: \mathbb{P}^1 \to \mathbb{C} \cup \{\infty\}$ given in the problem are in fact zeroes and poles of $f \circ \phi_1^{-1} = f|_{\mathbb{C}}$ on the chart $\phi_1 : \mathbb{P}^1 \setminus \{\infty\} = \mathbb{C} \xrightarrow{\text{id}} \mathbb{C}$. Also, $g: \mathbb{P}^1 \to \mathbb{C} \cup \{\infty\}$ given in the problem is actually $g \circ \phi_1^{-1} = g|_{\mathbb{C}}$, but obviously, you can determine the point $g(\infty)$ by the continuity of g (i.e. by taking the limit $z \to \infty$).

   To see that $f/g: \mathbb{P}^1 \to \mathbb{C} \cup \{\infty\}$ is meromorphic, we need to consider two charts of $\mathbb{P}^1$, namely ϕ_1 above and $\phi_2 : \mathbb{P}^1 \setminus \{0\} = \mathbb{C}^\times \cup \{\infty\} \xrightarrow{1/z} \mathbb{C}$. On the chart ϕ_1, we have

$$\left(\frac{f}{g}\right) \circ \phi_1^{-1}(z) = \frac{f(z)}{g(z)} = \frac{f(z) \prod_{j=1}^m (z - p_j)}{\prod_{i=1}^n (z - z_i)}$$

   which is obviously meromorphic since f is meromorphic, $\prod_{j=1}^m (z - p_j)$ and $\prod_{i=1}^n (z - z_i)$ are holomorphic, and $\prod_{i=1}^n (z - z_i)$ is not identically zero. On the chart ϕ_2, we have

$$\left(\frac{f}{g}\right) \circ \phi_2^{-1}(z) = \frac{f(1/z)}{g(1/z)} = \frac{f(1/z) \prod_{j=1}^m (1/z - p_j)}{\prod_{i=1}^n (1/z - z_i)} = \frac{f(1/z) z^n \prod_{j=1}^m (1 - p_j z)}{z^m \prod_{i=1}^n (1 - z_i z)}$$

   where $f(1/z)$ is meromorphic (since f is meromorphic on every chart), $z^n \prod_{j=1}^m (1 - p_j z)$ and $z^m \prod_{i=1}^n (1 - z_i z)$ are holomorphic, and $z^m \prod_{i=1}^n (1 - z_i z)$ is not identically zero. Hence f/g is meromorphic.

To understand the zeroes and poles of f/g, first consider f/g on the chart ϕ_1, i.e. on $\mathbb{C}$. Since $z_1, \ldots, z_n$ are all the zeros and $p_1, \ldots, p_m$ are all the poles of f (repeated as necessary) on $\mathbb{C}$, $f|_{\mathbb{C}}$ can be expressed as

$$f(z) = \frac{\prod_{i=1}^{n}(z - z_i)}{\prod_{j=1}^{m}(z - p_j)} h(z) = g(z)h(z)$$

where $h(z)$ has no zeroes and poles. Therefore $(f/g)(z) = h(z)$ on $\mathbb{C}$. This means f/g has no zeroes and poles on $\mathbb{C}$.

Finally, assume f/g is nonconstant. Since f/g is meromorphic, it can be seen as a map $f/g \colon \mathbb{P}^1 \to \mathbb{P}^1$ which is holomorphic (and not constantly equal to ∞). $\mathbb{P}^1$ is compact and connected, so by thebonus consequence at the end of problem 6, f/g is surjective, which is impossible since if $(f/g)(\infty) = 0$, then $(f/g)^{-1}(\infty) = \emptyset$ or if $(f/g)(\infty) = \infty$, then $(f/g)^{-1}(0) = \emptyset$, because f/g has no zeroes and poles on $\mathbb{C}$. Hence, f/g is constant (and not constantly equal to ∞ by meromorphicity).

This shows that any $f \in K(\mathbb{P}^1)$ can be written as $f = kg$ where $k \in \mathbb{C}$ is constant and g is as above, so $kg \in \mathbb{C}(z)$ (quotient of two polynomials). This shows $K(\mathbb{P}^1) \subset \mathbb{C}(z)$. Also, it is obvious to see $\mathbb{C}(z) \subset K(\mathbb{P}^1)$. Hence we get $K(\mathbb{P}^1) = \mathbb{C}(z)$.

Note: In fact, we have $(f/g)(\infty) \neq 0$ and $(f/g)(\infty) \neq \infty$, since otherwise we would have $(f/g)(z) = 0$ for all $z \in \mathbb{P}^1$ or $(f/g)(z) = \infty$ for all $z \in \mathbb{P}^1$ since f/g is constant, and this is impossible since f/g has no zeroes and poles on $\mathbb{C}$. So f/g has no zeroes and poles in whole $\mathbb{P}^1$.

9. Recall that if $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{C}$ with $\mathrm{Im}\tau > 0$, is a lattice, we have a Riemann surface structure on $\mathbb{C}/\Lambda$. Show that $K(\mathbb{C}/\Lambda)$ is isomorphic to the field of doubly periodic meromorphic functions on $\mathbb{C}$ with period $(1, \tau)$.

   Solution: Pick $f \in K(\mathbb{C}/\Lambda)$. Define $g := f \circ \pi \colon \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ where $\pi \colon \mathbb{C} \to \mathbb{C}/\Lambda$ is the projection. We want to show that g is a doubly periodic meromorphic function on $\mathbb{C}$ with period $(1, \tau)$. We have $g(z+1) = f([z+1]) = f([z]) = g(z)$ and $g(z+\tau) = f([z+\tau]) = f([z]) = g(z)$ for all $z \in \mathbb{C}$. Hence g is a doubly periodic with period $(1, \tau)$.

   To see that g is meromorphic, equivalently we can show that it is holomorphic as a map $g \colon \mathbb{C} \to \mathbb{P}^1$. Since $f \in K(\mathbb{C}/\Lambda)$, $f : \mathbb{C}/\Lambda \to \mathbb{P}^1$ is holomorphic. If we also show π is holomorphic, then f will be holomorphic since $f = g \circ \pi$. To show $\pi \colon \mathbb{C} \to \mathbb{C}/\Lambda$ is holomorphic, consider the charts of $\mathbb{C}/\Lambda$ (see Hw 1, Problem 5): Define $V'_{i,j} = \{s + t\tau : (s,t) \in (i, i+1) \times (j, j+1)\} \subset \mathbb{C}$ and $V_{i,j} = \pi(V'_{i,j})$ for $i, j \in \{0, 1/2\}$. We have the charts $\psi_{i,j} = (\pi|_{V'_{i,j}})^{-1} \colon V_{i,j} \to V'_{i,j}$ for $i, j \in \{0, 1/2\}$. We need to show

   $$\psi_{i,j} \circ \pi = (\pi|_{V'_{i,j}})^{-1} \circ \pi \colon \bigsqcup_{(k,l) \in \mathbb{Z}^2} V'_{i+k, j+l} \longrightarrow V'_{i,j}$$

   is holomorphic for $i, j \in \{0, 1/2\}$. But it is easy to see that $\psi_{i,j} \circ \pi$ is just translation on each connected component (to be explicit, $(\psi_{i,j} \circ \pi)(z) = z - (k + l\tau)$ on $V'_{i+k, j+l}$), hence it is holomorphic. Therefore π is holomorphic, and consequently $g \colon \mathbb{C} \to \mathbb{P}^1$ is holomorphic. This shows that g is a doubly periodic meromorphic function on $\mathbb{C}$ with period $(1, \tau)$.

   Conversely, pick a doubly periodic meromorphic function $g \colon \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ with period $(1, \tau)$. Define $f := g \circ \pi^{-1} : \mathbb{C}/\Lambda \to \mathbb{C} \cup \{\infty\}$. We want to show f is meromorphic, i.e. $f \in K(\mathbb{C}/\Lambda)$. First, observe that π^{-1} is not well-defined, because π has no inverse, so it is multi-valued. However, f is well-defined: For any $w_1, w_2 \in \pi^{-1}(z)$, $w_1 - w_2 \in \Lambda$, so $g(w_1) = g(w_2)$ since g is doubly periodic with period $(1, \tau)$. This shows $f(z) = g \circ \pi^{-1}(z)$ is single-valued, hence well-defined. Next, as a map $f : \mathbb{C}/\Lambda \to \mathbb{P}^1$, we want to show f is holomorphic. Equivalently, we can show that given any point $z \in \mathbb{C}$, there is a open neighbourhood U of z such that $f|_U$ is holomorphic: Pick $w \in \pi^{-1}(z)$ and a small enough open neighbourhood U' of w such that $\pi|_{U'}$ is a invertible (to be explicit, let $U' = \{w + (s + t\tau) : (s,t) \in (-1/2, 1/2) \times (-1/2, 1/2)\}$). Define $U = \pi(U')$, so U is an open neighbourhood of z. Since $\pi|_{U'}$ is a invertible, $(\pi|_{U'})^{-1} \colon U \to U'$ is well-defined. Then, obviously we have $f|_U = g \circ (\pi|_{U'})^{-1}$. We have shown that π is holomorphic, hence $\pi|_{U'}$ is holomorphic and its inverse $(\pi|_{U'})^{-1}$ is holomorphic. Since g is holomorphic also, then $f|_U$ is holomorphic. This shows $f \in K(\mathbb{C}/\Lambda)$.

So, we showed that $K(\mathbb{C}/\Lambda)$ is isomorphic to the field of doubly periodic meromorphic functions on $\mathbb{C}$ with period $(1, \tau)$.

10. **(i)** Let e_1, e_2, e_3 be the values of $\wp(1/2), \wp(\tau/2), \wp((1+\tau)/2)$ respectively. Show that e_1, e_2, e_3 are all distinct.

Solution: First, recall that we have $\wp'(1/2) = \wp'(\tau/2) = \wp'((1+\tau)/2) = 0$ (see the lecture notes). Also, we know $\wp, \wp' : \mathbb{C}/\Lambda \to \mathbb{P}^1$ are holomorphic maps between compact, connected Riemann surfaces, so we can talk about their degrees (so $\deg \wp = d_\wp(t)$ for any $t \in \mathbb{P}^1$). Moreover, $\wp$ has degree 2 (since it has only one pole with order 2), and $\wp'$ has degree 3 (since it has only one pole with order 3).

Also note that $v_\wp(1/2) = v_\wp(\tau/2) = v_\wp((1+\tau)/2) = 2$ as follows: Let's find $v_\wp(1/2)$, the others are similar to find. In this case we don't need to consider charts, since $\wp$ can be seen as a map from $\mathbb{C}$ to $\mathbb{P}^1$ (see the previous exercise). Consider the Taylor series of $\wp$ around $1/2$: The coefficient of z is $\wp'(1/2) = 0$. Therefore, the smallest $n \geq 1$ such that the coefficient of z^n is nonzero is at least 2, hence $v_\wp(1/2) \geq 2$. Since $\deg \wp = 2$, $v_\wp(1/2) \leq 2$, so we have $v_\wp(1/2) = 2$.

Now, assume $e_1 = e_2$. Then we have $\wp(1/2) = \wp(\tau/2) = e_1$. Hence,

$$2 = \deg \wp = d_\wp(e_1) = \sum_{s \in \wp^{-1}(e_1)} v_\wp(s) \geq v_\wp(1/2) + v_\wp(\tau/2) = 4$$

which is a contradiction. Hence, $e_1 \neq e_2$. Similarly, $e_1 \neq e_3$ and $e_2 \neq e_3$. So, all e_1, e_2, e_3 are distinct.

(ii) Show that for any $a \in \mathbb{C} \setminus \{e_1, e_2, e_3\}$, the equation $\wp(z) = a$ has exactly two distinct solutions.

Solution: Since $\deg \wp = d_\wp(a) = 2$, $\wp(z) = a$ has at most 2 solutions. Assume it has only one solution b, i.e. $\wp(b) = a$. Then,

$$2 = \deg \wp = d_\wp(a) = \sum_{s \in \wp^{-1}(a)} v_\wp(s) = v_\wp(b)$$

hence $v_\wp(b) = 2$. Therefore, we have $\wp'(b) = 0$. Note that $b \neq 1/2, \tau/2, (1+\tau)/2$, since $\wp(b) = a \in \mathbb{C} \setminus \{e_1, e_2, e_3\}$. However, we have

$$3 = \deg \wp' = d_{\wp'}(0) = \sum_{s \in (\wp')^{-1}(0)} v_{\wp'}(s) \geq v_{\wp'}(1/2) + v_{\wp'}(\tau/2) + v_{\wp'}((1+\tau)/2) + v_{\wp'}(b) \geq 4$$

since $v_{\wp'}(t) \geq 1$ for any t by definition. So we get a contradiction, hence $\wp(z) = a$ has exactly two distinct solutions.