Diagonalisation of quadratic forms

Definition 0.1. *Let k be any field, and let V be a k-linear vector space. A **symmetric bilinear pairing** on V is a map $\langle \cdot, \cdot \rangle : V \times V \to k$ such that for all $\alpha_1, \alpha_2 \in k$ and $v_1, v_2, v_3 \in V$,*

- $\langle \alpha_1 v_1 + \alpha_2 v_2, v_3 \rangle = \alpha_1 \langle v_1, v_3 \rangle + \alpha_2 \langle v_2, v_3 \rangle$,
- $\langle v_3, \alpha_1 v_1 + \alpha_2 v_2 \rangle = \alpha_1 \langle v_3, v_1 \rangle + \alpha_2 \langle v_3, v_2 \rangle$, *and*
- $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$.

Example 0.2. *Dot product in $\mathbb{R}^n$ over $\mathbb{R}$.*

Definition 0.3. *If $\mathrm{char}(k) \neq 2$, then the associated **quadratic form** to a symmetric bilinear pairing is*

$$\mathcal{Q}(v) = \langle v, v \rangle, \quad v \in V.$$

Remark 0.4. *The quadratic form $\mathcal{Q}$ is uniquely determined by $\langle \cdot, \cdot \rangle$, but the converse is also true, since*

$$\mathcal{Q}(v_1 + v_2) = \langle v_1 + v_2, v_1 + v_2 \rangle = \langle v_1, v_1 \rangle + \langle v_1, v_2 \rangle + \langle v_2, v_1 \rangle + \langle v_2, v_2 \rangle = \mathcal{Q}(v_1) + \mathcal{Q}(v_2) + 2\langle v_1, v_2 \rangle,$$

by bilinearity and symmetry, so

$$\langle v_1, v_2 \rangle = \frac{1}{2} \left(\mathcal{Q}(v_1 + v_2) - \mathcal{Q}(v_1) - \mathcal{Q}(v_2) \right).$$

Example 0.5. *Let $V = k^n$, and let A be a symmetric $n \times n$ matrix over k. Then*

$$\langle v_1, v_2 \rangle = v_1^\top A v_2 \in k, \quad v_1, v_2 \in V,$$

is a symmetric bilinear pairing. More generally, let V be any finite-dimensional vector space, so $V = \mathrm{span}_k\{e_1, \ldots, e_n\}$ for $\{e_i\}$ a k-basis, and let the (i, j)-th entry of A be $\langle e_i, e_j \rangle$. Under the unique isomorphism

$$\phi : V \to k^n, \quad e_i \mapsto \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix},$$

we get a symmetric bilinear pairing

$$\langle v, w \rangle = \phi(v)^\top A \phi(w) \in k, \quad v, w \in V.$$

Definition 0.6. *A **quadratic space** over k is an ordered pair $(V, \langle \cdot, \cdot \rangle)$ for V a finite-dimensional k-linear vector space, and $\langle \cdot, \cdot \rangle : V \times V \to k$ a symmetric bilinear pairing. Two quadratic spaces $(V, \langle \cdot, \cdot \rangle)$ and $(W, \langle\langle \cdot, \cdot \rangle\rangle)$ are **isometric** if there exists $\phi : V \to W$ an isomorphism such that $\langle v, w \rangle = \langle\langle \phi(v), \phi(w) \rangle\rangle$ for all $v, w \in V$, so any quadratic space is isometric to a specimen from the example.*

Remark 0.7. *Change of basis has the following effect. Let A be the matrix of the symmetric bilinear pairing $\langle \cdot, \cdot \rangle$ in the basis $e_1, \ldots, e_n$. If the matrix of the change of basis is B, in the new basis the matrix of the symmetric bilinear pairing is $B^\top AB$, since $(Bv)^\top A(Bw) = v^\top(B^\top AB)w$.*

Theorem 0.8 (Gram-Schmidt orthogonalisation process)**.** *If $(V, \langle \cdot, \cdot \rangle)$ is a quadratic space, then V has a basis $e_1, \ldots, e_n$ in which the matrix of $\langle \cdot, \cdot \rangle$ is diagonal.*

Proof. Two cases. If $\mathcal{Q} \equiv 0$, then $\langle \cdot, \cdot \rangle \equiv 0$. Otherwise there exists $v \in V$ such that $\mathcal{Q}(v) \neq 0$. Let $e_1 = v$, and

$$v^\perp = \{w \in V \mid \langle v, w \rangle = 0\}.$$

This is a k-linear subspace. This is trivial as $w \mapsto \langle v, w \rangle$ is k-linear. Then $v \notin \ker\langle v, \cdot \rangle$, so $\dim v^\perp = \dim V - 1$. We apply the process to $(v^\perp, \langle \cdot, \cdot \rangle|_{v^\perp})$, by using induction on the dimension. □

Theorem 0.9. *If $\mathrm{char}(k) \neq 2$, then for $F \in k[X_0, \ldots, X_n]$ homogeneous of degree two, there exists a linear transformation, such that after the change of variables, F is of the form*

$$\alpha_0 X_0^2 + \cdots + \alpha_n X_n^2, \quad \alpha_0, \alpha_1, \alpha_2 \in k.$$

Proof. Let $F(X_0, \ldots, X_n) = \sum_{i \leq j} a_{ij} X_i X_j$ for $a_{ij} \in k$. It is the quadratic form on k^{n+1} associated to the bilinear pairing in the standard basis with matrix $A = (b_{ij})$, where

$$b_{ij} = \begin{cases} \frac{1}{2} a_{ij} & i \leq j, \\ \frac{1}{2} a_{ji} & i > j. \end{cases}$$

Now apply the Gram-Schmidt theorem. □