Course on Representation Theory

Yankı Lekili, Spring 2015

1 Outline (and notation):

Fix G a complex semisimple connected Lie group with Lie algebra $\mathfrak{g}$. We will think of $\mathfrak{g}$ as the tangent space T_eG of G at the identity. The action of G on itself by conjugation $g : h \to ghg^{-1}$ naturally induces an action on T_eG. This is the adjoint action, which we denote by

$$Ad : G \to GL(\mathfrak{g}).$$

Differentiating this action, we get the adjoint action of $\mathfrak{g}$ given by

$$ad(x) : y \to [x, y].$$

Let B be a Borel subgroup, .i.e., a maximal solvable subgroup of G and let T be a maximal torus contained in B. Let $U = [B, B]$ be the unipotent radical of B so that $B = T \cdot U$, in particular B is connected.

Let $\mathfrak{b}$, $\mathfrak{h}$ and $\mathfrak{n}$ denote Lie algebras of B, T and U respectively. Then $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ and $\mathfrak{h}$ is called a Cartan subalgebra of $\mathfrak{g}$. The dim$\mathfrak{h}$ is called the *rank* of $\mathfrak{g}$.

We also consider the normalizer $N_G(T)$ of T. The quotient $W_T := N_G(T)/T$ is called the *Weyl group* of G. As we shall see, its isomorphism type does not depend on T. So, in general we simply write W for the Weyl group.

Example 1.1. *If $G = SL_n(\mathbb{C})$ then we can take B to be the upper-diagonal matrices. T to be the diagonal matrices and U to be the upper diagonal matrices with all of its diagonal entries are equal to* 1. *$W \cong \mathfrak{S}_n$ is isomorphic to the symmetric group.*

A basic fact from representation theory of finite groups W is as follows:

Proposition 1.2. *The number of irreducible representations of W on finite dimensional complex vector spaces is equal to the number of conjugacy classes in W.*

Indeed, both numbers are equal to dimension of vector space of class functions on W. However, this count does not actually exhibit an explicit bijection between the two sets. In other words, given a conjugacy class, we don't know how to construct an irreducible representation.

In the case W is the Weyl group, there is a construction of Springer, which gives an effective bijection between these two sets.

Springer constructed representations of the Weyl group W on the cohomology of certain varieties $\mathcal{B}_X$ - the so-called Springer fiber over a nilpotent element $X \in \mathfrak{g}$. Here, we say that an element $X \in \mathfrak{g}$ is nilpotent if it acts nilpotently via the adjoint representation. In other words, $ad(X)$ acts nilpotently as an endomorphism of the vector space $\mathfrak{g}$. [1] The *nilpotent cone* $\mathcal{N}$ is defined as the set of all nilpotent elemetns in $\mathfrak{g}$:

$$\mathcal{N} := \{X \in \mathfrak{g} : X \text{ nilpotent } \}$$

In good characteristic (such as char=0, or see page 19 of [J]) and if G is simply connected, a theorem of Springer that says that there is a G-equivariant isomorphism between the unipotent variety U and the nilpotent cone $\mathcal{N}$. (See [H]. Section 6.20).

A quick way to define the Springer fiber over the nilpotent element $X \in \mathfrak{g}$ is as follows:

$$\mathcal{B}_X := \{gB \in G/B : g^{-1}Xg \in \mathfrak{n}\}$$

Note that $\mathcal{B}_0 = G/B$. As the course progresses, we will study the geometry of these fibers in more depth.

As the name indicates, these are the fibers of the Springer resolution:

$$p : \tilde{\mathcal{N}} \to \mathcal{N}$$

where

$$\tilde{\mathcal{N}} := \{(gB, X) \in G/B \times \mathcal{N} : g^{-1}Xg \in \mathfrak{n}\}$$

It turns out that $\tilde{\mathcal{N}}$ can be identified with the cotangent bundle $T^*(G/B)$. As such, it is an example of a *symplectic variety*, i.e. it carries a holomorphic 2-form Ω which is non-degenerate and closed. (Note that this should not be confused with C^∞ symplectic forms, which appear for ex. in Kähler geometry). There are many interesting symplectic varieties, such as ADE resolution of $\mathbb{C}^2/\Gamma$ where Γ is a finite subgroup of $SL_2(\mathbb{C})$, Hilbert schemes of points on $\mathbb{C}^2$, and more generally *Nakajima quiver varieties.*

The actual construction of the W-representations on the cohomology of Springer fibers can be done in many different but equivalent ways and all of them require some machinery. (See, for example, the discussion in Section 9.5 of [H]). This will be one of our main goals in this course. A remarkable point is that the Weyl group W does NOT act on the variety $\mathcal{B}_X$ for $X \neq 0$. Nonetheless, it does act on the cohomology of the varieties $\mathcal{B}_X$.

Example 1.3. *In the case $G = SL_n(\mathbb{C})$, one can identify G/B with full flag variety, $\mathcal{F}l(\mathbb{C}^n)$. The points of this space are complete flags:*

$$\{0\} \subset V_1 \subset V_2 \dots V_{n-1} \subset V_n = \mathbb{C}^n$$

[1] If we work over fields of finite characteristic, the definition nilpotency is more complicated. We need to first fix G such that $Lie(G) = \mathfrak{g}$. Then, we define $X \in \mathfrak{g}$ to be nilpotent, if it acts nilpotently on each finite dimensional G-module (under the derived representation). Note that this definition depends on both $\mathfrak{g}$ and G. For a semisimple Lie algebra $\mathfrak{g}$ in $char = 0$, this is equivalent to the simpler definition that we have given.

where V_i are subspace of $\mathbb{C}^n$ of dimension i. We can think of $X \in \mathfrak{sl}_n$ as an $n \times n$ matrix. Then, we can identify the Springer fiber $\mathcal{B}_X$ with the set of flags fixed by X, i.e. :

$$\mathcal{B}_X = \{V_\bullet : XV_i \subset V_i\}$$

In this case, Springer constructs an irreducible representation of $\mathfrak{S}_n$ for every unipotent conjugacy class in $SL_n(\mathbb{C})$. Note that the theory of Jordan normal form gives a bijection between the conjugacy classes of unipotent elements in $SL_n(\mathbb{C})$ and partitions of n. It is also well-known that conjugacy classes in $\mathfrak{S}_n$ correspond to cycle decompositions of permutations, hence to partitions of n.

Springer originally worked over finite characteristic and used l-adic cohomology groups $H^*(\mathcal{B}_X, \mathbf{Q}_l)$. Later on, he has found a way to carry out his construction over $\mathbb{C}$. In the first part of this course, we will work over $\mathbb{C}$ and use Borel-Moore homology following the treatment in [CG].

Another important part of the picture is the adjoint quotient map:

$$\chi : \mathfrak{g} \to \mathfrak{g}//G = Spec\, \mathbb{C}[\mathfrak{g}^*]^G$$

which comes from the map of algebras $\mathbb{C}[\mathfrak{g}^*]^G \to \mathbb{C}[\mathfrak{g}^*]$. Note we can also identify the target with $\mathfrak{h}//W$ via the Chevalley isomorphism of the ring of invariants:

$$\mathbb{C}[\mathfrak{g}^*]^G \cong \mathbb{C}[\mathfrak{h}^*]^W$$

Springer theory can be seen as study of singularities of the adjoint quotient map.

The resolution $p : \tilde{\mathcal{N}} \to \mathcal{N}$ extends to a map $q : \tilde{\mathfrak{g}} \to \mathfrak{g}$ where $\tilde{\mathfrak{g}} = \{(X, \mathfrak{b}) : X \in \mathfrak{b} \subset \mathfrak{g}\}$. More substantially, we have a simultaneous resolution, introduced by Grothendieck:

$$\begin{array}{ccccc} \tilde{\mathcal{N}} & \hookrightarrow & \tilde{\mathfrak{g}} & \xrightarrow{\tilde{\chi}} & \mathfrak{h} \\ {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & \downarrow \\ \mathcal{N} & \hookrightarrow & \mathfrak{g} & \xrightarrow[\chi]{} & \mathfrak{h}//W \end{array}$$

This means that $\tilde{\chi}$ is a smooth morphism (in fact, a C^∞ fiber bundle) such that for each $\tilde{t} \in \mathfrak{h}$, $\tilde{\chi}^{-1}(\tilde{t})$ is a resolution of singularities of the corresponding adjoint fibre $\chi^{-1}(t)$.

We end this outline with the simplest example. If $\mathfrak{g} = \mathfrak{s}l_2$. In this case, for a suitable choice of coordinate $\chi : \mathbb{C}^3 \to \mathbb{C}$ given by $(a, b, c) \to a^2 + b^2 + c^2$. The nilpotent cone $\mathcal{N} = \chi^{-1}(0)$ has an ordinary node singulariy. The simultaneous resolution consists of doing a base extension: $t^2 = a^2 + b^2 + c^2$ and then blowing up the singular point, which replaces it with $\mathbb{C}P^1 \cong SL_2(\mathbb{C})/B$.

2 Symplectic geometry (basics)

An important geometric property of the simultaneous resolution map

$$\tilde{\chi} : \tilde{\mathfrak{g}} \to \mathfrak{h}$$

is that it is a smooth fibre bundle (topologically trivial) with fibres *diffeomorphic* to G/T and the total space $\tilde{\mathfrak{g}}$ has a canonical *Poisson structure* such that the fibres $\tilde{x}^{-1}(t)$ are the *symplectic leaves*, i.e. the restriction of the Poisson bracket to the fibres make them into (holomorphic) symplectic manifolds. [2]

We now give a basic treatment of these structures following [[CG], Chapter 1].

Definition 2.1. *Let X be a C^∞-manifold, a smooth complex manifold or an algebraic variety. A symplectic structure on X is a non-degenerate regular (in the corresponding category) 2-form ω such that $dw = 0$.*

At least in the beginning, we will mostly work with algebraic varieties and ω will be a holomorphic non-degenerate closed 2-form.

Note that non-degeneracy of ω implies that X be even-dimensional.

The most standard example is $\mathbb{C}^{2n}$ with coordinates $q_1, \dots q_n, p_1, \dots, p_n$ and

$$\omega = dp_1 \wedge dq_1 + \ldots + dp_n \wedge dq_n.$$

The roots of symplectic geometry are in classical mechanics. I will review this shortly after we define Poisson structures as well.

An important difference between symplectic and Riemannian geometries is that there is no local invariants of symplectic structure, unlike for ex. the curvature of a Riemannian metric. Given a symplectic manifold, one can always find a local chart in which the symplectic form takes the canonical form as the one in $\mathbb{C}^{2n}$ above. This theorem is known as the *Darboux theorem* can be found in any standard symplectic geometry book (such as the book of Ana Cannas da Silva's).

We next give another standard construction of a symplectic manifold, which will be relevant for our purposes.

Example 2.2. *Let M be any manifold (say, a complex smooth algebraic variety), then T^*M has a canonical symplectic structure. Let $\pi : T^*M \to M$ be the projection. Consider the 1-form λ on T^*M given on $T^*_\alpha M$ by*

$$\lambda_\alpha(v) = \alpha(\pi_*(v)) \in \mathbb{C}$$

We set $\omega = d\lambda$. Clearly, this is a closed 2-form. To check, non-degeneracy, let us express λ in local coordinates. Let q_i be local coordinates on M, and p_i are coordinates in T^*_α dual to $\frac{\partial}{\partial q_i}$ on $T_\alpha M$. Then the 1-form $\lambda = \sum_i p_i dq_i$ and $\omega = \sum_i dp_i \wedge dq_i$, so ω agrees with the standard example in these coordinates, hence it is non-degenerate.

[2] In fact, the work of Kronheimer and Biquard shows that the fibres have hyperkähler structures, but that's harder result, which uses moduli spaces of solutions to Nahm's equations.

We have mentioned in the outline that there is a resolution $p : \tilde{\mathcal{N}} \to \mathcal{N}$ of the nilpotent cone, which is isomorphic as an algebraic variety to $T^*(G/B)$. The above example equips it with a canonical symplectic form. This is also the fibre $\tilde{\chi}^{-1}(0)$. We will see later on that over an open dense subset, the fibres of $\tilde{\chi}$ correspond to coadjoint orbits. These also have canonical symplectic forms.

Recall that the adjoint action $Ad : G \to End(\mathfrak{g})$. Dually, we have the *coadjoint action*, $Ad^* : G \to End(\mathfrak{g}^*)$ given by

$$\langle Ad^*(g)\xi, x\rangle = \langle \xi, Ad(g^{-1})x\rangle$$

where $\langle , \rangle$ is the natural pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{C}$. We also have the linearization:

$$ad^*_x(\xi)(y) = \xi([y, x])$$

(Check that this is a Lie algebra homomorphism! [3])

One feature of coadjoint orbits, when you consider a few examples, is that they are always even-dimensional. This is no accident. Indeed, a very important class of symplectic manifolds are the orbits of these coadjoint actions.

Proposition 2.3. *Let $\mathbb{O} \subset \mathfrak{g}^*$ be any coadjoint orbit. There is a canonical G-invariant symplectic structure on $\mathbb{O}$. (It is called the Kirillov-Kostant-Souriau symplectic structure.)*

Proof. Let $\alpha \in \mathfrak{g}^*$ be an arbitrary element. The G-orbit $\mathbb{O}$ through α can be naturally identified with G/G^α where G^α is the stabilizer - this is always a closed subgroup. Let us write $\mathfrak{g}^\alpha = LieG^\alpha$, then we have that $T_\alpha\mathbb{O} = \mathfrak{g}/\mathfrak{g}^\alpha$. We define a skew-symmetric form on $T_\alpha\mathbb{O}$ by:

$$\omega_\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \quad \omega_\alpha(x, y) = \alpha([x, y]).$$

Let us prove that this descends to a non-degenerate form $\mathfrak{g}/\mathfrak{g}^\alpha \times \mathfrak{g}/\mathfrak{g}^\alpha \to \mathbb{C}$. We need to check that for $y \in \mathfrak{g}^\alpha$, $\omega_\alpha(x, y) = 0$ for all $x \in \mathfrak{g}$. Now, notice that $\alpha[x, y] = ad^*_y(\alpha)(x)$. Hence, if this is zero for all $x \in \mathfrak{g}$, it means that $ad^*_y(\alpha) = 0$, but this means $y \in \mathfrak{g}^\alpha$. G-invariance can be seen from the identity:

$$(Ad^*(g)\alpha)([x, y]) = \alpha([ad(g^{-1})x, ad(g^{-1})y])$$

It remains to prove that the induced 2-form ω on $\mathbb{O}$ is closed. Given $x, y, z \in \mathfrak{g}$, ξ_x, ξ_y, ξ_z be the corresponding G-invariant vector fields on $\mathbb{O}$. For example, $(\xi_x)(\alpha) = \frac{\partial}{\partial t} exp(tx) \cdot \alpha|_{t=0}$.

We use the following well-known Cartan formula:

$$dw(\xi_x, \xi_y, \xi_z) = \xi_x \cdot \omega(\xi_y, \xi_z) + \xi_z \cdot \omega(\xi_x, \xi_y) + \xi_y \cdot \omega(\xi_z, \xi_x) - \omega([\xi_x, \xi_y], \xi_z) - \omega([\xi_z, \xi_x], \xi_y]) - \omega([\xi_y, \xi_z], \xi_x)$$

Evaluating at α, we have the formulae: $(\xi_x \cdot \omega(\xi_y, \xi_z))(\alpha) = -\alpha([x, [y, z]])$ and $\omega([\xi_x, \xi_y], \xi_z)(\alpha) = \alpha([[x, y], z])$. The result now follows by applying Jacobi identity. □

[3] There is a sign error in [CG]'s computation of ad^* in Prop. 1.1.5.

As we shall see, there is a dense set of points $\alpha \in \mathfrak{g}^*$, for which G^α is a maximal torus and $\mathbb{O}$ is a closed subvariety of $\mathfrak{g}^*$, in particular it is affine (this probably fails in general over finite characteristic, see [H] and please let me know if you find an example). Other orbits are smaller dimensional and in general locally closed subvarieties.

The symplectic structure on the orbits of the coadjoint action plays an important role in the orbit method approach to representation theory. (See Kirillov's "Lectures on the orbit method")

Exercise: *Adjoint orbits and coadjoint orbits for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ can be identified via the non-degenerate bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ given by $(x, y) \to Tr(xy)$.*

Identify $\mathfrak{sl}_2$ with $\mathfrak{sl}_2^*$. Take any semisimple element in $\mathfrak{sl}_2$, i.e. a non-zero 2×2-matrix with zero trace that is diagonalizable. Show that the adjoint orbit through this element is isomorphic to the affine quadric in $\mathbb{C}^3$. Explicitly compute the Kostant-Kirillov-Souriau symplectic from Ω on this affine quadric. In suitable coordinates, such an orbit can be identified with a regular fibre $\chi^{-1}(t)$ of $\chi : \mathbb{C}^3 \to \mathbb{C}$, given by $\{(a, b, c) \in \mathbb{C}^3 : a^2 + b^2 + c^2 = t \neq 0\}$ and then we have:

$$\Omega = \frac{da \wedge db \wedge dc}{d\chi}$$

If all the eigenvalues of some (hence all) element in the orbit is real, then show that the imaginary part $Im(\Omega)$ coincides with the canonical (real) symplectic form on T^*S^2.

A more conceptual proof of the previous proposition is based on the notion of Poisson manifold.

Definition 2.4. *A commutative associative algebra A over $\mathbb{C}$ is called a* Poisson algebra *if it is equipped with a $\mathbb{C}$-bilinear Lie bracket $\{,\} : A \otimes A \to A$ satisfying the Leibniz identity:*

$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$

A Poisson manifold (or variety) is a smooth manifold with a Poisson bracket on its algebra of regular functions.

We now explain that regular functions $\mathcal{O}(M)$ of a symplectic manifold has a natural Poisson bracket defined in the following way: Let $f \in \mathcal{O}(M)$ be any regular function. As ω is non-degenerate, we have an associated vector field X_f defined by the formula:

$$\omega(\cdot, X_f) = df$$

We then set:

$$\{f, g\} = \omega(X_f, X_g)$$

We leave the verification that this defines a Poisson bracket as an exercise but note that the Jacobi identity reduces to the relation:

$$X_{\{f,g\}} = [X_f, X_g]$$

Note that to define the bracket operation, we merely used that ω is a non-degenerate form. To check that the bracket satisfies the Jacobi identity, one has to use that ω is a closed form.

In the case of the standard symplectic structure on $\mathbb{C}^{2n}$, the polynomial algebra $\mathbb{C}[q_1, \ldots, q_n, p_1, \ldots, p_n]$ has a Poisson bracket given by:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

Exercise: *If $f, g \in \mathbb{C}[q_1, \ldots, q_n, p_1, \ldots, p_n]$ are homogeneous element of degree 2, then their Poisson bracket $\{f, g\}$ is also homogeneous of degree 2. Show that elements of degree 2 form a Lie algebra canonically isomorphic to the symplectic Lie algebra $\mathfrak{sp}_{2n}$.*

Poisson structures arise in the context of *quantization*. Let us briefly touch on this subject.

Let B be an associative filtered (non-commutative) $\mathbb{C}$-algebra with unit. Spelling this out, we have an increasing filtration by $\mathbb{C}$-vector spaces:

$$\mathbb{C} \subset B_0 \subset B_1 \subset \ldots$$

such that $B = \bigcup_{i=0}^{\infty} B_i$ and $B_i \cdot B_j \subset B_{i+j}$ for all $i, j \geq 0$.

Let us consider the associated graded algebra $A = grB = \oplus_i B_i/B_{i-1}$. We have the following proposition:

Proposition 2.5. *Suppose B is a associative filtered algebra as above, and $grB = A$ is commutative. Then, there is a natural Poisson bracket on A.*

Proof. We define a bilinear pairing

$$\{,\} : B_i/B_{i-1} \times B_j/B_{j-1} \to B_{i+j-1}/B_{i+j-2}$$

as follows:

$$\{[b_i], [b_j]\} = b_i b_j - b_j b_i \ (mod\ B_{i+j-2})$$

Commutativity of A makes this well-defined. Axioms of Poisson algebra is straightforward to verify. □

The problem of quantization is to go the other way. One usually starts with a commutative algebra, such as the algebra of regular functions $\mathcal{O}(M)$ on a variety, together with a Poisson bracket on it, and seeks to find a non-commutative deformation algebra B such that $grB = A$. Often one considers formal deformations, that is, an associative algebra structure on the $\mathbb{C}[[h]]$-algebra $B = A[[h]]$. The Poisson algebra $(A, \{,\})$ is called the *quasiclassical limit* of the *quantization* B.

We will next discuss a quantization in the special case of $\mathcal{O}(T^*M)$ equipped with its Poisson structure coming from the canonical symplectic structure on T^*M.

Let $\mathcal{T}_M$ denote the sheaf of vector fields on M given on an open set U by:

$$\mathcal{T}_M(U) := \{\theta \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_M)(U) : \theta(fg) = \theta(f)g + f\theta(g) \text{ for } \ f, g \in \mathcal{O}_M(U)\}$$

We identify $\mathcal{O}_M$ with a subsheaf of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_M)$ via

$$\{f \in \mathcal{O}_M(U)\} \to \{m_f \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_M)(U), m_f(g) = fg\}$$

The sheaf of *differential operators* $\mathcal{D}_X$ is the $\mathbb{C}$-subalgebra of $\mathcal{E}nd_{\mathbb{K}}(\mathcal{O}_M)$ generated by $\mathcal{O}_M$ and $\mathcal{T}_M$.

For any point of M, we can find an affine open neighborhood U and a local coordinate system $\{x_i, \partial_i\}$ on it such that

$$x_i \in \mathcal{O}_M(U), \mathcal{T}_U = \bigoplus_{i=1}^{n} \mathcal{O}_U \partial_i, \quad [\partial_i, \partial_j] = 0, \quad [\partial_i, x_j] = \delta_{ij}$$

Hence, we have:

$$\mathcal{D}_U = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_U \partial^\alpha$$

where $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n}$. On such a local coordinate system, we can define a filtration by:

$$\mathcal{F}_l \mathcal{D}_U = \bigoplus_{|\alpha| \leq l} \mathcal{O}_U \partial^\alpha$$

where $|\alpha| = \sum_i \alpha_i$.

Alternatively, for arbitrary U, we can use Grothendieck's inductive definition of differential operators via the formula:

$$\mathcal{F}_l \mathcal{D}_U = \{P \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_M)(U) : [P, m_f] \in \mathcal{F}_{l-1} D_U \text{ for all } f \in \mathcal{O}_M(U)\}$$

and $\mathcal{F}_0 \mathcal{D}_U = \mathcal{O}_U$.

We define $gr\mathcal{D}_M = \bigoplus_{l=0}^{\infty} \mathcal{F}_l \mathcal{D}_M / \mathcal{F}_{l-1} \mathcal{D}_M$. Note that if $P \in \mathcal{F}_l \mathcal{D}_M(U)$ and $Q \in \mathcal{F}_m \mathcal{D}_M(U)$, then $[P, Q] \in \mathcal{F}_{l+m-1} \mathcal{D}_M(U)$ (Check this!). Thus $gr\mathcal{D}_M$ is a sheaf of *commutative* algebras finitely generated over $\mathcal{O}_M$. Indeed, over a local coordinate system $\{x_i, \partial_i\}$ as above, we have

$$gr_l \mathcal{D}_U = \bigoplus_{|\alpha| = l} \mathcal{O}_U \partial_i^\alpha$$

Hence, $gr\mathcal{D}_U = \mathcal{O}_U[\partial_1, \ldots, \partial_n]$. Now, we can identify $\partial_1, \ldots, \partial_n$ as a coordinate system of the cotangent space $\oplus_{i=1}^n \mathbb{C} dx_i$. Therefore, letting $\pi : T^*M \to M$, there is a canonical identification

$$gr\mathcal{D}_U \cong \pi_* \mathcal{O}_{T^*M}|_U$$

Taking global sections, we get that the non-commutative algebra of global differential operators on M is a quantization of the algebra of regular functions on T^*M. The natural map $\sigma : \mathcal{D}_M \to gr\mathcal{D}_M$ is called the *principal symbol.* It is not too hard to show that the induced Poisson bracket on $\mathcal{O}(T^*M)$ coincides with the one coming from the symplectic structure of T^*M. Indeed, it suffices to check this on $gr_1(\mathcal{D}_M)$ which generates $\mathcal{O}(T^*M)$ over $\mathcal{O}_M$.

Let us next turn to another important example of a quantization.

Definition 2.6. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The quotient of the tensor algebra $T\mathfrak{g}$ by the ideal generated by expressions $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in \mathfrak{g}$ is called the* universal enveloping algebra *and denoted by $\mathcal{U}\mathfrak{g}$.*

$\mathcal{U}\mathfrak{g}$ has a canonical filtration:

$$\mathbb{C} = \mathcal{U}_0\mathfrak{g} \subset \mathcal{U}_1\mathfrak{g} \subset \dots$$

where $\mathcal{U}_j\mathfrak{g}$ is the $\mathbb{C}$-linear span of all monomials of degree $\leq j$. It is the image $\mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes g) \oplus \dots \oplus (\mathfrak{g})^{\otimes j}$ under the canonical projection $T\mathfrak{g} \to \mathcal{U}\mathfrak{g}$.

The following theorem is one of the fundamental results in Lie algebra theory. There are many proofs (if you are algebraically oriented than see Bergman's Diamond lemma paper or if you are geometrically oriented see the book of daSilva-Weinstein; I recommend both!):

Theorem 2.7. *(Poincaré-Birkhoff-Witt) There is a canonical graded algebra isomorphism :*

$$gr\mathcal{U}\mathfrak{g} \cong S\mathfrak{g} = \mathbb{C}[\mathfrak{g}^*]$$

Hence, we see that $\mathcal{U}\mathfrak{g}$ is quantization of a Poisson algebra structure on $\mathbb{C}[\mathfrak{g}^*]$.

Let us calculate the corresponding Poisson bracket. Consider a basis $e_1, \dots e_n$ of $\mathfrak{g}$ as a $\mathbb{C}$-vector space. We can then find a set of structure constants $c_{ij}^k \in \mathbb{C}$ via:

$$[e_i, e_j] = \sum_k c_{ij}^k e_k$$

Now, let V be a vector space over $\mathbb{C}$ and $x_1, \dots, x_n$ be coordinates on V. Let c_{ij}^k for $i, j, k = 1, \dots n$ be arbitrary complex numbers such that

$$c_{ij}^k = -c_{ji}^k$$

Consider the bivector field on V given by:

$$\Pi = \sum_{i,j,k} c_{ij}^k x_k \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}$$

Using this, we define a pairing on $\mathbb{C}[V]$ via:

$$\{f, g\} = df \otimes dg(\Pi) = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

Proposition 2.8. *The pairing on $\mathbb{C}[V]$ is a Poisson bracket if and only if c_{ij}^k form a collection of structure constants for some Lie algebra $\mathfrak{g}$.*

Proof. Note that bilinearity and anti-symmetry is already built-in to the definition. In view of Leibniz rule, the only thing to check is the Jacobi identity for linear functions. A natural basis of linear functions V^* is given by the coordinate functions $x_i : V \to \mathbb{C}$. For these we have:

$$\{x_i, x_j\} = \sum_k c_{ij}^k x_k$$

Therefore, Jacobi identity for the bracket on $\mathbb{C}[V]$ is equivalent to a Lie algebra structure on $\mathfrak{g} = V^*$. □

Taking $V = \mathfrak{g}^*$ and identifying $V^* = (\mathfrak{g}^*)^* \cong \mathfrak{g}$, we see that the above construction gives the Poisson structure on $\mathbb{C}[\mathfrak{g}^*]$ induced from $\mathcal{U}\mathfrak{g}$. One can express this bracket without appealing to coordinates as follows. For $f, g \in \mathbb{C}[\mathfrak{g}^*]$ and $\alpha \in \mathfrak{g}^*$ we have:

$$\{f, g\} : \alpha \to \langle \alpha, [d_\alpha(f), d_\alpha(g)] \rangle$$

where $d_\alpha(f) \in (\mathfrak{g}^*)^* = \mathfrak{g}$ is the differential of f at a point α.

Note that because of the anti-symmetry condition $c_{ij}^k = -c_{ji}^k$, the Poisson bivector field Π can really be seen as a section in $\Gamma(V, \Lambda^2 TV)$. In general, it is an instructive exercise to show that for any Poisson structure on regular functions $\mathcal{O}(M)$ of a variety M, can be described by a Poisson bivector field $\Pi \in \Gamma(M, \Lambda^2 TM)$.

One can now give an alternative construction of the symplectic structure on coadjoint orbits. Given a Poisson manifold $(M, \{,\})$, we have Hamiltonian vector fields $\{f, \cdot\} : \mathcal{O}(M) \to \mathcal{O}(M)$, for every function $f \in \mathcal{O}(M)$. These vector fields span a completely integrable (in general) singular foliation, each of whose maximal integral sub-manifolds inherits a symplectic structure. These are called *symplectic leaves* of the Poisson manifold.

Carrying out this construction in the case of $\mathfrak{g}^*$ with its canonical Poisson bracket defined above shows that the symplectic leaves are precisely the coadjoint orbits in $\mathfrak{g}^*$.

2.1 Physical background

In Hamiltonian formulation of classical mechanics, one considers a manifold M with local coordinates $q_1, \ldots q_n$ as the configuration space and T^*M as the phase space where the coordinates $p_1, \ldots, p_n$ in the cotangent direction corresponds to momentum variables. Let $M = \mathbb{R}$ for simplicity. One has a Hamiltonian function $H : T^*\mathbb{R} \to \mathbb{R}$ given by:

$$H(x, p) = \frac{p^2}{2} + V(x)$$

where $V(x) \in C^\infty(\mathbb{R})$ is the potential. The *equations of motion* is given by

$$\frac{df}{dt} = \{f, H\}$$

Unwinding this, one recovers *Newton's equation.* Conservation of energy is the statement that

$$\{H, H\} = 0.$$

Quantization, replaces the phase space with a Hilbert space $\mathcal{H}$ (usually a space of L^2 functions on a Lagrangian submanifold) and the observables $C^\infty(T^*M)$ are replaced by a non-commutative operator algebra B of (unbounded) operators on $\mathcal{H}$. The Hamiltonian H is now an element of B, and *Schrödinger's equation* is:

$$-i\hbar \frac{db}{dt} = [b, H]$$

Here $\hbar$ is the Planck constant. Classical mechanics is the quasiclassical limit of quantum mechanics. The algebra B is a quantization of $C^\infty(T^*M) = B/\hbar B$ with the Poisson bracket:

$$\{\,,\,\} = \lim_{\hbar \to 0} \frac{i[\,,\,]}{\hbar}$$

2.2 Submanifolds

First, let's study this at the linear level. Let V be a vector space with a symplectic form Ω. For any vector subspace $W \subset V$, we have its symplectic complement defined by:

$$W^\Omega = \{v \in V : \Omega(v, w) = 0 \text{ for all } \; w \in W\}$$

The non-degeneracy of Ω ensures that $(W^\Omega)^\Omega = W$ and that when V is finite-dimensional,

$$\text{dim}W + \text{dim}W^\Omega = dimV$$

Unlike the case of a symmetric form, $W \cap W^\Omega$ can be non-empty. The following form special classes of subspaces in a symplectic vector space:

Definition 2.9. *A linear subspace $W \subset V$ is:*

1. *Isotropic if $W \subset W^\Omega$.*
2. *Coisotropic if $W^\Omega \subset W$*

Isotropic subspaces can be equivalently defined as subspaces W such that $\Omega_{|W} \equiv 0$. Coisotropic subspaces can be equivalently defined by subspaces cut out by linear forms $l_1 = l_2 = \ldots = l_k = 0$ such that their pair-wise Poisson bracket $\{l_i, l_j\} = 0$.

Subspaces which are isotropic and coisotropic at the same time are called *Lagrangian*. The dimension of Lagrangian subspaces is equal to half the dimension of the symplectic space. Lagrangian subspaces are maximal isotropic spaces and minimal coisotropic spaces.

Definition 2.10. *Let M be a symplectic variety. A possibly singular subvariety Z of M is called isotropic (resp. coisotropic, Lagrangian) subvariety of M, if for any smooth point $z \in Z$, $T_z Z$ is isotropic (resp. coisotropic, Lagrangian) subspace of $T_z M$.*

In a cotangent bundle T^*X, the zero-section X is a Lagrangian. More generally, if $f \in \mathcal{O}(X)$ is a function on X, then the image of the section $df : X \to T^*X$ is a Lagrangian.

A fibre of a cotangent bundle is also a Lagrangian. This corresponds to the graph of the "delta function" at a point. More generally, the covectors applied at the points of a submanifold $Y \subset X$ and vanishing on the tangent vectors to Y form a Lagrangian submanifold $T^*_Y X \subset T^*X$, called the *conormal bundle* to Y. It is the graph of the "delta function" of the submanifold Y.

On a cotangent bundle T^*X there is a natural $\mathbb{C}^*$ action that corresponds to scaling along the cotangent fibre. In a local coordinate chart, this action is generated by the Euler vector field:

$$Z = \sum_i p_i \frac{d}{dp_i}$$

Under the isomorphism given by $\Omega : T^*X \to TX$, the canonical 1-form $\lambda = \sum_i p_i dq_i$ is sent to the Euler vector field, i.e., $\iota_Z \Omega = \lambda$.

Therefore, if a submanifold is invariant under the $\mathbb{C}^*$ action and isotropic, then λ vanishes on it identically. The Lagrangian subvarieties of T^*X that are invariant under $\mathbb{C}^*$ action can be characterized by the following result of Kashiwara:

Lemma 2.11. *(Kashiwara) Let X be a smooth variety and let $\Lambda \subset T^*X$ be a closed irreducible (possibly singular) algebraic $\mathbb{C}^*$ invariant Lagrangian subvariety. Let Y be the smooth part of the image of Λ under the projection $\pi : T^*X \to X$, then $\Lambda = \overline{T^*_Y X}$.*

(Here $\overline{T^*_Y X}$ means closure in the Zariski topology. A proof in analytic category also exists, though one has to prove that the closure in classical topology gives an analytic subset. See [HTT, Appendix E] for details.)

Proof. Y is dense open in closed variety $\pi(\Lambda)$, hence $\Lambda \subset \pi^{-1}(\overline{Y})$. Next, since the Euler vector field Z is tangent to Λ^{reg}, the smooth part of Λ and Λ is isotropic, we have

$$0 = \Omega(Z, v) = \lambda(v), \text{ for all } v \in T\Lambda^{reg}$$

Therefore $\lambda_{|\Lambda^{reg}} \equiv 0$. Now, if $\alpha \in \Lambda^{reg} \cap \pi^{-1}(Y)$ is any covector, and let $\pi(\alpha) = y \in Y$. By definition, for $v \in T_\alpha(\Lambda)$,

$$0 = \lambda(v) = \alpha(\pi_*(v))$$

Hence, α vanishes on the image of the map

$$\pi_* : T_\alpha \Lambda \to T_y Y$$

By Bertini-Sard's lemma, there is an open dense set $U \subset \Lambda^{reg} \cap \pi^{-1}(Y)$ such that, for all $\alpha \in U$, the map π_* is surjective. Since $\pi(U)$ is open dense in Y, it follows that $\alpha(T_y Y) = 0$ for all $y \in Y$ and $\alpha \in U$. Therefore, we have:

$$U \subset T^*_Y X$$

Now, both sets are varieties of the same dimension and U is open dense in Λ, hence $\Lambda = \overline{T^*_Y X}$ by irreducibility of Λ, as required. ☐

Note that our proof appeals to Bertini's theorem (which holds only in characteristic zero. In fact, the statement fails in finite characteristic - see Kleiman's paper Tangency and duality).

More generally, if Λ is any $\mathbb{C}^*$ invariant Lagrangian subvariety, we can find a Whitney stratification of $X = \bigsqcup_{\alpha \in A} X_\alpha$ such that Λ has irreducible components $T^*_{X_\alpha} X$ for some $\alpha \in A$.

An application: In case, you are tired of being introduced to too many new notions (you shouldn't be!), let us now give an application. (We follow the book Tevelev on projective duality, though the discussion is very similar to the one given in [CG]).

Let V be a finite dimensional vector space, and $G \subset PGL(V)$ be an algebraic subgroup. We then have a natural action of G on $\mathbb{P}(V)$ as well as on $\mathbb{P}(V^*)$. We have the following duality result for these actions:

Theorem 2.12. *(Pjaseckii) Suppose that G has finitely many orbits on $\mathbb{P}(V)$. There is a natural bijection between G-orbits on $\mathbb{P}(V)$ and the G-orbits on $\mathbb{P}(V^*)$.*

Before, we give the proof, let us first give the following simple result as preparation:
Lemma 2.13. *Let G be a connected algebraic group action on an algebraic variety X with finitely many orbits. Then any irreducible G-stable closed subvariety of X is the closure of a G-orbit.*

Proof. Let Y be an irreducible G-stable closed subvariety. Let $\mathbb{O}$ be any orbit in Y. Let us call points of $\overline{\mathbb{O}}\backslash\mathbb{O}$ boundary points of $\mathbb{O}$ where we take the closure in Zariski topology. Now, notice that the boundary of any orbit is invariant with respect to G action, and of dimension less than that of the orbit itself. Therefore, the boundary consists of orbits of lesser dimension. Now, let $\mathbb{M}$ be an orbit in Y of maximal dimension. Because of what we have just said, it can not intersect the closure of other orbits in Y. Since there are only finitely many orbits in Y, it follows that $\mathbb{M}$ is open in Zariski topology. Therefore, $\overline{\mathbb{M}}$ is an irreducible component of Y. Since Y is itself irreducible, it follows that $Y = \overline{\mathbb{M}}$. □

Exercise: Consider the action of $GL_2(\mathbb{C})$ on $\mathbb{C}^2$. Observe that $\mathbb{C}^2\backslash\{0\}$ is G-stable but not the closure of any G-orbit. Why does this not contradict with the conclusion of the lemma?

In fact, we'll obtain the proof of Pjaseckii's result as a corollary to projective duality theorem. Recall that for $X \subset \mathbb{P}(V)$ be any irreducible variety. Let $C(X) \subset V$ be the affine cone over X. For any point $x \in X$, we define the embedded tangent space $T_{x,X} \subset \mathbb{P}(V)$ to be the $\mathbb{P}(T_{v,C(X)})$ where v is any non-zero point on the line x. A hyperplane $H \subset \mathbb{P}(V)$ is a tangent hyperplane of X if $T_{x,X} \subset H$ for some $x \in X_{reg}$. The closure of the set of all hyperplanes to X is called the projective dual subvariety $X^* \subset \mathbb{P}(V^*)$.
Theorem 2.14. *For any irreducible projective variety $X \subset \mathbb{P}(V)$, if $z \in X_{reg}$ and $H \in X^*_{reg}$, then H is tangent to X at z if and only if regarded as a hyperplane in $\mathbb{P}(V^*)$, z is tangent to X^* at H. Hence,*

$$X^{**} = X$$

Proof. Consider the incidence variety $I^0_X \subset \mathbb{P}(V) \times \mathbb{P}(V^*)$ defined by (z, H) such that $z \in X_{reg}$ and H is a hyperplane tangent to z. I^0_X can be idenitified with the projectivization of conormal bundle $T^*_{X_{reg}}V$. Indeed, an equation of a hyperplane H in the projective space $\mathbb{P}(V)$ containing $T_z X_{reg}$ is precisely an element of V^* that vanishes along $T_z X_{reg}$, hence an element of $T^*_{X_{reg}}V$ by definition.

The conormal variety is the closure I_X of I^0_X. To prove the theorem it suffices to show that $I_X = I_{X^*}$. By what we said above, this is equivalent to showing that

$$\overline{T^*_{X_{reg}}V} = \overline{T^*_{X^*_{reg}}V^*} \subset T^*(V) = V \times V^* = T^*(V^*)$$

Here we have an identification of T^*V and $T^*(V^*)$ as symplectic manifolds, after switching the sign of the canonical symplectic form on one of them.

Now note that $\overline{T^*_{X_{reg}}(V)}$ is a closed irreducible Lagrangian subvariety of $V \times V^*$ that is invariant under dilations of both V and V^*. Hence, viewing it as a conical Lagrangian variety in $T^*(V^*)$, we conclude from our Lemma 2.11 above that it is $\overline{T^*_{X^*_{reg}}}$ as X^*_{reg} is the smooth part of the image of its projectition under $\pi : T^*(V^*) \to V^*$. This concludes the proof. □

Proof of Pjaseckii: Take any G-orbit $\mathbb{O} \subset \mathbb{P}^N$ whose closure is not all of $\mathbb{P}^N$, then the projective dual variety of its closure, $\overline{\mathbb{O}}^*$, is G-invariant, irreducible and non-empty. Therefore, its the closure of some orbit $\tilde{\mathbb{O}}$ from our lemma. By the previous theorem $\overline{\mathbb{O}} = \overline{\tilde{\mathbb{O}}}^*$. □

Remark 2.15. *If the number of G-orbits is infinite, then in general there is no natural bijection.*

2.3 Moment map

Let (M, ω) be a symplectic manifold, and G be a Lie group acting on M via *symplectomorphisms*, that is,

$$\omega(g \cdot x, g \cdot y) = \omega(x, y) \quad \text{for all} \quad x, y \in T_m M, m \in M, g \in G$$

The linearization of the G-action gives a map of Lie algebras:

$$\mathfrak{g} \to Vect^s(M) \subset Vect(M)$$

where $Vect^s(M)$ are the *symplectic* vector fields on M, i.e. $\xi \in Vect(M)$ such that $\mathcal{L}_\xi \omega = 0$.

By the Cartan's homotopy formula, we have:

$$\mathcal{L}_\xi \omega = d(\iota_\xi \omega) + \iota_\xi(d\omega) = d(\iota_\xi \omega)$$

Therefore, a vector field ξ is symplectic if and only if the 1-form $\iota_\xi \omega$ is closed.

Recall that any function $f \in \mathcal{O}(M)$ we have an associated vector field ξ_f defined via the formula $\iota_{\xi_f} \omega = -df$. Since $-df$ is exact, hence closed, the vector fields of the form ξ_f are always symplectic. Furthermore, this assignment leads to a Lie algebra homomorphism:

$$\mathcal{O}(M) \to Vect^s(M)$$

where $\mathcal{O}(M)$ has the Lie algebra structure coming from the Poisson bracket on M.

Definition 2.16. *A symplectic G-action on (M, ω) is said to be* Hamiltonian, *if there exists a Lie algebra homomorphism $H : \mathfrak{g} \to \mathcal{O}(M)$ making the following diagram of Lie algebras commute:*

$$\begin{array}{ccc} & \mathfrak{g} & \\ {}^{H}\swarrow & & \searrow \\ \mathcal{O}(M) & \longrightarrow & Vect^s(M) \end{array}$$

If such an H exists, the dual map $\mu : M \to \mathfrak{g}^*$ defined by:

$$\mu(m)(x) = H_x(m)$$

is called the moment map.

For $\xi \in \mathfrak{g}$, let ξ_M denote the corresponding vector field on M induced by the G-action, then a convenient way to compute the moment map is the formula:

$$\iota_{\xi_M}\omega = -d\langle \mu, \xi \rangle$$

Here, we have substituted $H_\xi = \langle \mu, \xi \rangle$. In particular, note that existence of H_ξ depends on whether $\iota_{\xi_M}\omega$ is exact. However, even this obstruction vanishes, it is not immediate that H_ξ can be chosen so that the resulting map $\mathfrak{g} \to \mathcal{O}(M)$ is a *Lie algebra homomorphism.* For $x, y \in g$, we have:

$$\{H_x, H_y\} = H_{[x,y]} + C(x, y)$$

The action is said to be Hamiltonian if we can take $C(x, y) = 0$ for all $x, y \in \mathfrak{g}$.

In terms of the moment map, this corresponds to the requirement that $\mu : M \to \mathfrak{g}^*$ be equivariant (with respect to coadjoint action of G on $\mathfrak{g}^*$) for the action of the identity component $G_0 \subset G$. In other words, we have:

$$\mu(g \cdot m) = Ad^*(g)(\mu(m)), \text{ for all } \ g \in G_0$$

In practice, G will almost always be connected, in which case $G = G_0$.

In the general case, the function $C(x, y)$ is bilinear, skew-symmetric, and satisfies the identity :

$$C([x, y], z) + C([y, z], x) + C([z, x], y) = 0$$

That is, it is a 2-cocycle in the Lie algebra cohomology of $\mathfrak{g}$. Different choices of constants in the Hamiltonians H_x leads to a modification of the cocycle C by an exact term, in particular we get a well-defined class $[C] \in H^2(\mathfrak{g}, \mathbb{C})$ (say, we are working over $\mathbb{C}$), and the symplectic action is Hamiltonian if and only if this class vanishes. If the action is Hamiltonian, the functions H_x are determined up to addition of a 1-cocycle, i.e. a map from $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to \mathbb{C}$.

In summary, first we need to check the obstructions coming from $[\iota_{\xi_M}\omega] \in H^1(M; \mathbb{C})$ vanish. (Note that since $\iota_{\xi_M}\omega$ is closed 1-form of type $(1, 0)$, any primitive of it would be holomorphic.) Second, we need to check whether the cocycle C vanishes in $H^2(\mathfrak{g}, \mathbb{C})$, and finally the choices that we have for H_x are parametrized by $H^1(\mathfrak{g}, \mathbb{C})$.

Remark 2.17. *Let $\mathfrak{g}$ be a Lie algebra and let $C^k = \Lambda^*\mathfrak{g} \to \mathbb{C}$ be alternating $\mathbb{C}$-linear maps from $\mathfrak{g} \times \ldots \times \mathfrak{g} \to \mathbb{C}$. Define the linear operator $\delta : C^k \to C^{k+1}$ via:*

$$\delta c(x_0, x_1, \ldots, x_k) = \sum_{i<j} (-1)^{i+j} c([x_i, x_j], x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_k)$$

Exercise: Check that $\delta^2 = 0$. The cohomology of this complex is the Lie algebra cohomology defined by Chevalley and Eilenberg.

For a semisimple Lie algebra $\mathfrak{g}$ over a field of characteristic zero (such as $\mathbb{C}$), Whitehead's first and second lemma implies that $H^1(\mathfrak{g},\mathbb{C}) = H^2(\mathfrak{g},\mathbb{C}) = 0$. Exercise: Show that if $H^1(\mathfrak{g},\mathbb{C}) = H^2(\mathfrak{g},\mathbb{C}) = 0$, any symplectic G-action is Hamiltonian (use the fact that commutator of symplectic vector fields is Hamiltonian).

Examples: 1) Consider the action of a vector space V on itself by translation:

$$t : q \to q + t$$

This induces a symplectic action on $T^*V = V \times V^*$. Note that we have an identification of $\mathfrak{g} \cong V$. It is straightforward to calculate that the corresponding moment map $V \times V^* \to V^*$ is the projection $(q,p) \to p$, hence the terminology moment map.

2) Here is a slightly less trivial example:

Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Equip it with the (real) rotationally invariant symplectic 2-form

$$\omega = \iota_v(dx \wedge dy \wedge dz) = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$$

where

$$v = x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz} \in Vect(\mathbb{R}^3)$$

Consider the action of $G = S^1$ on S^2 given by (clockwise) rotation around the z-axis. The generating vector field of the action of S^1 is $\xi = -x\frac{d}{dy} + y\frac{d}{dx}$. One then computes:

$$\iota_\xi \omega = -dz$$

Hence, $\mu : S^2 \to \mathbb{R}$ given by $(x,y,z) \to z$ is the moment map for this action, where we identified $\mathfrak{g}^*$ with $\mathbb{R}$ in the obvious way.

From this we can also deduce the moment map for the action of $SO(3)$ on S^2. Identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ so that the infinitesimal rotation around the j^{th} basis vector e_j maps to e_j. This, in turn, induces an identification of $\mathfrak{so}(3)^*$ with $\mathbb{R}^3$ and under this identification the moment map $\mu : S^2 \to \mathbb{R}^3$ is just the inclusion map.

3) Let $M = \mathbb{C}^2$ with coordinates p, q and $\Omega = dp \wedge dq$. Consider the action of $G = SL_2(\mathbb{C})$ on M. We have

$$\mathfrak{g} = \mathfrak{sl}_2 = \{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} | a, b, c \in \mathbb{C}\}$$

The generating vector fields of the action are given as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to q\frac{\partial}{\partial p}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to p\frac{\partial}{\partial q}$$

The correspond Hamiltonian functions are pq, $q^2/2$ and $-p^2/2$ respectively. Identifying $\mathfrak{sl}_2$ with $\mathfrak{sl}_2^*$ via the invariant non-degenerate bilinear form $(A, B) \to tr(A \cdot B)$ allows us to write the moment map as:

$$\mu : (p, q) \to \frac{1}{2} \begin{pmatrix} pq & q^2 \\ -p^2 & -pq \end{pmatrix}$$

Note that this map sends $\mathbb{C}^2$ to the nilpotent cone of $\mathfrak{sl}_2$ ramified at the origin.

4) We next discuss another example of interest to us. Suppose a Lie group G acts on X by diffeomorphisms. Then, we get an induced *symplectic* action of G on $(M = T^*X, \omega = d\lambda)$. To be pedanting, suppose $g \in G$ acts on X by $\phi_g : X \to X$, then the action on M is given by:

$$(q, p) \to (\phi_g(q), (d\phi_g^*)^{-1}p)$$

Clearly, this action preserves $\lambda = pdq$, hence is symplectic.

We next show that such an action is always Hamiltonian. Let $\xi \in \mathfrak{g}$ be a generator. Let $u_\xi \in Vect(X)$ and $\tilde{u}_\xi \in Vect(T^*X)$ be the corresponding vector fields on X and T^*X induced by the G-action. Clearly,

$$\pi_*(\tilde{u}_\xi) = u_\xi$$

where $\pi : T^*X \to X$ is the projection. Now, observe that:

$$\mathcal{L}_{\tilde{u}_\xi}\lambda = 0$$

Indeed, as we saw above, λ is invariant under all automorphisms of T^*X that arise from an automorphism of X.

Next, consider the function $h_\xi : T^*X \to \mathbb{C}$ defined on a covector α by :

$$h_\xi(\alpha) = \alpha(u_\xi)$$

We have $\lambda_\alpha(\tilde{u}_\xi) = \alpha(u_\xi)$. Therefore, $h_\xi = \iota_{\tilde{u}_\xi}\lambda$. On the other hand,

$$0 = \mathcal{L}_{\tilde{u}_\xi}\lambda = d\iota_{\tilde{u}_\xi}\lambda + \iota_{\tilde{u}_\xi}\omega = dh_\xi + \iota_{\tilde{u}_\xi}\omega$$

Hence, the assignment

$$H : \xi \to h_\xi = \lambda(\tilde{u}_\xi)$$

is a Hamiltonian for the G-action on T^*X.

Exercise: Check that $H : \mathfrak{g} \to \mathcal{O}(T^*X)$ gives a Lie algebra homomorphism. Conclude that the induced map $\mathbb{C}[\mathfrak{g}^*] \to \mathcal{O}(T^*X)$ is a map of Poisson algebras.

Remark 2.18. *Let X be a G-manifold, then the Lie algebra morphism $\mathfrak{g} \to Vect(X)$ canonically induces a map*

$$\mathcal{U}\mathfrak{g} \to \mathcal{D}_X$$

This map is clearly filtration preserving and one gets an induced map on the associated graded algebras:

$$\mathbb{C}[\mathfrak{g}^*] \to \mathcal{O}(T^*X)$$

It can be shown that this is the dual to the moment map $\mu : T^*X \to \mathfrak{g}^*$. Hence, one can think of the map $\mathcal{U}\mathfrak{g} \to \mathcal{D}_X$ as a quantization of the moment map.

Next, let us specialize to the case $X = G/P$ for $P \subset G$ an algebraic subgroup of G. Recall that X has a unique algebraic structure such that X has a canonical G-action (see for ex. Springer- Linear algebraic groups, Sec. 5.5.) and we thus get a Hamiltonian G-action on $T^*(G/P)$. We will work out an explicit description of the moment map:

$$\mu : T^*(G/P) \to \mathfrak{g}^*$$

Let $\mathfrak{p}$ be the Lie algebra of P and $\mathfrak{p}^\perp \subset \mathfrak{g}^*$ be the annihilator of $\mathfrak{p} \subset \mathfrak{g}$.

Observe that, if we restrict the coadjoint action of G on $\mathfrak{g}^*$ to P, then we have, for $x \in \mathfrak{p}$, $\xi \in \mathfrak{p}^\perp$ and $p \in P$:

$$\langle Ad^*(p)\xi, x\rangle = \langle \xi, Ad(p^{-1})x\rangle = 0$$

Hence, $\mathfrak{p}^\perp$ is stable under the coadjoint action of P. We can then construct an associated fibre bundle $G \times_P \mathfrak{p}^\perp$ as the quotient of $G \times \mathfrak{p}^\perp$ by the free P-action :

$$(g, z) \to (gp, Ad^*(p^{-1})z)$$

This quotient is an algebraic variety and projection to the first component turns it into a vector bundle over G/P, whose fibres are isomorphic to $\mathfrak{p}^\perp$ (see Springer- Linear algebraic groups, 5.5.8). Indeed, it is obtained by gluing together varieties $U \times \mathfrak{p}^\perp$ where $U \subset G/P$ are open sets over which the principal P-bundle $\pi : G \to G/P$ have sections, so that $\pi^{-1}(U) \cong U \times P$.
Proposition 2.19. *There is a natural G-equivariant isomorphism:*

$$T^*(G/P) \cong G \times_P \mathfrak{p}^\perp$$

Proof. Let $e = 1 \cdot P/P$ be the base point. The stabilizer of the G-action on G/P at the point $g \cdot e$ is the group gPg^{-1}, hence we have, $T_{g\cdot e}(G/P) = \mathfrak{g}/Ad(g)\mathfrak{p}$. It follows that for any $g \in G$, we have an identification:

$$T^*_{g\cdot e}(G/P) = (\mathfrak{g}/Ad(g)\mathfrak{p})^* = Ad^*(g) \cdot \mathfrak{p}^\perp \subset \mathfrak{g}^*$$

To give an isomorphism of bundles, consider $G/P \times \mathfrak{g}$ as a trivial $\mathfrak{g}$ bundle over G/P. The infinitesimal action of $\mathfrak{g}$ on G/P induces a bundle map:

$$G/P \times \mathfrak{g} \to T(G/P)$$

with kernel the subbundle E whose fiber at a point $gP \in G/P$ is the stabilizer $Ad(g)\mathfrak{p}$. It is now easy to write a bundle map isomorphism $G \times_P \mathfrak{g}/\mathfrak{p}$ given by :

$$[g, v] \to (gP, Ad(g)v)$$

Note that $(gP, Ad(g)v) = (gpP, Ad(gp)Ad(p^{-1})v)$, hence the map is well-defined, and the bundle isomorphism follows from isomorphism of fibers.

This gives an identification of $T(G/P)$ with $G \times_P \mathfrak{g}/\mathfrak{p}$. The dual construction gives an identification $G \times_P \mathfrak{p}^\perp \to T^*(G/P)$ given by:

$$[g, z] \to (gP, Ad^*(g)z)$$

where $T^*(G/P)$ is viewed as a sub-bundle of the trivial bundle $G/P \times \mathfrak{g}^*$.

$\square$

Proposition 2.20. *Under the isomorphism $T^*(G/P) \cong G \times_P \mathfrak{p}^\perp$, the moment map $\mu : T^*(G/P) \to \mathfrak{g}^*$ is given by:*

$$(g, z) \to Ad^*(g) \cdot z =: gzg^{-1}, \; g \in G, z \in \mathfrak{p}^\perp$$

Note that this map defined on $G \times \mathfrak{p}^\perp$ descends to the quotient $G \times_P \mathfrak{p}^\perp$.

Proof. Write $\pi : T^*(G/P) \to G/P$ for the projection. Let $[g, z]$ represent the equivalence class of $(g, z) \in G \times_P \mathfrak{p}^\perp = T^*(G/P)$. For $\xi \in \mathfrak{g}$, let us write u_ξ and $\tilde{u}_\xi$ the induced vector fields on G/P and $T^*(G/P)$ as before. We have

$$\mu([g, z])(\xi) = \lambda_{[g,z]}(\tilde{u}_\xi) = ([g, z])(u_\xi) = (Ad^*(g) \cdot z)(\xi)$$

$\square$

In particular, we have:

Corollary 2.21. *Let G be a finite-dimensional semisimple Lie group and $B \subset G$ is a Borel subgroup, then there is a G-equivariant isomorphism*

$$T^*(G/B) = G \;\times_B\; \mathfrak{n}$$

Proof. Under the isomorphism $\mathfrak{g}^* \cong \mathfrak{g}$ given by an invariant pairing, $\mathfrak{b}^\perp$ goes to $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$. $\square$

There are two special classes of vector fields on $T^*(G/P)$. Given a 1-form α on G/P, there is an associated vertical vector field $\tilde{\alpha}$ whose restriction to any fibre $T^*_q(G/P)$ is the constant vector field α_q, the value of α at q. In local coordinates (q_i, p_i), the vertical vectors are of the form $\sum a_i(q)\partial/\partial p_i$, corresponding to the 1-form $\alpha = \sum_i a_i(q)dq_i$ on G/P. In particular, note that $\lambda = pdq$ vanishes on these vertical vector fields.

On the other hand, for each $x \in \mathfrak{g}$ there are vector fields $\tilde{u}_x$ induced by the action of G on $T^*(G/P)$.

Lemma 2.22. *The vector field $\tilde{u}_x$ is tangent to the fibre $T^*_{gP}(G/P)$ if and only if $x \in Ad(g)\mathfrak{p}$.*

Proof The vector field is $\tilde{u}_x$ is vertical at gP if and only if the vector field u_x is tangent to the stabilizer of gP for the G-action on G/P, which is if and only if $x \in Ad(\mathfrak{g})\mathfrak{p}$. $\square$

In particular, note that the dimension of the non-vertical vectors of the form $\tilde{u}_x$ at gP is $\dim(G) - \dim(P)$. Therefore, together with vertical vector fields coming from 1-forms on G/P, these generate all the tangent vectors to $T^*(G/P)$.

In other words, we get a G-equivariant Ehressmann connection on $T(T^*(G/P))$ where the horizontral bundle is given by the vectors of the form $\tilde{u}_x$ which are not vertical. (Compare this to the Maurer-Cartan connection on T^*G.)

The canonical symplectic form ω on $T^*(G/P)$ can be evaluated on these vector fields easily via the following formulae (whose proofs we omit):

For any vertical vector fields $\tilde{\alpha}, \tilde{\beta}$:

$$\omega(\tilde{\alpha}, \tilde{\beta}) = 0$$

This is a consequence of the fact that canonical 1-form λ vanishes on any vertical vector field $\tilde{\alpha}$ since $\pi_*(\tilde{\alpha})$. We can see here that cotangent fibres are Lagrangians, hence $T^*(G/P) \to G/P$ is in fact a Lagrangian fibration.

For induced vector fields $\tilde{u}_x$, $\tilde{u}_y$, $x, y \in \mathfrak{g}$, and $[g, z] \in T^*_{gP}(G/P)$:

$$\omega(\tilde{u}_x, \tilde{u}_y)|_{[g,z]} = z(Ad(g^{-1})([x, y]))$$

For any vertical vector $\beta \in Ad^*(g)\mathfrak{p}^\perp \cong T^*_{gP}(G/P)$ viewed as a tangent vector at $\alpha \in T^*_{gP}(G/P)$:

$$\omega(\tilde{\beta}, \tilde{u}_x))|_\alpha = \beta(Ad(g)x)$$

There is a natural generalization of cotangent bundles, namely *twisted cotangent bundles*.
Proposition 2.23. *Suppose P is connected. Let $\chi \in \mathfrak{g}^*$. Then the subspace $\chi + \mathfrak{p}^\perp \subset \mathfrak{g}^*$ is invariant under adjoint action of P if and only if $\chi|_{[\mathfrak{p},\mathfrak{p}]} = 0$.*

Proof. Suppose $\chi + \mathfrak{p}^\perp$ is P-invariant. The tangent space to $\chi + \mathfrak{p}^\perp$ gets identified with $\mathfrak{p}^\perp$, hence the linearization of the adjoint action of P at identity gives a map:

$$\mathfrak{p} \to End(\mathfrak{p}^\perp)$$

Concretely, this means that for any $p \in \mathfrak{p}^\perp$

$$ad^*_x(\chi + p)(y) = \chi([y, x]) = 0 \text{ for all } \ x, y \in \mathfrak{p}.$$

In other words, this is if and only if $\chi|_{[\mathfrak{p},\mathfrak{p}]} = 0$. Conversely, suppose the linearization of the adjoint action at identity preserves the tangent space $\mathfrak{p}^\perp$ to $\chi + \mathfrak{p}^\perp$. This means that $\chi + \mathfrak{p}^\perp$ is preserved under the action of a small neighborhood of the identity in P, but since P is connected this implies the result. □

Definition 2.24. *For $\chi \in \mathfrak{g}^*$ such that $\chi|_{[\mathfrak{p},\mathfrak{p}]} = 0$, the χ-twisted cotangent bundle of G/P is defined to be the associated bundle $T^\chi(G/P) := G \times_P (\chi + \mathfrak{p}^\perp)$. The projection $\pi : T^\chi(G/P) \to G/P$ has the structure of an affine fibration.*

The twisted cotangent bundle has a canonical symplectic structure ω defined via the formulae analogous to those for the cotangent bundle.

For vertical $\tilde{\alpha}, \tilde{\beta}$:

$$\omega(\tilde{\alpha}, \tilde{\beta}) = 0$$

For induced vector fields $\tilde{u}_x$, $\tilde{u}_y$, $x, y \in \mathfrak{g}$, and $(g, z) \in G \times_P (\chi + \mathfrak{p}^\perp)$:

$$\omega(\tilde{u}_x, \tilde{u}_y)|_{(g,z)} = z(Ad(g^{-1})([x, y]))$$

For any vertical vector $\beta \in Ad^*(g)\mathfrak{p}^\perp$ at gP:

$$\omega(\tilde{\beta}, \tilde{u}_x) = \beta(Ad(g)x)$$

In particular, note that the fibres of the projection are affine Lagrangian subspaces.

Exercise: Check that the condition $\chi_{[\mathfrak{p},\mathfrak{p}]} = 0$ makes the 2-form ω well-defined. Show that $d\omega = 0$ and that G acts symplectically on $T^\chi(G/P)$.

Remark 2.25. *Arnold-Liouville theorem (which you can read in the book of Arnold-Givental on symplectic geometry) that any smooth fibration $p : M \to B$ of a symplectic manifold M with fibers smooth connected and simply connected Lagrangians is isomorphic as a Lagrangian fibration to an open subset of a twisted cotangent bundle.*

2.4 Coisotropic subvarieties

Let (M, ω) be a symplectic manifold. Recall that $\Sigma \subset M$ is coisotropic if for each smooth point $m \in \Sigma$,

$$T_m\Sigma^\omega \subset T_m\Sigma$$

where, $T_m\Sigma^\omega$ is the symplectic orthogonal of the tangent space $T_m\Sigma$.

Suppose $\mathcal{J}_\Sigma \subset \mathcal{O}(M)$ is the defining ideal of Σ, we have:

Proposition 2.26. *The subvariety Σ is coisotropic if and only if*

$$\{\mathcal{J}_\Sigma, \mathcal{J}_\Sigma\} \subset \mathcal{J}_\Sigma.$$

Proof. For any $f \in \mathcal{J}_\Sigma$, we have the associated Hamiltonian vector field $X_f \in T_m\Sigma^\omega$ for $m \in \Sigma^{reg}$ since df vanishes on $T_m(\Sigma)$. Moreover, since $\dim T\Sigma + \dim T\Sigma^\omega = \dim M$, the vector fields of the form X_f span $T_m\Sigma^\omega$. Now, clearly, $T_m\Sigma$ is coisotropic if and only if $T_m\Sigma^\omega$ is isotropic. On the other hand, we have $\{\mathcal{J}_\Sigma, \mathcal{J}_\Sigma\} \subset \mathcal{J}_\Sigma$ if and only if

$$\text{For all } \; f, g \in \mathcal{J}_\Sigma, \;\; \omega_q(X_f, X_g) = 0 \text{ for all } \; q \in \Sigma^{reg}$$

□

Suppose $\Sigma \subset M$ is a coisotropic variety, the restriction of ω to Σ fails to be non-degenerate on $T_m\Sigma$, and its radical is precisely $Tm\Sigma^\omega$. In fact, $T\Sigma^\omega \subset T\Sigma$ is a vector subbundle. The next proposition shows that this is integrable subbundle, hence coistropic subvarieties are foliated by isotropic leaves.

Proposition 2.27. *There exists a foliation on Σ such that, for any $m \in \Sigma$, the space $T_m\Sigma^\omega$ is equal to the tangent space at m to the leaf of the foliation.*

Proof. Recall that by Frobenius theorem, a subbundle $E \subset T\Sigma$ is integrable, if and only if for any sections $X, Y : \Sigma \to E$, $[X, Y]$ is also a section lying in E.

As in the previous proposition, the vector fields of the form X_f for $f \in \mathcal{J}_\Sigma$ span the $T_m\Sigma^\omega$ for any m. But now recall that

$$[X_f, X_g] = X_{\{f,g\}}$$

and $\{f, g\} \in \mathcal{J}_\Sigma$ by the previous proposition. □

The following result will play an important role later on. Recall that a solvable group A is one whose derived series terminates at the trivial group. Derived series of subgroups are defined recursively as setting $G^{(0)} = G$ and

$$G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$$

Note any subgroup and any quotient group of a solvable group is solvable. One of the most important properties of a solvable algebraic group is that it has a connected codimension 1 normal subgroup. Namely, note that $G/[G, G]$ is abelian and non-trivial. Therefore, it has a connected subgroup of codimension 1. Now, consider the preimage of that subgroup by the surjective map $\pi : G \to G/[G, G]$. This gives a connected codimension 1 normal subgroup in G. The existence of such normal subgroups is helpful in making inductive arguments.

For us, the most important example of a solvable group is the group B of invertible upper triangular $n \times n$ matrices. In fact, Lie's theorem implies that any solvable subgroup of a matrix group is conjugate to a subgroup of B.

A connected Lie group G is solvable if and only if its Lie algebra $\mathfrak{g}$ is solvable.
Theorem 2.28. *Suppose that a solvable algebraic group A acts on a symplectic variety M in a Hamiltonian fashion. Let $\mu : M \to \mathfrak{a}^*$ be the corresponding moment map. Then for a coadjoint orbit $\mu^{-1}(\mathbb{O})$ is either empty or a coadjoint subvariety of M.*

Here $\mu^{-1}(\mathbb{O})$ stands for the *reduced* scheme associated to the scheme theoretic inverse image.

We will defer the proof of this theorem until later. Partly because the proof is technical and I am not sure whether you will appreciate it until after you see where it is used. On the other hand, the following special case is easy to prove:
Proposition 2.29. *Suppose that an algebraic group A acts on a symplectic variety M in a Hamiltonian fashion. Let $\mu : M \to \mathfrak{a}^*$ be the moment map. Let $\mathbb{O}$ be a coadjoint orbit consisting of regular values of μ. Then $\mu^{-1}(\mathbb{O})$ is a coadjoint subvariety of M.*

Note that since μ is a G-equivariant map, all the points of $\mathbb{O}$ are regular values if and only if one of them is.

Proof. Let $\mathcal{J}_{\mathbb{O}} \subset \mathbb{C}[\mathfrak{a}^*]$ be the defining ideal of the orbit. (By definition, this is the ideal of regular functions vanishing on the orbit $\mathcal{O}$, but of course, the associated variety is the closure of the orbit $\mathcal{O}$. Note that every coadjoint orbit is both open and closed in $\mathfrak{a}^*$, hence its closure uniquely determines the orbit)

Now, recall that the Poisson bracket on $\mathcal{O}$ coming from its symplectic structure was induced from the Poisson bracket on $\mathbb{C}[\mathfrak{a}^*]$. Therefore, for $f, g \in \mathcal{J}_{\mathbb{O}}$, we have:

$$\{f, g\}|_{\mathbb{O}} = \{f|_{\mathbb{O}}, g|_{\mathbb{O}}\} = 0$$

Hence $\mathcal{J}_{\mathbb{O}}$ is closed under the Poisson bracket. Finally, recall that the dual of μ gives a map of Poisson algebras:

$$\mu^* : \mathbb{C}[\mathfrak{a}^*] \to \mathcal{O}(M)$$

hence $\mu^*(\mathcal{J})_{\mathbb{O}}$ is closed under the Poisson bracket. Since $\mu^{-1}(\mathbb{O})$ is assumed to be reduced, and its regular functions $\mu^*(\mathcal{J})_{\mathbb{O}})$ is stable under the Poisson bracket, it follows from what we proved above that $\mu^{-1}(\mathbb{O})$ is coisotropic. □

Note that for the above proposition, we did not need that $\mathfrak{a}$ is solvable. In general, one concludes as above that $\mu^{-1}(\mathbb{O})$ is stable under the Poisson bracket, but the ring of regular functions on $\mu^{-1}(\mathbb{O})$ is, in general, the radical $\sqrt{\mu^*(\mathcal{J}_{\mathbb{O}})}$. This may not be closed under the Poisson bracket unless $\mathfrak{a}$ is solvable as the example 3) above shows. Recall that we have $SL_2(\mathbb{C})$ acting on $\mathbb{C}^2$, and the moment map is sending $(p, q) \to \frac{1}{2}\begin{pmatrix} pq & q^2 \\ -p^2 & -pq \end{pmatrix}$, the preimage of 0 is just $(0, 0)$, but this is clearly not a coisotropic.

Remark 2.30. *The following important theorem about coisotropic varieties is of similar nature but it takes place in the setting of quantization. Suppose B is a non-commutative associative algebra with a filtration such that the associated graded $grB = A$ is a commutative algebra. Let I be a finitely generated left-ideal of B and consider $grI \subset A$. This is an ideal in A and moreover it is closed under the Poisson bracket ($x, y \in I$ implies $xy - yx \in I$).*

Theorem 2.31. *(Integrability of characteristics) $\sqrt{grI}$ is stable under Poisson bracket on A, hence defines a coisotropic subvariety in Spec(A).*

In the case B is the ring $\mathcal{D}_X$ of differential operators on , say, smooth affine variety X, and I is a $\mathcal{D}_X$-module, Spec$(\sqrt{grI}) \subset T^*X$ is called the characteristic subvariety *(or singular support).*

3 Complex semisimple Lie Algebras (basics)

3.1 Recollections

NB: When I am preparing the lecture for this part of the course, I do look up all the proofs of the statements which I don't remember how to prove. I strongly encourage you to do the same as we don't have time to cover all the elementary classical part of the theory in class.

Let G be a complex semisimple connected Lie group with Lie algebra $\mathfrak{g}$. Let B be a Borel subgroup - a maximal solvable subgroup. Today, we will prove Borel fixed point theorem:

Theorem 3.1. *Let B be connected solvable algebraic group action on a complete variety X. Then B has a fixed point.*

Proof. Recall that the fixed point set X^G of a G-action on a variety is closed in X. Indeed, the fixed point set of a single element $g \in G$ is the preimage of the diagonal under the map $X \to X \times X$ given by $x \to (x, gx)$. Now, X^G is the union of closed sets X^g, hence is closed.

To prove the theorem, we argue by induction. The derived subgroup $[B, B]$ is a proper subgroup and is solvable, hence by induction $X^{[B,B]}$ is non-empty. Furthermore, since $[B, B]$ is a normal subgroup of B, $X^{[B,B]}$ is stable under the action of B. Indeed, for any $g \in [B, B]$, and $b \in B$, $bgx = gg^{-1}bgx = gx$.

Now, recall that G-orbits are locally closed [4]. Therefore, the minimal dimensional orbits have to be closed, hence complete. Now consider such a minimal dimensional orbit O for the action of B on $X^{[B,B]}$. Let $O = B/B_x$ be such an orbit where $x \in O$ and $B_x \subset B$ is the stabilizer of x. Now, B_x contains $[B, B]$. Therefore, B_x is a normal subgroup of B. Recall another basic fact that the quotient of an complex algebraic group by a closed *normal* subgroup is always an affine algebraic group. (Without the normal condition, in general, one gets a quasi-projective variety). Hence B/B_x is and irreducible affine variety. But O is complete, therefore O consists of one point which is the fixed point for the B-action on X. □

This theorem allows one to reduce the study of solvable algebraic groups to subgroups of upper triangular matrices. Here is a list of some basic facts that you may find in standard textbooks on Lie theory (see, for ex., Onishchik and Vinberg). Most of these are corollaries of the Borel fixed point theorem.

Any two Borel subgroups are conjugate, and each Borel subgroup is its own normalizer, i.e. $N_G(B) = B$. Each maximal torus (a.k.a. Cartan subgroup) is its own centralizer, and its normalizer is $N_G(T)$ is such that the Weyl group $W = N_G(T)/T$ is finite. The number of Borel subgroups B containing a maximal torus T is finite and the Weyl group permutes them simply transitively. The union of all Cartan subgroups of contains an Zariski open subset (but is not, in general, the whole of G - i.e. not every matrix is diagonalizable but most are).

The radical R of an algebraic group G is the unique maximal closed connected normal subroup of G. It is also given by the intersection of all Borel subroups of G. An algebraic group is said to be *semisimple* if its radical is the trivial group; $SL_n(\mathbb{C})$ is an example of a semisimple group. An algebraic group is said to be *reductive* if its radical is an algebraic torus. $GL_n(\mathbb{C})$ is an example of a reductive group. Equivalently, G is reductive if and only if its unipotent radical is trivial. Recall that the unipotent radical of G is the unique closed connected normal unipotent subgroup.

The unipotent radical $U = [B, B]$ of B is a maximal connected unipotent subgroup of G. If T is a maximal torus in B, then we have the Levi decomposition $B = TU$.

[4]Here we are using the action is algebraic. For complex Lie group actions, this is not true. For ex. consider the action of $B = \mathbb{C}$ on $X = \mathbb{C}^* \times \mathbb{C}^*$ given by $(z, (u, v)) \to (e^z u, e^{\alpha z} v)$ for $\alpha \in \mathbb{R}\backslash\mathbb{Q}$

For any element $x \in \mathfrak{g}$ we write $C_G(x)$ for the centralizer in G. Recall that this is the stabilizer of the adjoint action of G on $\mathfrak{g}$. Lie algebra of the stabilizer subgroup is $C_{\mathfrak{g}}(x)$ which is the centralizer of x in $\mathfrak{g}$. We can also identify $C_{\mathfrak{g}}(x)$ with $Ker(ad(x))$.

Recall that the rank of a Lie algebra $\mathfrak{g}$ is the defined to be the dimension of any maximal torus (which are all conjugate).

It turns out that for any $x \in \mathfrak{g}$ one has the following inequality:
Proposition 3.2. *$dim(C_G(x)) \geq rk\ \mathfrak{g}$.*

Proof. Let B be any Borel subgroup containing x and $U = [B, B]$ be its unipotent radical. In particular, $[B, x] \subset U$, hence $C_B(x) \subset Ux$. Therefore, $\dim C_B(x) \geq \text{codim}_B U = \text{rk}\ \mathfrak{g}$. This clearly implies $\dim C_G(x) \geq \text{rk}\ \mathfrak{g}$. □

Definition 3.3. *Work over $\mathbb{C}$, an element $x \in \mathfrak{g}$ is*

- regular *if $dim(C_G(x)) = rk\ \mathfrak{g}$, i.e., the stabilizer of $ad(x)$ is of minimal possible dimension.*
- semi-simple *if $ad(x) \in GL(\mathfrak{g})$ is diagonalizable.*
- nilpotent *if $ad(x) \in GL(\mathfrak{g})$ is nilpotent.*

Let $\mathfrak{h} \subset \mathfrak{g}$ denote a Cartan subalgebra, i.e., Lie algebra of a maximal torus. Any element of $\mathfrak{h}$ is semisimple. Any semisimple element of $\mathfrak{g}$ is conjugate to an element of $\mathfrak{h}$.

Exercise: For x regular semisimple, $C_{\mathfrak{g}}(x)$ is a Cartan subalgebra. (Hint: Let $\mathfrak{h} = C_{\mathfrak{g}}(x)$, show that $x \in \mathfrak{h}$ is regular semisimple, conclude that $\mathfrak{h}$ is nilpotent. Finally, show that $\mathfrak{h}$ is self-normalizing.)

Recall the basic fact that any element $x \in \mathfrak{g}$ has a unique Jordan decomposition:

$$x = x_s + x_n$$

where x_s is semi-simple and x_n is nilpotent and $[x_s, x_n] = 0$. (This is a consequence of the basic linear algebra result called Jordan decomposition. First embed $\mathfrak{g} \subset GL(V)$ faithfully using a representation V and apply the Jordan decomposition theorem).

For $x \in \mathfrak{g}$, consider the characteristic polynomial of the endomorphism adx :

$$P_x(t) = det(tI - ad(x))$$

If $n = \dim\mathfrak{g}$, we can write:

$$P_x(t) = \sum_{i=0}^{n} a_i(x) t^i$$

where we can view $a_i(x) \in \mathbb{C}[\mathfrak{g}]$ as a G-invariant polynomial on $\mathfrak{g}$ of degree $n - i$. Clearly, the polynomial $P_x(t)$ is invariant under the adjoint action of G. Therefore, $a_0 = a_1 = \ldots = a_{r-1} = 0$ identically, where $r = \text{rk}\mathfrak{g}$.

Indeed, we have:

Lemma 3.4. *x is regular semisimple if and only if $a_r(x) \neq 0$.*

Proof. Note that $a_r(x) \neq 0$ if and only if $t = 0$ is a zero of $P_x(t)$ of order exactly r. In paritcular, x is regular. Let's show next that x is semisimple. Any $x = x_s + x_n$ since, x_s and x_n commute, we can simultaneously put $ad(x_s)$ and $ad(x_n)$ upper triangular form. Since $ad(x_n)$ is nilpotent, its diagonal entries are identically zero. Hence, the characteristic polynomial of $ad(x_s)$ and $ad(x_s + x_n)$ coincide. Therefore, if x is regular, it follows that x_s is regular. On the other hand, x_s being regular and semisimple, implies tha t it's centralizer is a Cartan subalgebra. But, if $a_r(x) \neq 0$, the centralizer already includes the a subspace of dimension r consisting of diagonal matrices. In addition, x_n is also in the centralizer, but a Cartan subalgebra has dimension r, therefore $x_n = 0$. Conversely, if x is semisimple regular, then after bringing $ad(x)$ to a diagonal form, there would be precisely r non-zero entries, hence this implies that $a_r(x) \neq 0$. □

In particular, note that the union $\mathfrak{g}^{sr}$ of regular semisimple elements are open in $\mathfrak{g}$.

Example 3.5. *For $\mathfrak{g} = \mathfrak{sl}_2 = \{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} | a, b, c \in \mathbb{C}\}$ We have*

$$P_x(t) = t^3 - 4(bc + a^2)t$$

Hence $rk(\mathfrak{g}) = 1$ and the regular semisimple elements are those such that $a^2 + bc \neq 0$, i.e. , regular semisimple elements coincide with non-nilpotent elements.

Note however that a nilpotent element can be regular. For example, show that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is regular.

Recall the upper semicontinuity of dimension theorem states that if $f : X \to Y$ is dominant morphism of irreducible varieties, and $d(x)$ is the largest dimension of any component of $f^{-1}(f(x))$ containing x, then for all n the set $\{x \in X | d(x) \geq n\}$ is closed in X.

Let us apply this to action of G on a variety X:

Proposition 3.6. *Let G be action on an irreducible variety X, then let $X_n = \{x \in X | dim(G_x) \geq n\}$ where G_x is the stabilizer subgroup. Then, X_n is closed in X for each $n \geq 0$.*

Proof. Consider the inverse image of the (closed) diagonal in $X \times X$ under the morphism $(g, x) \to (gx, x)$, that is:

$$Z = \{(g, x) : gx = x\}$$

This is closed in $G \times X$. Let $\pi : Z \to X$ is the projection map. This is certain surjective (hence dominant) as (e, x) maps to x. The fibre $\pi^{-1}(\pi(g, x)) = \pi^{-1}(x) = G_x \times \{x\}$ is equidimensional. Semi-continuity of dimension gives that

$$Z_n = \{(g, x) \in Z : \dim \pi^{-1}(x) \geq n\}$$

is closed. The image of Z_n under π is X_n. Z_n is union of fibers of π and π is an open map, therefore, $\pi(Z_n) = X_n$ is closed. □

One can think of this alternatively as saying that the set

$$\{x \in X | \dim G \cdot x \leq n\}$$

is closed.

In particular, applying this to the adjoint action of G on $X = \mathfrak{g}$, we recover that the set of regular semisimple elements is open in $\mathfrak{g}$.

3.1.1 Bruhat decomposition

Next, let us denote by $\mathcal{B}$ the set of all Borel subalgebras of $\mathfrak{g}$. By definition $\mathcal{B}$ is a closed subvariety of the Grassmanian of $\dim\mathfrak{b}$-dimensional subspaces in $\mathfrak{g}$ formed by solvable Lie subalgebras. Therefore $\mathcal{B}$ is a projective variety.

Choosing a Borel subgroup $B \subset G$, we can write a map:

$$G/B \to \mathcal{B}$$

by sending $[g] \to Ad(g)\mathfrak{b}$. Since any two Borel subalgebras are conjugate and $N_G(B) = B$, this gives a bijection. One can show that this bijection is a G-equivariant isomorphism of algebraic vvarieties.

Notice that the following are immediate from this: i) $\mathfrak{b} \in \mathcal{B}$ is a fiexed point of the adjoint action of $g \in G$ if and only if $g \in B$. ii) $\mathfrak{b} \in \mathcal{B}$ is the zero-point of the vector field on $\mathcal{B}$ associated to $x \in \mathfrak{g}$ if and only if $x \in \mathfrak{b}$.

We have seen that by Borel fixed point theorem any solvable algebraic group action on a projective variety has fixed points. We next recall the following result about $\mathbb{C}^*$ actions on projective varieties:

Theorem 3.7. *(Bialynicki-Birula) Let $\mathbb{C}^*$ act on a projective variety X with finitely many fixed points W. For $w \in W$, define the attracting set:*

$$X_w = \{x \in X | lim_{t \to 0} t \cdot x = w\}$$

The attracting sets give a decomposition

$$X = \sqcup_{w \in W} X_w$$

where X_w is a smooth locally closed algebraic subvariety. Furthermore, there are natural isomorphisms of algebraic varieties :

$$X_w \cong T_w(X_w) \cong T_w^+ X$$

which commute with the $\mathbb{C}^*$ action.

Above, $T_w^+ X$ is the positive part of the weight decomposition

$$T_w X = T_w^+ X \oplus T_w^- X$$

induced by the linearization of the $\mathbb{C}^*$ action on X at w.

Example 3.8. *Consider the $\mathbb{C}^*$ action on $\mathbb{P}^2$ given by $t \cdot [x : y : z] = [tx : t^{-1}y : z]$. This has 3 fixed points: $[0:0:1], [1:0:0], [0:1:0]$. Show that the Bialynicki-Birula decomposition gives the standard cell decomposition of $\mathbb{P}^2$.*

We will next apply B-B decomposition to obtain a cell decomposition of $\mathcal{B}$. We will need the following lemma:

Lemma 3.9. *Let $T \subset B$ be a maximal torus, then the map $N_G(T)/T \to \mathcal{B}$ given by $x \to x\mathfrak{b}x^{-1}$ gives a bijection between the Weyl group and the Borel subalgebras containing the Cartan subalgebra $\mathfrak{h} = LieT$. This implies that the fixed points of the T action on $\mathcal{B}$ are in one-to-one correspondence with the Weyl group $W = N_G(T)/T$.*

Proof. Indeed, let $\mathfrak{b}'$ be a Borel subalgebra containing $\mathfrak{h}$. Choose x such that $x\mathfrak{b}x^{-1} = \mathfrak{b}'$. Therefore, $\mathfrak{h} \subset x\mathfrak{h}x^{-1}$. But, since Cartan subalgebras are maximal, this implies $\mathfrak{h} = x\mathfrak{h}x^{-1}$ hence $x \in N_G(T)$. This gives surjectivity. To see the injectivity suppose $x \in N_G(T)$, such that $x\mathfrak{b}x^{-1} = \mathfrak{b}$, then, since $N_G(B) = B$, $x \in B$. But we also know that $N_B(T) = T$, hence $x \in T$.

To see the second part, since $N_G(B) = B$, the set of fixed points of the adjoint action of T on B correspond to Borel subalgebras that contain $\mathfrak{h}$, but we saw that these are parametrized by the finite group $N_G(T)/T$. □

Now, choose a one-parameter subgroup $\mathbb{C}^* \subset T$ which is in general position in the sense that $Lie\mathbb{C}^* \subset \mathfrak{h}$ is spanned by a regular semisimple element $h \in \mathfrak{h}$. The action of $ad(h)$ on $\mathfrak{b} = LieB$ can be diagonalized. Let us insist (for later use) that the eigenvalues of this action are all non-negative. (We can arrange this easily, by modifying the one-parameter subgroup if necessary.)

We let $\mathbb{C}^*$ act on $\mathcal{B}$ by conjugation. The fixed points of this action are the Borel subalgebras $\mathfrak{b}$ containing h.

Claim: regular semisimple $h \in \mathfrak{b}$ implies $\mathfrak{h} \in \mathfrak{b}$.

Indeed, since h is regular semisimple, it's contained in a unique Cartan subalgebra $\mathfrak{h} = C_{\mathfrak{g}}(h)$. Hence, any Borel $\mathfrak{b}$ containing h must contain $\mathfrak{h}$.

Therefore, the set of $\mathbb{C}^*$ fixed points in $\mathcal{B}$ is equal to the set of T-fixed points in $\mathcal{B}$.

But, we have seen that T-fixed points are in correspondence with the Weyl group elements. Therefore, Bialynicki-Birula decomposition with respect to the $\mathbb{C}^*$ action above gives us a cell decomposition:

$$\mathcal{B} = \bigsqcup_{w \in W} \mathcal{B}_w$$

We next claim that under the identification of G/B with $\mathcal{B}$ the cells $\mathcal{B}_w$ correspond to B-orbits on G/B, i.e we have a double coset decomposition

$$G = \bigsqcup_{w \in W} BwB$$

such that $\mathcal{B}_w = BwB/B$.

We first observe that under the adjoint action of h on $\mathfrak{g}$ we obtain a decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ (as we arranged the weights on $\mathfrak{b}$ to be positive).

Next, let us observe that the action of G on $\mathcal{B}$ decomposes the tangent space $T_w\mathcal{B}$ into positive and negative pieces according to the weights of action of $\mathbb{C}^* \subset G$:

$$T_w\mathcal{B} = T_w^+\mathcal{B} \oplus T_w^-\mathcal{B}$$

The tangent space to $\mathcal{B}_w$ is by definition $T_w^+\mathcal{B}$. Hence, we see that $\dim(\mathcal{B}_w) = \dim U$, where $U \subset B$ is the unipotent subgroup of B.

We claim that indeed, $U \cdot w \subset \mathcal{B}_w$. We need to check that an element uwB/B is in the attracting set of w. Let's view this point as the Borel subalgebra $uw \cdot \mathfrak{b} = Ad(uw)\mathfrak{b} \subset \mathfrak{g}$. The action of $t \in \mathbb{C}^*$ is given by: $tuw \cdot \mathfrak{b}$. It suffices to show that this goes to $w \cdot \mathfrak{b}$ as $t \to 0$. Now, note that since w is a fixed point for the action of t:

$$(tut^{-1}t) \cdot w = (tut^{-1}) \cdot w$$

On the other hand $tut^{-1} \to 1 \in U$ as $t \to 0$ (as we have arranged that t action has positive weights on $\mathfrak{b}$ so also on $\mathfrak{n}$).

This proves $U \cdot w \subset \mathcal{B}_w$. Our next claim is $U \cdot w = \mathcal{B}_w$.

For this we view $\mathcal{B}_w$ with the U-action and use the fact that any orbit of a unipotent group on an affine algebraic variety is closed. Therefore, $U \cdot w$ is closed subvariety in $\mathcal{B}_w$. (Strictly speaking, this requires the justification that U action on $\mathcal{B}$ preserves $\mathcal{B}_w$ but this is similar to the argument given above in showing $U \cdot w \subset \mathcal{B}_w$).

But now $U \cdot w$ is an closed subvariety of $\mathcal{B}_w$ of the same dimension, therefore it has to be the case that $U \cdot w = \mathcal{B}_w$.

Thus, since T fixes w, we have finally arrived at the Bruhat cell decomposition:

$$G/B = \bigsqcup_{w \in W} UwB/B = \bigsqcup_{w \in W} BwB/B$$

Let us just see that when $G = GL(n, \mathbb{C})$. This result is actually something you know well. namely given an invertible matrix g, you can perform column operations in the form of multplying a column by a scalar and subtracting from it a multiple of any column to its left. This amounts to multiplying g with a upper triangular matrix $b^{-1} \in B$ way, this gives the *reduced echelon form*, so it looks like

$$gb^{-1} = \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Each column ends with 1 and the part of the row to the right of any such 1 is zero.

We can then apply a permutation w to get to a upper triangular matrix:

$$u = \begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence, $gb^{-1} = uw$, hence $g = uwb$.

Definition 3.10. *$C_w := BwB/B \subset G/B$ is called a Schubert cell, and its closure $X_w := \overline{C}_w$ is called a Schubert variety.*

Note that the isotropy group of U action is:

$$U_w := U \cap wUw^{-1}$$

In the case $G = GL(n, \mathbb{C})$, these are the matrices a_{ij} such that $a_{ij} = 0$ if $i < j$ and $w^{-1}(i) < w^{-1}(j)$. The complementary subspace in U is given by:

$$U^w := U \cap wU^-w^{-1}$$

where U^- is lower-triangular matrices in $GL(n, \mathbb{C})$ case and U^w are the matrices a_{ij} such that $a_{ij} = 0$ if $i < j$ and $w^{-1}(i) > w^{-1}(j)$.

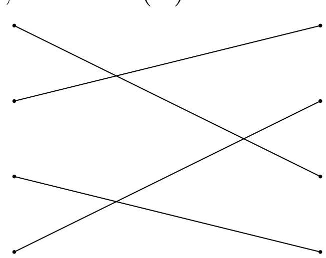
In general, let w_0 be the "longest element" in W, i.e., Bw_0B/B is the dense orbit in G/B. Then $U^- = w_0Uw_0$.

Since $U_w \times U^w \to U$ is an isomorphism of varieties, we get that $U^w \to C_w$, $g \to gwB$ is an isomorphims as well.

Hence, C_w is an affine space of dimension :

$$l(w) = \#\{(i,j) : 1 \leq i < j \leq n : w(i) > w(j)\}$$

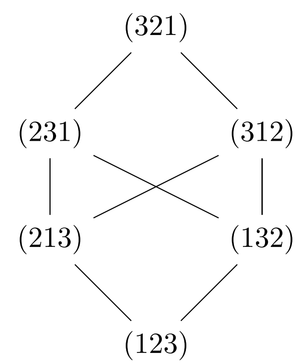
This number can be calculated as follows calculating the number of crossings in. For example, for the permutation $w = (1, 3, 4, 2)$, we have $l(w) = 3$.



In general, $l(w)$ is the length of the element $w \in W$ as an element of the Coxeter group. (We will see later that W is generated by reflections and $l(w)$ is the least number of decomposition into reflections).

There is also a partial order on W which encodes what is contained in what.
Definition 3.11. *There is a partial order on W, so-called the* Bruhat order*, defined by declaring $v \leq w$ if $X_v \subset X_w$ for $v, w \in W$.*

For example, for $Fl_3 = SL_3(\mathbb{C})/B$, we have:



There is also a Bruhat decomposition for G/P where P is a parabolic subgroup (Recall this means that P contains a Borel subgroup). Each such P determines a subgroup $W_P \subset W$ (Recall W is generated by reflections, any W_P is generated by a subset of theses reflections. In fact, this gives a 1-1 correspondence between parabolic subgroups P and Weyl subgroups W_P). Now, we have a generalized Bruhat cell decomposition:

$$G/P = \bigsqcup_{w \in W^P} BwP/P$$

where $W^P = W/W_P$ is the coset space. The cells are of dimension $l(w)$ if w is chosen to be of minimal length in its coset in W/W_P. We then get a partial order on W^P by the same principle as above.

Exercise: Express the Grassmannian of 2-planes in $\mathbb{C}^4$ as $SL(4, C)/P$ and describe the Bruhat order on the corresponding Weyl group cosets W^P.

It is instructive to consider the case $Gl_n(\mathbb{F}_q)$, where $\mathbb{F}_q$ is a field with q elements. Then the order of G is

$$(q^n - 1)(q^n - q) \dots (q^n - q^{n-1})$$

The order of the subgroup of upper triangular matrices is $(q-1)q^{\frac{1}{2}n(n-1)}$ and U^w has order $q^{l(w)}$. The Bruhat decomposition gives us the identity:

$$|G/B| = \sum_{w \in \mathfrak{S}_n} |U^w|$$

i.e.

$$\prod_{k=1}^{n} \frac{q^k - 1}{q - 1} = \sum_{w\mathfrak{S}_n} q^{l(w)}$$

Next, we would like to give a hint of a Springer representation in a simple case. Namely, consider the affine bundle:

$$G/T \to G/B$$

This is a locally trivial fibration. The fibres are isomorphic to $B/T \cong U$. But U is unipotent, hence contractible. Therefore, G/T and G/B are homotopy equivalent. Therefore, we get an isomorphism singular homology groups:

$$H_*(G/T) \cong H_*(\mathcal{B})$$

On the other hand, we can act on G/T by the Weyl group via:

$$w : gT \to gTw = gwT$$

Since w normalizes T, this gives an action. Through the above isomorphism, we then get an action of W on $H_*(\mathcal{B})$. It turns out that this action does not depend on the choices of T and B and so it is absolutely canonical. Furthermore, it turns out that this representation of W is isomorphic to its regular representation. (We might see a proof of this later.)

Note that there is no canonical action of W on the space $\mathcal{B}$, though one can construct an noncanonical actions via expressing G/B as the quotient of the maximal compact subgroup K modulo the maximal compact torus.

3.2 Springer diagram : The SL_n case

We will first concentrate on $G = SL_n(\mathbb{C})$. Let us recall the basic fact that in this case the space $\mathcal{B}$ can be identified with the variety $\mathcal{F}l_n$ of full flags in $\mathbb{C}^n$:

Proposition 3.12. *The space $\mathcal{B}$ is identified naturally with the variety $\mathcal{F}l_n$ of flags*

$$\{0\} \subset V_1 \subset V_2 \ldots V_{n-1} \subset V_n = \mathbb{C}^n$$

where V_i are subspace of $\mathbb{C}^n$ of dimension i.

Proof. We construct a map from $\mathcal{F}l_n \to \mathcal{B}$ by assigning to a flag F the Borel subalgebra

$$\mathfrak{b}_F = \{x \in \mathfrak{sl}_n : x(F_i) \subset F_i \quad \forall i\}$$

If we let $F = \{0 \subset \mathbb{C}^1 \subset \mathbb{C}^2 \ldots \subset \mathbb{C}^n\}$ be the standard flag then clearly $\mathfrak{b}_F$ is the Borel subalgebra of the upper triangular matrices in $\mathfrak{sl}_n$. Now, any flag in $\mathcal{F}l_n$ is conjugate to the standard one by the action of SL_n. On the other hand, the above map is equivariant with respect to the SL_n-action and $N_G(B) = B$ means that the indeed the map we defined is an embedding of $\mathcal{F}l_n \to \mathcal{B}$. The surjetivity is ensured by Lie's theorem, which says that any Borel subalgebra preserves a flag. □

Next, let us consider the variety $\mathbb{C}^n/\mathfrak{S}_n$ of all unordered n-tuples of complex numbers. We next show that this is isomorphic to n-dimensional vector space as an algebraic variety. Namely, consider the space $\mathbb{C}[\lambda]_{n-1}$ of polynomials in λ of degree less than or equal to n. This is clearly an n-dimensional complex vector space. Consider the map:

$$\pi : \mathbb{C}^n \to \mathbb{C}[\lambda]_{n-1}$$
$$(x_1, \ldots, x_n) \to \lambda^n - \prod(\lambda - x_i)$$

Since the right hand side does not depend on the order of the x_i, we get a well-defined map $\mathbb{C}^n/\mathfrak{S}_n \to \mathbb{C}[\lambda]_{n-1}$ which is a bijection by the fundamental theorem of algebra!

Remark 3.13. *Chevalley-Shephard-Todd theorem states that a finite group acting on a complex vector space is a complex reflection group (i.e. generated by reflections along hyperplanes) if and only if its ring of invariants is a polynomial ring*

Next, consider any linear map $x : \mathbb{C}^n \to \mathbb{C}^n$, we then have its characteristic polynomial:

$$det(\lambda 1 - x) \in \mathbb{C}[\lambda]$$

We can then consider its roots as an unordered set $\{x_1, \ldots, x_n\}$. If $x \in \mathfrak{sl}_n$, then we have $x_1 + x_2 + \ldots x_n = 0$. Consider the hyperplane:

$$\mathbb{C}^{n-1} \cong \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_1 + x_2 + \ldots + x_n = 0\}$$

This hyperplane is clearly invariant the action of $\mathfrak{S}_n$ which permutes the coordinates. Hence, we have a well-defined map:

$$\chi : \mathfrak{sl}_n \to \mathbb{C}^{n-1}/\mathfrak{S}_n$$
$$x \to \{x_1, \ldots, x_n\}$$

sending a matrix x to its eigenvalues. This map is called the *adjoint quotient map.*

Next, we define the incidence variety :

$$\tilde{\mathfrak{g}} = \{(x, F) \in \mathfrak{sl}_n \times \mathcal{B} : x(F_i) \subset F_i \quad \forall i\}$$

where $F = (F_0 = 0 \subset F_1 \subset \ldots \subset F_n = \mathbb{C}^n)$ is a complete flag. In terms of the Borel subalgebras description of $\mathcal{B}$:

$$\tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \in \mathfrak{sl}_n \times \mathcal{B} : x \in \mathfrak{b}\}$$

Hence, $\tilde{\mathfrak{g}}$ is clearly a rank $dimB$ vector bundle over $\mathcal{B}$.

We can construct a map:

$$\tilde{\chi} : \tilde{g} \to \mathbb{C}^{n-1}$$

which should be though as assigning a pair $(x, F) \in \tilde{g}$ the order n-tuple of eigenvalues of x. Indeed, since x preservers F_i, it induces a linear map

$$x : F_i/F_{i-1} \to F_i/F_{i-1}$$

We write x_i for the eigenvalue of action of x on this 1-dimensional vector space. This gives a map $(x, F) \to (x_1, \ldots, x_n) \in \mathbb{C}^n$. Since $x \in \mathfrak{sl}_n$, we must again have $x_1 + \ldots + x_n = 0$, hence we get the desired map: $\tilde{\chi} : \tilde{g} \to \mathbb{C}^{n-1} \cong \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_1 + x_2 + \ldots + x_n = 0\}$.

Thus, we used the flag F to order the eigenvalues of x.

We now have $\pi, \chi, \tilde{\chi}$, the fourth map of the square is the *Springer map*:

$$\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$$
$$(x, F) \to x$$

These all fit into the famous commutative diagram:

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \xrightarrow{\tilde{\chi}} & \mathfrak{h} \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathfrak{g} & \xrightarrow[\chi]{} & \mathfrak{h}//W \end{array}$$

where we wrote $\mathfrak{h} = \mathbb{C}^{n-1}$ for a Cartan subalgebra of $\mathfrak{sl}_n$ and $\mathfrak{h}//W$ for $\mathbb{C}^{n-1}/\mathfrak{S}_n$.

Note that the fiber $\mu^{-1}(x)$ gets identified with the Springer fiber:

$$\mathcal{B}_x = \{F \in \mathcal{B} : x(F_i) \subset F_i \quad \forall i\}$$

Here is the most important result of today:

Proposition 3.14. *For $x \in \mathfrak{g}^{sr}$ (semisimple regular), $\mathcal{B}_x$ consists of $n!$ points, and there is a canonical $\mathfrak{S}_n$ action on $\mathcal{B}_x$ making it a principal homogeneous $\mathfrak{S}_n$-space.*

Proof. The set $\mathfrak{g}^{sr}$ of regular semisimple elements consists of linear maps $x : \mathbb{C}^n \to \mathbb{C}^n$, which have zero trace and n distinct eigenvalues. For $x \in \mathfrak{g}^{sr}$, let us decompose :

$$\mathbb{C}^n = \bigoplus_i V_i, \quad \dim(V_i) = 1$$

where the V_i are the n distinct eigenspace of x. Note that any subspace fixed by x is a direct sum of these eigen spaces. Now, the set $\mathcal{B}_x$ of all complete flags fixed by x are of the form:

$$\mathcal{B}_x = \{F = (V_{i_1} \subset V_{i_1} \oplus V_{i_2} \subset \ldots)\}$$

so that there is a canonical bijection between $\mathcal{B}_x$ and the set of orderigns of the set $\{1, \ldots, n\}$). Hence, the set $\mathcal{B}_x$ consists of $n!$ points. Moreover, we can define an action $\mathfrak{S}_n$ on $\mathcal{B}_x$ as follows: Given $F \in \mathcal{B}_x$, choose the orderings of the eigenspaces such that $F_i = V_1 \oplus \ldots \oplus V_i$ for all i (again a flag gives us a way to order the eigenvalues). Then for any element $w \in W \cong \mathfrak{S}_n$, define the flag

$$w(F) = V_{w^{-1}(1)} \subset \ldots \subset V_{w^{-1}(1)} \oplus \ldots V_{w^{-1}(n)}$$

(The inverses is there so as to get an action from the *right.*)

Clearly, there were no choices made to define the action, hence it is canonical.

□

In the commutative diagram above, we now see that both $\tilde{\mathfrak{g}}$ and $\mathfrak{h}$ have actions of the Weyl group W, and $\tilde{\chi}$ is an equivariant map with respect to these $\mathfrak{S}_n$ actions.

We have now seen that general fibres of μ are discrete and carry the standard action of W. We next want to go deeper in the stratification of $\mathfrak{g}$ given by (co-)adjoint orbits and see the W action on the cohomology of all he fibers of μ. Note that we have also seen the homology of the zero-fiber $\mu^{-1}(0) = \mathcal{B}$, the fiber above the deepest stratum, carries an action of W.

But first, we want to generalize the above picture to other semisimple Lie algebras.

3.3 Root systems

First, we need to clarify what we call *the* Weyl group W of a Lie algebra $\mathfrak{g}$. So far, we have taken the lazy approach of picking a maximal torus $T \subset G$ and calling $N_G(T)/T$ the Weyl group. Even though, isomorphism type of this group does not depend on T (or G), its concrete construction depends on these choices. We will now construct W in an abstract way starting from $\mathfrak{g}$, which does not depend on any choices.

Indeed, the way to do this is to define W as the symmetry group of the root system associated to $\mathfrak{g}$. Let us start somewhat more abstractly. Strictly speaking we will discuss slightly more general situation than the case of finite-dimensional semisimple Lie algebras.

Definition 3.15. *A generalized Cartan matrix is a matrix $(a_{ij})_{i,j\in I}$ with integer entries and for some finite set I, such that*

- $a_{ii} = 2$ *for all* $i \in I$,
- $a_{ij} \leq 0$ *for all* $i \neq j$,
- $a_{ij} = 0$ *if and only if* $a_{ji} = 0$.

A is called a symmetrizable Cartan matrix if it can be written as:

$$A = D \times S$$

where D is diagonal and S is symmetric, where D and S are both square matrices with entries in $\mathbb{Q}$.

We will see that finite-dimensional semisimple Lie algebras have *positive definite* Cartan matrices associated to them. For example, the Cartan matrix of $\mathfrak{sl}_5$ is :

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

Definition 3.16. *A Kac-Moody root datum consists of*

- *A generalized Cartan matrix* $A = (a_{ij})_{i,j \in I}$*,*
- *Two finitely generated free* $\mathbb{Z}$*-modules* $h_{\mathbb{Z}}$ *and* $h_{\mathbb{Z}}^{\vee}$*, called coweight and weight lattices repectively. These come with a perfect pairing*

$$\langle , \rangle : \mathfrak{h}_{\mathbb{Z}}^{\vee} \times \mathfrak{h}_{\mathbb{Z}} \to \mathbb{Z}$$

- *A pair of maps*

$$I \to h_{\mathbb{Z}}^{\vee} : i \to \alpha_i$$
$$I \to h_{\mathbb{Z}} : i \to \alpha_i^{\vee}$$

  such that

$$\langle \alpha_i, \alpha_j^{\vee} \rangle = a_{ij}$$

The set of α_i's is called the (positive) simple roots and the set of $\alpha_i^{\vee}$'s are called simple coroots.

The sublattice

$$R = \bigoplus_i \mathbb{Z}\alpha_i \subset \mathfrak{h}_{\mathbb{Z}}^{\vee}$$

is called the *root lattice*, and the sublattice

$$R^{\vee} = \bigoplus_i \mathbb{Z}\alpha_i^{\vee} \subset \mathfrak{h}_{\mathbb{Z}}$$

is called the *coroot lattice*.

One usually considers the vector spaces $\mathfrak{h}_{\mathbb{Z}} \otimes \mathbb{R}$ (or $\mathfrak{h}_{\mathbb{Z}} \otimes \mathbb{C}$) and regards $\mathfrak{h}_{\mathbb{Z}}$ as a lattice in this vector space (similarly for $\mathfrak{h}_{\mathbb{Z}}^{\vee}$).

Given two distinct positive roots α_i, α_j, we define the multiplicities: $m_{ij} = 2, 3, 4, 6, \infty$ respectively, if $a_{ij}a_{ji} = 0, 1, 2, 3$ or ≥ 4.

In the finite dimensional case, where one has a non-degenerate Killing form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ given by $(x, y) = Tr(adx \cdot ady)$, roots and coroots can be identified and then by the cosine rule

$$4\cos^2(\phi) = \langle \alpha_i, \alpha_j^{\vee} \rangle \langle \alpha_j, \alpha_i^{\vee} \rangle = a_{ij}a_{ji}$$

measures the unique angle $0 < \phi < \pi$ between the two simple roots α_i, α_j. Right-hand side is a non-negative integer < 4, so in this case it can only be $0, 1, 2, 3$. Hence, in the finite-dimensional case m_{ij} can only be $2, 3, 4$ or 6.

Definition 3.17. *The Weyl grooup of root system is defined by generator* s_i *for* $i \in I$ *and relations*

$$s_i^2 = 1 \quad i \in I,$$
$$(s_i s_j)^{m_{ij}} = 1 \quad i, j \in I, i \neq j.$$

For example, for $\mathfrak{sl}_n$, the angles between roots are $2\pi/3$, hence $m_{ij} = 3$, and we get the braid relations:

$$s_i s_j s_i = s_j s_i s_j \quad i, j \in \{1, \ldots, n-1\}, i \neq j$$

It is an exercise to check that together with the relations s_i^2, these define the symmetric group $\mathfrak{S}_n$.

In general, we refer to the Weyl group defined via a root datum by the *abstract Weyl group*.

W acts on the weight and coweight lattices via simple reflections:

$$s_i\lambda = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i \quad \lambda \in \mathfrak{h}_\mathbb{Z}$$
$$s_i h = h - \langle h, \alpha_i \rangle \alpha_i^\vee \quad h \in \mathfrak{h}_\mathbb{Z}^\vee$$

These clearly preserve the roots R and coroots $R^\vee$. These two actions preserve the pairing, that is.

$$\langle w \cdot \lambda, w \cdot h \rangle = \langle \lambda, h \rangle$$

Furthermore, the actions on weight and coweight lattices are faithful.

We now describe the way to obtain a root system starting from a Lie algebra $\mathfrak{g}$. We first need a lemma:

Lemma 3.18. *For any Borel subalgebra $\mathfrak{b}, \mathfrak{b}' \in \mathcal{B}$, there is a canonical isomorphism*

$$\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \cong \mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}']$$

Proof. Choose $g \in G$ be such that $\mathfrak{b}' = g\mathfrak{b}g^{-1}$. This gives a map $\mathfrak{b} \to \mathfrak{b}'$ sending $x \to gxg^{-1}$. For a different choice $\tilde{g}$ with the property $\mathfrak{b}' = \tilde{g}\mathfrak{b}\tilde{g}^{-1}$, we get a new map from $\mathfrak{b} \to \mathfrak{b}'$. However, these maps agree on the quotients $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \to \mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}']$ as the adjoint action of B on $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ is trivial and $\tilde{g}^{-1}g \in N_G(B) = B$. Therefore, we have canonical isomorphisms as required. □

We identify all the quotient spaces $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ and call it the *abstract Cartan subalgebra* and denote it by $\mathfrak{H}$. Note that $\mathfrak{H}$ is not a subalgebra of $\mathfrak{g}$ but for any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a Borel subalgebra containing $\mathfrak{b} \supset \mathfrak{h}$, the composite $\mathfrak{h} \to \mathfrak{b} \to \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \cong \mathfrak{H}$ is an isomorphism.

Now, choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and decompose $\mathfrak{g}$ into simultaneous eigenspaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. We get:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \neq \{0\}} \mathfrak{g}_\lambda$$

where $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} : ad(h)x = \lambda(h)x \ , \ \forall h\}$ are called the weight spaces for the adjoint action. Each of them are 1-dimensional. Note that only finitely many

$$\lambda \in \mathfrak{h}^*$$

occur in the above decomposition as we assume $\mathfrak{g}$ is finite dimensional. The span of the non-zero weights λ is called the root lattice $R \in \mathfrak{h}^*$. Using the Killing form $(x, y) = Tr(adx \cdot ady)$, we also obtain coroots

$$R^\vee \in \mathfrak{h}$$

On the other hand, we are missing a choice of a $\mathbb{Z}$-basis for R and $R^\vee$. For this reason, pick a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$ and consider weights of adjoint $\mathfrak{h}$-action on $\mathfrak{b}$ (Note that since $\mathfrak{h} \subset \mathfrak{b}$, the adjoint action of $\mathfrak{h}$ preserves $\mathfrak{b}$ - recall that isomorphism between $\mathcal{B}$ and G/B.). We take the weights of this action to be the positive weights. Another way to say is this is that, once we choose o Borel subalgebra $\mathfrak{b}$, then we get the adjoint action of $\mathfrak{h}$ on $\mathfrak{b}$, we then get an induced action on $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ and the weights of this action give us simple (positive) roots. Changing from $\mathfrak{b}$ to $\mathfrak{b}'$ comes with a canonical map $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ to $\mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}']$ and we get a conjugate choice of simple roots. (Recall that since $\mathfrak{b}$ and $\mathfrak{b}'$ both contain $\mathfrak{h}$, They are related by an element $w \in W$, via $\mathfrak{b}' = w\mathfrak{b}w$, hence the choice different $\mathfrak{b}$'s correspond to action of the Weyl group on the roots).

This specifies the set:

$$\alpha_i, \quad i \in I$$

of (positive) simple roots. The duals of these with respect to the Killing form are called the coroots $\alpha_i^\vee$.

Exercise: Use the filtraion $\mathfrak{b} \supset [\mathfrak{b}, \mathfrak{b}] \supset [\mathfrak{b}, [\mathfrak{b}, \mathfrak{b}]] \supset \ldots \supset \{0\}$, to show that positive span of α_i generate the positive roots.

We also define the weight lattice $\mathfrak{h}_Z \in \mathfrak{h}^*$ and coweight lattice $\mathfrak{h}_\mathbb{Z}^\vee \in \mathfrak{h}$ as :

$$\mathfrak{h}_Z = \{h \in \mathfrak{h} : h(R) \in \mathbb{Z}\}$$
$$\mathfrak{h}_Z^\vee = \{\alpha \in \mathfrak{h}^* : \alpha(R^\vee) \in \mathbb{Z}\}$$

Finally, the image of $\mathfrak{h}_\mathbb{Z}$ and $\alpha_i^\vee$'s under the composition $\mathfrak{h} \to \mathfrak{b} \to \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ gives a lattice $\mathfrak{H}_\mathbb{Z}$ with a positive basis in the vector space $\mathfrak{H}$, and again by using the Killing form we get a dual lattice $\mathfrak{H}_\mathbb{Z}^\vee$ with basis α_i in $\mathfrak{H}^*$.

We leave it as an exercise to check that this gives a root datum as defined above and importantly it does not depend on the choice of $\mathfrak{h}$ or $\mathfrak{b}$.

Let us give a concrete example to see how this works:

Consider $\mathfrak{sl}_n(\mathbb{C})$. Let $\mathfrak{h}$ be the diagonal matrices of trace 0. Then we have the Cartan decomposition given as:

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{h} \oplus \Sigma_{i \neq j} \mathbb{C} E_{ij}$$

where E_{ij} is the matrix with 1 in the i, j position and 0 elsewhere.

Indeed, for a diagonal matrix

$$x = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

with $\lambda_1 + \ldots + \lambda_n = 0$, we have:

$$[x, E_{ij}] = (\lambda_i - \lambda_j) E_{ij}$$

Therefore, the root lattice is spanned by

$$\alpha_{ij} : \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \lambda_n \end{pmatrix} \rightarrow \lambda_i - \lambda_j$$

Thare $n(n-1)$ weights arising this way. Clearly, these are redundant. Now, if we choose the Borel subalgebra $\mathfrak{b}$ of upper triangular matrices, we get the positive basis:

$$\alpha_i : x \rightarrow \lambda_i - \lambda_{i+1}$$

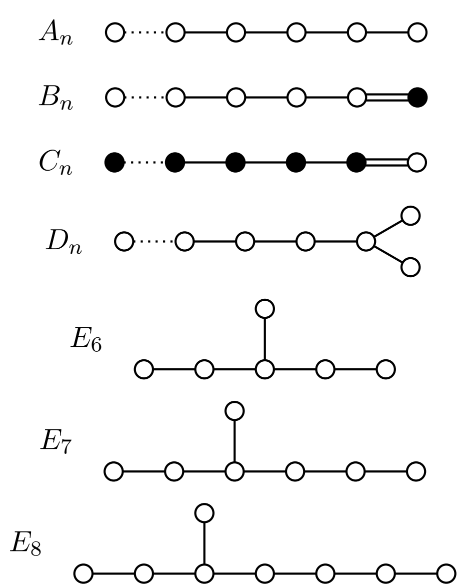
The coweight lattice has a basis h_i given by diagonal matrices with 1 on i^{th} entry and -1 on $(i+1)^{th}$ entry.

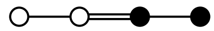
The Cartan matrix has entries:

$$a_{ij} = \alpha_i(h_j) = \begin{cases} 2 & \text{if } \ i = j \\ -1 & \text{if} j = i \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Finally, we end this discussion by mentioning the following important classification theorem:
Theorem 3.19. *For any root system with a positive definite Cartan matrix, there exists a unique up to isomorphism semisimple Lie algebra over $\mathbb{C}$ with the given root system.*

It is an easy linear algebra exercise to show that there are only finitely many isomorphism types of irreducible root systems with positive definite matrices. These are usually displayed via the Dynkin diagrams:



F_4 

G_2 

We next have the following lemma in preparation for later.

Lemma 3.20. *Given two Borel subalgebras $\mathfrak{b}, \mathfrak{b}' \subset \mathfrak{g}$. There exists a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b} \cap \mathfrak{b}'$*

Proof. Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ and $\mathfrak{n}' = [\mathfrak{b}', \mathfrak{b}']$. Let $\mathfrak{p} = \mathfrak{b} \cap \mathfrak{b}'$ and $\mathfrak{c}$ be a complementary subspace to $\mathfrak{b} + \mathfrak{b}'$. Let $\mathfrak{h}$ be Cartan subalgebra of $\mathfrak{b}$ (hence of $\mathfrak{g}$). Let us write $\dim(\mathfrak{h}) = r$, $\dim(\mathfrak{n}) = n$ and $\dim(\mathfrak{p}) = p$.

We have $dim \mathfrak{b} = dim \mathfrak{b}' = r + n$ and $dim \mathfrak{c}^{\perp} = dim(\mathfrak{b} + \mathfrak{b}') = 2(r + n) - p$. On the other hand, $dim(\mathfrak{c}^{\perp} \cap \mathfrak{p}) \geq dim \mathfrak{c}^{\perp} + dim \mathfrak{p} - dim \mathfrak{g} = 2(r + n) - p + p - (r + 2n) = r$.

We also have $\mathfrak{n} = \mathfrak{b}^{\perp}$ and $\mathfrak{n}' = \mathfrak{b}'^{\perp}$, and $\mathfrak{p} \cap \mathfrak{n} \subset \mathfrak{n}'$. Hence, it follows that $\mathfrak{p} \cap \mathfrak{n} \subset \mathfrak{n} \cap \mathfrak{n}' = \mathfrak{b}^{\perp} \cap (\mathfrak{b}')^{\perp}$. Therefore,

$$\mathfrak{c}^{\perp} \cap \mathfrak{p} \cap \mathfrak{n} = \{0\}$$

We deduce that $\mathfrak{b} = \mathfrak{n} \oplus (\mathfrak{c}^{\perp} \cap \mathfrak{p})$ and $\mathfrak{b}' = \mathfrak{n}' \oplus (\mathfrak{c}^{\perp} \cap \mathfrak{p})$. Take $z \in \mathfrak{h} \cap \mathfrak{g}^{rs}$. There exist $y \in \mathfrak{n}$ such that $x = y + z \in \mathfrak{c}^{\perp} \cap \mathfrak{p}$. Hence $ad_{\mathfrak{g}} x$ and $ad_{\mathfrak{g}} z$ have the same characteristic polynomial. so x is generic simultaneously for $\mathfrak{b}$, $\mathfrak{b}'$ and $\mathfrak{g}$. It follows that its centralizer $C_{\mathfrak{g}}(x)$ is a Cartan subalgebra in $\mathfrak{b} \cap \mathfrak{b}'$. □

Now, given two Borel subalgebras $\mathfrak{b}$, $\mathfrak{b}'$, find a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b} \cap \mathfrak{b}'$ which always exists by the above lemma (though, it is not unique in general). Since $\mathfrak{b}$ and $\mathfrak{b}'$ contain the same Cartan subalgebra $\mathfrak{h}$, there is an element $w \in W$ of the Weyl group associated to $\mathfrak{h} \subset \mathfrak{g}$, such that $w \mathfrak{b} w^{-1} = \mathfrak{b}'$. Let us write $w(\mathfrak{b}, \mathfrak{b}')$ for the element of the abstract Weyl group that w maps to under the isomorpshism of W with the abstract Weyl group element induced by

$$\mathfrak{h} \to \mathfrak{b} \to \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$$

Exercise: The element $w(\mathfrak{b}, \mathfrak{b}')$ is independent of the choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b} \cap \mathfrak{b}'$.

We say that two Borel subalgebras $\mathfrak{b}$ and $\mathfrak{b}'$ are in the same relative position $w(\mathfrak{b}, \mathfrak{b}')$.

Proposition 3.21. *Two pairs $(\mathfrak{b}_1, \mathfrak{b}_1')$ and $(\mathfrak{b}_2, \mathfrak{b}_2')$ are in the same relative position w in the abstract Weyl group, if and only if the points $(\mathfrak{b}_1, \mathfrak{b}_1')$ and $(\mathfrak{b}_2, \mathfrak{b}_2') \in \mathcal{B} \times \mathcal{B}$ belong the same G-orbit under the diagonal G action on $\mathcal{B} \times \mathcal{B}$. In other words, there exists a canonical bijeciton:*

$$\{G - orbits\ in\ \mathcal{B} \times \mathcal{B} \cong W\}$$

where W is the abstract Weyl group.

Proof. This is essentially a reformulation of Bruhat decomposition. Choose a Borel subalgebra $\mathfrak{b}$ and let T be the maximal torus corresponding to $\mathfrak{h} \subset \mathfrak{b}$. Let $w \in W_T$, then by definition $\mathfrak{b}$ and $w \mathfrak{b} w^{-1}$ are in relative position $w \in W$. Note also that if $w(\mathfrak{b}_1, \mathfrak{b}_1') = w(g \mathfrak{b}_1 g^{-1}, g \mathfrak{b}_1' g^{-1})$ since if $\mathfrak{b}_1$ and $\mathfrak{b}_1'$ are related by a Cartan element w for the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, then $g \mathfrak{b}_1 g^{-1}$ and $g \mathfrak{b}_1' g^{-1}$ are related by the Cartan element gwg^{-1} for the Cartan subalgebra $g \mathfrak{h} g^{-1} \subset \mathfrak{g}$.

Finally, to prove the claimed bijection, we must show that G-diagonal orbit on $\mathcal{B}\times\mathcal{B}$ contains a single poing $(\mathfrak{b}, w\mathfrak{b}w^{-1})$, $w \in W_T$. Now, recall the Bruhat decomposition gives us the following bijections:

$$W_T \cong B\backslash G/B \cong \{B - \text{orbits on } \mathcal{B}\} \cong \{G - \text{diagonal orbits on } \mathcal{B}\times\mathcal{B}\}$$

□

Example: Let $G = SL_2(\mathbb{C})$ and $W = \{1, s\}$. Recall that in this case, we have $\mathcal{B} = \mathbb{C}P^1$, thus $\mathcal{B} = \mathbb{C}P^1 \times \mathbb{C}P^1$. The two G-orbits are the diagonal in $\mathcal{B}\times\mathcal{B}$ and its complement.

We are now in a position to extend Springer's commutative diagram to the case of general semisimple Lie algebras.

Definition 3.22. *Let* $\tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \in \mathfrak{g}\times\mathcal{B} : x \in \mathfrak{b}\}$*, and let* $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ *and* $p : \tilde{\mathfrak{g}} \to \mathcal{B}$ *be projections.*

The fibre of $\pi : \tilde{\mathfrak{g}} \to \mathcal{B}$ over $\mathfrak{b}$ is the set of elements of $\mathfrak{b}$. Hence π is a vector bundle over $\mathcal{B}$ of rank equal to $\dim(\mathcal{B})$.

One way to understand $\tilde{\mathfrak{g}}$ is to fix a Borel B with Lie algebra $\mathfrak{b}$ and consider the B-action $G\times\mathfrak{b}$ given by $b\cdot(g, x) \to (gb^{-1}, bxb^{-1})$, the orbit space of this action is the associated vector space to the principal B-space G, and the adjoint representation of B on $\mathfrak{b}$, so we write it as $G\times_B \mathfrak{b}$. It is a simple exercise to see:

Proposition 3.23. *The projection* $p : \tilde{\mathfrak{g}} \to \mathcal{B}$ *makes* $\tilde{\mathfrak{g}}$ *a* G*-equivariant vector bundle over* $\mathcal{B} = G/B$ *with fibre* $\mathfrak{b}$*, and there is a* G*-equivariant isomorphism between* $\tilde{\mathfrak{g}}$ *and* $G\times_B\mathfrak{b}$*.*

As for the map $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$, all we observe at the moment is that it is a proper map, as it is the restriction of the first projection $\mathfrak{g}\times\mathcal{B} \to \mathfrak{g}$ which is proper. The fibers $\mu^{-1}(x) = \mathcal{B}_x$ consists of all Borel subalgebras that contain x. When $x = 0$, this is all of $\mathcal{B}$ and when $x \in \mathfrak{g}^{sr}$ then $\mathcal{B}_x$ is discrete and it contains $|W|$ many points.

We next introduce the analogue of $\tilde{\chi} : \tilde{\mathfrak{g}} \to \mathfrak{H}$. This is, by definition:

$$\begin{gathered}\tilde{\mathfrak{g}} \to \mathfrak{H} \\ (x, \mathfrak{b}) \to x + [\mathfrak{b}, \mathfrak{b}] \in \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]\end{gathered}$$

Exercise: Show that for each $x \in \mathfrak{g}^{sr}$, there is a canonical free W-action on $\mu^{-1}(x)$ making the projection $\tilde{\mathfrak{g}}^{sr} \to \mathfrak{g}^{sr}$ a principal W-bundle. (Here is how to define the W-action: Given $x \in \mathfrak{g}^{sr}$, let $\mathfrak{b} \in \mu^{-1}(x)$. The centralizer $C_{\mathfrak{g}}(x)$ is a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Given $w \in W$, let $w(\mathfrak{b})$ be the unique Borel subalgebra containing $\mathfrak{h}$ that is relative position w with respect to $\mathfrak{b}$.)

We will next discuss Chevalley Restriction Theorem, which says:

Theorem 3.24. *For any Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *the restriction map gives a canonical graded algebra isomorphism:*

$$\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{h}]^W$$

.

Proof. Observe that there is an obvious restriction map $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^W$. Let us first see that this map is an injective map. Indeed, recall that any $x \in \mathfrak{g}^{sr}$ is G-conjugate to an element of $\mathfrak{h}$. If $P \in \mathbb{C}[\mathfrak{g}]^G$ such that its restriction to $\mathfrak{h}$ is identically zero, it follows that P is identically zero on $\mathfrak{g}^{sr}$. But, $\mathfrak{g}^{sr} \subset \mathfrak{g}$ is dense, therefore P has to be identically zero.

Next, we will prove surjectivity. Let $P \in \mathbb{C}[\mathfrak{h}]$ which is W-invariant, we need to find a G-invariant polynomial R in $\mathbb{C}[\mathfrak{g}]$ whose restriction to $\mathfrak{h}$ is P. Since the isomorphism $\mathfrak{h} \to \mathfrak{b}/[\mathfrak{b},\mathfrak{b}] = \mathfrak{H}$ is a canonical isomorphism that intertwines the W-actions, we can view P as a W-invariant polynomial on $\mathfrak{H}$.

Now, let $\tilde{P}$ be the composition $P \circ \tilde{\chi}$. It is clear from the description of $\tilde{\mathfrak{g}}$ as $G \times_B \mathfrak{b}$ that $\tilde{P}$ is a G-invariant polynomial.

Next, restrict $\tilde{P}$ to the open dense set $\tilde{\mathfrak{g}^{sr}} = \mu^{-1}(\mathfrak{g}^{sr})$. Note that $\tilde{\chi} : \tilde{\mathfrak{g}^{sr}} \to \mathfrak{h}$ commutes with the W-actions and since P is W-invariant, it follows that $\tilde{P}$ is W-invariant.

Now, we have seen that $\mu : \tilde{\mathfrak{g}}^{sr} \to \mathfrak{g}^{sr}$ is a Galois covering. Therefore, by a general result about Galois coverings, μ^* gives an identification at the level of function fields:

$$\mathbb{C}(\tilde{\mathfrak{g}}^{sr})^W \cong \mathbb{C}(\mathfrak{g}^{sr})$$

Hence, we obtain a rational function $R \in \mathbb{C}(\mathfrak{g}^{sr})$ as the image of P under this identification, such that $\tilde{P} = R \circ \mu$.

Finally, we note that since $\tilde{P}$ is regular and μ is proper, it follows that in fact R is a polynomial (has no poles).

□

Remark 3.25. *Another theorem of Chevalley says that $\mathbb{C}[\mathfrak{h}]^W$ is a free polynomial algebra. Hence the algebraic variety Spec$(\mathbb{C}[\mathfrak{h}]^W) = \mathfrak{h}/W$ is isomorphic to a vector space of dimension equal to* rk$\mathfrak{g}$ *as an algebraic variety.*

Given a pair $h \subset \mathfrak{b}$, consider the diagram:

$$\mathfrak{g} \leftarrow \mathfrak{h} \to \mathfrak{b} \to \mathfrak{b}/[\mathfrak{b},\mathfrak{b}] = \mathfrak{H}$$

These induce isomorphisms $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^W \leftarrow \mathbb{C}[\mathfrak{H}]^W$. Furthermore, one can chekc that the isomorphism:

$$\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{H}]^W$$

does not depend on the choice of $(\mathfrak{h}, \mathfrak{b})$. Therefore, we get a canonical $\mathbb{C}$-algebra embedding:

$$\mathbb{C}[\mathfrak{H}]^W \to \mathbb{C}[\mathfrak{g}]$$

and we write

$$\chi : \mathfrak{g} \to \mathfrak{H}/W$$

for the corresponding map. Therefore, we have finally completed the description of all the maps in the commutatitive diagram:

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \xrightarrow{\tilde{\chi}} & \mathfrak{H} \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathfrak{g} & \xrightarrow[\chi]{} & \mathfrak{H}/W \end{array}$$

Once again, I would like to emphasize that the maps or spaces in this diagram are canonical, does not depend on the choice of a $\mathfrak{h}$ or a $\mathfrak{b}$.

Exercise: Check commutative of the diagram to make sure you understand all the maps.

Corollary 3.26. *Let $\mathfrak{b}$ be a Borel subalgebra with nilradical $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$. Then, for any G-invariant polynomial P on $\mathfrak{g}$ and any $x \in \mathfrak{b}$ the restriction $P_{|x+\mathfrak{n}}$ is constant.*

Proof. Let $x \in \mathfrak{b}$ and $y = x + n$ for some $n \in \mathfrak{n}$. Observe that $\tilde{x} = (x, \mathfrak{b})$ and $\tilde{y} = (y, \mathfrak{b})$ map to the same element in $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ under $\tilde{chi}$ hence, the commutativity of the diagram above implies the result. □

Another interpretation of this result can be given in view of the following:

Proposition 3.27. *Let $x \in \mathfrak{b}$ be a semisimple regular element. Then $x + \mathfrak{n} = B \cdot x$ is a single B-orbit.*

Proof. First, observe that the affine space $x + \mathfrak{n}$ is B-stable. Indeed, since B is connected, it suffices to show that the tangent space $\mathfrak{n}$ is invariant under the linearization of the B-action. This means, we need to check:

$$[\mathfrak{b}, \mathfrak{n}] \subset \mathfrak{n} \quad \text{and} \quad [\mathfrak{b}, x] \subset \mathfrak{n}$$

The first is clear, since $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is an ideal. Furthermore, since $\mathfrak{b}/\mathfrak{n}$ is an abelian Lie algebra, for any $x \in \mathfrak{b}$, $[\mathfrak{b}, x] = 0$ modulo $\mathfrak{n}$, hence $[\mathfrak{b}, x] \in \mathfrak{n}$.

Next, let $U \subset B$ be the unipotent radical. Since $x + \mathfrak{n}$ is B-stable, it is also U-stable. Now, note that $[\mathfrak{n}, x] = \mathfrak{n}$ since x is regular semisimple (which implies $C_{\mathfrak{g}}(x)$ is a Cartan subalgebra, hence $C_{\mathfrak{g}}(x) \cap \mathfrak{n} = \{0\}$. Therefore, the linearization of the action of U on $x + \mathfrak{n}$ is a submersion. Then, the implicit function theorem implies that the U-orbit of x is open in $x + \mathfrak{n}$. On the other hand, any orbit of an action of a unipotent group on an affine variety is closed, thus it follows that $U \cdot x = x + \mathfrak{n}$, hence $B \cdot x = U \cdot x = x + \mathfrak{n}$. □

The previous corollary can be obtained from this proposition. Namely, if P is a G-invariant polynomial on $\mathfrak{g}$. Then P is constant on any B-orbit. In particular, it is constant on $x + \mathfrak{n}$ where x is regular semisimple. By continuity, it is constant on $x + \mathfrak{n}$ for any $x \in b$ since regular semisimple elements are dense in $\mathfrak{b}$.

3.4 Nilpotent cone

Recall from the introduction that we have $\mathcal{N} := \{x \in \mathfrak{g} : x \text{ nilpotent } \}$. Using the projection $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$, we also define;

$$\tilde{\mathcal{N}} := \mu^{-1}(\mathcal{N}) = \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} | x \in \mathfrak{b}\}$$

Now, if $\mathfrak{b} \in \mathcal{B}$ and $x \in \mathfrak{b}$ such that $ad(x)$ is nilpotent, that means that x has no Cartan component in a decomposition $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ where $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is the nilradical. Hence, an element of $\mathfrak{b}$ is nilpotent if and only if it belongs to $\mathfrak{n}$. Thus $\tilde{\mathcal{N}}$ is a vector bundle over $\mathcal{B}$ with fibre over $\mathfrak{b}$ given by $[\mathfrak{b}, \mathfrak{b}]$. Furthermore, since any two Borel subalgebras (hence their nilradicals) are G-conjugate, if we fix a Borel subgroup B with $LieB = \mathfrak{b}$, then we get a G-equivariant vector bundle isomorphism:

$$\begin{aligned} G \times_B \mathfrak{n} &\to \tilde{\mathcal{N}} \\ (g, x) &\to (Ad(g) \cdot x, Ad(g) \cdot \mathfrak{b}) \end{aligned}$$

In particular, observe that $\tilde{\mathcal{N}}$ is a smooth variety, while $\mathcal{N}$ is always singular at the origin.

In fact, as we saw before, under the isomorphism $\mathfrak{g}^* \cong \mathfrak{g}$ given by an invariant pairing, we have that $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ gets identified with $\mathfrak{b}^\perp$. Moreover, $G \times_B \mathfrak{b}^\perp$ is G-equivariantly isomorphic to the cotangtent bundle $T^*(G/B)$. Furthemore, we have also seen that the G-action on $T^*(G/B)$ is always Hamiltonian, and the moment map is given by:

$$\begin{aligned} \mu :\cong G \times_B \mathfrak{b}^\perp &\to \mathfrak{g}^* \\ (g, z) &\to Ad^*(g) \cdot z \end{aligned}$$

or equivalently using the identifications given by $\mathfrak{g}^* \cong \mathfrak{g}$, we can view this as a map

$$\begin{aligned} \mu : G \times_B \mathfrak{n} &\to \mathfrak{g} \\ (g, x) &\to Ad(g) \cdot x. \end{aligned}$$

Hence, innded, the moment map takes values on $\mathcal{N}$ and as a map

$$\mu : \tilde{\mathcal{N}} \to \mathcal{N}$$

is just the restriction of $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ to $\tilde{\mathcal{N}}$.

Definition 3.28. *The map $\mu : T^*\mathcal{B} \to \mathcal{N}$ is called the Springer resolution.*

We note that the map μ is surjective, since any nilpotent element of $\mathfrak{g}$ is known to be contained in a nilradical $\mathfrak{n}$ of some $\mathfrak{b}$. Thus, we now have constructed the diagram that was mentioned in the introduction:

$$\begin{array}{ccccc} \tilde{\mathcal{N}} & \hookrightarrow & \tilde{\mathfrak{g}} & \xrightarrow{\tilde{\chi}} & \mathfrak{H} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \pi} \\ \mathcal{N} & \hookrightarrow & \mathfrak{g} & \xrightarrow[\chi]{} & \mathfrak{H}/W \end{array}$$

Let us write $\mathbb{C}[\mathfrak{g}]_+^G$ to be the G-invariant polynomials on $\mathfrak{g}$ without constant term. The following result of Kostant gives intrinsic definition of what it means to be for an element in $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) is nilpotent without referring to the adjoint representation (resp. to an invariant pairing).

Proposition 3.29. *An element $x \in \mathfrak{g}$ is nilpotent if and only if for every $P \in \mathbb{C}[\mathfrak{g}]_+^G$, we have $P(x) = 0$. In other words, $\mathcal{N} = \chi^{-1}(0)$.*

Proof. Since we have $\mathbb{C}[\mathfrak{H}]^W \cong \mathbb{C}[\mathfrak{g}]^G$, proving the isomorphism amounts to showing that $\mathcal{N} = \chi^{-1}(0)$. Now, let $x \in \mathcal{N}$, and choose a point $(x, \mathfrak{b}) \in \mu^{-1}(x)$. Since x is nilpotent, we have $x \in [\mathfrak{b}, \mathfrak{b}]$, it follows, by definition, that $\tilde{\chi}(x, \mathfrak{b}) = 0 \in \mathfrak{H}$. By the commutativity of the diagram, we conclude that $\chi(x) = 0$.

Conversely, if $\chi(x) = 0$, then choose $(x, \mathfrak{b}) \in \mu^{-1}(x)$. We have, $\pi(\tilde{\chi})(x, \mathfrak{b}) = \chi(x) = 0$. Hence, $(x, \mathfrak{b}) \in \tilde{\chi}^{-1}(0)$ (since 0 is a fixed point of the W action on $\mathfrak{H}$), but that means $x \in \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. □

Corollary 3.30. *$\mathcal{N}$ is an irreducible variety of dimention $2dim\ \mathfrak{n}$.*

Proof. Since $T^*\mathcal{B}$ is smooth and connected (hence irreducible), and map $\mu : T^*\mathcal{B} \to \mathcal{N}$ is surjetive, the irreducibility of $\mathcal{N}$ is clear. Furthermore, we have the inequality:

$$\dim\mathcal{N} \leq \dim T^*\mathcal{B} = 2\dim\ \mathfrak{n}$$

On the other hand, $\mathcal{N}$ is the zero-fiber of the map $\chi : \mathfrak{g} \to \mathfrak{h}/W$. Therefore, its dimension satisfies the inequality:

$$\dim\mathcal{N} \geq \dim\ \mathfrak{g} - \dim\ \mathfrak{h} = 2\dim\ \mathfrak{n}$$

Hence, we conclude that $\dim\mathcal{N} = 2\dim\ \mathfrak{n}$.

□

We next state the following proposition without proof (we will give the proof later on over char 0):

Proposition 3.31. *There are only finitely many nilpotent orbits in $\mathfrak{g}$.*

Note that in the case $g = \mathfrak{sl}_n$, this is just the Jordan normal form theorem.

Proposition 3.32. *The regular nilpotent elements form a single Zariski-dense orbit in $\mathcal{N}$.*

Proof. Since $\mathcal{N}$ is irreducible and there are only finitely many orbits, it contains a unique open-dense orbit $\mathbb{O}$. Then, $\dim\mathcal{N} = \dim G - \dim C_G(x)$ for $x \in \mathbb{O}$. Hence, it follows that $\dim C_G(x) = \mathrm{rk}\mathfrak{g}$, which means, by definition, that x is regular. □

We next want to show that $\mu : T^*(\mathcal{B}) \to \mathcal{N}$ is 1-1 over the orbit $\mathbb{O}$ of regular nilpotent elements. We will need a little bit of preparation for this.

Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is a Borel subalgebra, and $\alpha_1, \ldots, \alpha_r$ be the set of positive simple roots and let $\bar{\alpha}_i \in \bar{\mathfrak{n}} = \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ be their projection to the Cartan subalgebra $\mathfrak{h}$. We define:

$$\mathfrak{n}^{reg} := \{x \in \mathfrak{n} : \bar{x} = \sum a_i \bar{\alpha}_i \text{ for } \ a_i \in \mathbb{C}^*\}$$

Clearly $\mathfrak{n}^{reg}$ is a Zariski dense open subset of $\mathfrak{n}$.

Proposition 3.33. $\mathfrak{n}^{reg}$ *is a single* B*-orbit consisting of regular nilpotent elements in* $\mathfrak{g}$*.*

Proof. Let $x \in \mathfrak{n}$ be a regular nilpotent element. Since $x \in \mathfrak{n}$, we have $[x, \mathfrak{n}] \subset [\mathfrak{n}, \mathfrak{n}]$ hence $x + [\mathfrak{n}, \mathfrak{n}]$ is stable under the adjoint U-action (as U is connected; this argument is as before), where U is the unipotent subgroup of G corresponding to $\mathfrak{n}$. Since x is also regular, we have

$$Ad(U) \cdot x \geq \dim \mathfrak{n} - \text{rk } \mathfrak{g} = \dim [\mathfrak{n}, \mathfrak{n}]$$

Hence, the U-orbit of x is open in $x + [\mathfrak{n}, \mathfrak{n}]$ but U is a unipotent group acting on the affine space $x + [\mathfrak{n}, \mathfrak{n}]$ hence its orbits are closed. Therefore, we conclude that $x + [\mathfrak{n}, \mathfrak{n}]$ is a single U-orbit.

Now, consider the projection $\mathfrak{n} \to \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$, this is a T-equivariant map with respect to the natural adjoint action T-actions. Furthermore, the image of $\mathfrak{n}^{reg}$ under this quotient is dense. Therefore, in particular $\mathfrak{n}^{reg}$ contains a regular nilpotent element $x \in \mathfrak{n}$ and hence $x + [\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}^{reg}$. As we have seen $x + [\mathfrak{n}, \mathfrak{n}]$ is a single U-orbit. Furthermore, any other element of $\mathfrak{n}^{reg}$ is obtaines by $T \cdot (x + [\mathfrak{n}, \mathfrak{n}])$, so it follows that $\mathfrak{n}^{reg} = B \cdot x$. □

Corollary 3.34. *The element* $n = \alpha_1 + \alpha_2 + \ldots + \alpha_r$ *is a regular nilpotent element in* $\mathfrak{g}$*.*

Proof. Since $n \in n^{reg}$ it follows from the above proposition that n is regular. □

Proposition 3.35. *Any regular nilpotent element is contained in a unique Borel subalgebra.*

Proof. We have seen that $\dim \mathcal{N} = 2\dim \mathfrak{n}$ and since we have $\dim \tilde{\mathcal{N}} = \dim T^*\mathcal{B} = 2\dim \mathfrak{n}$, the map $\mu : T^*\mathcal{B} \to \mathcal{N}$ is a surjective map of irreducible varieties of the same dimension. Therefore, it follows that generic fibres are 0-dimensional.

Now, from the previous proposition, all regular nilpotents form a single dense conjugacy class in $\mathcal{N}$. Since $\mu : T^*\mathcal{B} \to \mathcal{N}$ is a G-equivariant map, it suffices to prove the result for a single regular nilpotent element. Note that in particular, we know that the fibre of μ over a regular nilpotent element has to be discrete but we are aiming to show that the fibre consists of a unique element.

By picking a $\mathfrak{b}$ and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, we can find a regular nilpotent element $n = \alpha_1 + \ldots + \alpha_r$ as above. Thus, it suffices to show that $\mu^{-1}(n) = \mathcal{B}_n$ is a unique element. Let $h \in \mathfrak{h}$ be a regular semisimple element such that $-ad(h) \cdot n = [h, n] = n$. Note that this is easy to arrange, since n is the sum of simple roots α_i so there is a basis $\{h_i\}$ of the coweight lattice in $\mathfrak{h}$ such that $[h_i, \alpha_j] = \delta_{ij}$. Now, the equation $[h, n] = n$ implies that the Springer fibre $\mathcal{B}_n \subset \mathcal{B}$ of Borel subalgebras containing n is h-stable. Consider the corresponding $\mathbb{C}^*$ action on $\mathcal{B}$ induced by $\mathfrak{h}$. We then have the Bialynicki-Birula decomposition with respect to this $\mathbb{C}^*$-action as in the proof of the Bruhat decomposition. The fixed points of this $\mathbb{C}^*$ action are in correspondence with the Weyl group elements, so there are $|W|$ Borel subalgebras $\mathfrak{b} \subset \mathcal{B}$ that are fixed. Namely, the Borel algebras that contain h. Among these only one of them contains n. This follows because, any borel subalgebra containing h contains the Cartan subalgebra $C_{\mathfrak{g}}(h)$, if two borel subalgebra contain $C_{\mathfrak{g}}(h)$, they are related by the action of the Weyl group $N_G(T)/T$. But the Weyl group reflects the simple roots, hence, the stabilizer of the element n under the action of the Weyl group is just the identity element.

□

It follows from this proposition that $\mu : T^*(\mathcal{B}) \to \mathcal{N}$ is an isomorphism over the Zariski open part of $\mathcal{N}$ formed by regular nilpotent elements. Thus μ is a resolution of singularities of $\mathcal{N}$, since $T^*(\mathcal{B})$ is a smooth variety.

3.5 Steinberg variety

We will now discuss an important variety in geometric representation theory. This will give rise to "correspondences" when we later construct representations of the Weyl group.

Recall that we have the Springer resolution $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$. We write:

$$Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} = \{(x, \mathfrak{b}), (x', \mathfrak{b}) \in \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} : x = x'\}$$

or equivalently, we have an isomorphism:

$$Z = \{(x, \mathfrak{b}, \mathfrak{b}') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} : x \in \mathfrak{b} \cap \mathfrak{b}'\}$$

Therefore, there is a natural map $i : Z \to \mathcal{B} \times \mathcal{B}$. Now, recall that G-diagonal orbits on $\mathcal{B} \times \mathcal{B}$ are canonically parametrized by the elements of the abstract Weyl group. We have:

$$\mathcal{B} \times \mathcal{B} = \bigsqcup_{w \in W} Y_w = \bigsqcup_{w \in W} G \cdot (B, wB)$$

where we have written Y_w for the orbit corresponding to the element w in the abstract Weyl group.

Let $Z_w := i^{-1}(Y_w) \subset Z \subset T^*(\mathcal{B}) \times T^*(\mathcal{B})$. Now, we have an isomorphism:

$$T^*(\mathcal{B}) \times T^*(\mathcal{B}) \cong T^*(\mathcal{B} \times \mathcal{B})$$

given by

$$((x_1, \mathfrak{b}_1), (x_2, \mathfrak{b}_2)) \to ((x_1, -x_2), (\mathfrak{b}_1, \mathfrak{b}_2))$$

where we have inserted a sign so that the standard symplectic form on $T^*(\mathcal{B} \times \mathcal{B})$ corresponds under the isomorphism to $p_1^*\omega_1 - p_2^*\omega_2$, where ω_1 and ω_2 are the standard symplectic forms on the first and second factor of $T^*\mathcal{B} \times T^*\mathcal{B}$.

We then have the following proposition:

Proposition 3.36. *The Steinberg variety Z is the union of the conormal bundles to all G-orbits in $\mathcal{B} \times \mathcal{B}$. Indeed, Z_w is the cornomal bundle to Y_w, and we have $Z = \sqcup_w T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$. Furthermore, every irreducible component of Z is the closure of $T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$.*

Proof. Recall that we have the identification $T^*\mathcal{B} = G \times_B \mathfrak{b}^\perp$. Hence, an element of $T^*\mathcal{B}$ can be written as $(x, \mathfrak{b})$ such that $x \in \mathfrak{b}^\perp$.

The fibre of the the conormal bundle to Y_w at a point $\alpha = (\mathfrak{b}_1, \mathfrak{b}_2) \in \mathcal{B} \times \mathcal{B}$ consists of

$$((x_1, \mathfrak{b}_1), (x_2, \mathfrak{b}_2)) \in \mathfrak{g}^* \times \mathcal{B} \times \mathfrak{g}^* \times \mathcal{B}$$

such that $x_1 \in \mathfrak{b}_1^\perp$ and $x_2 \in \mathfrak{b}_2^\perp$ and in addition (x_1, x_2) annihilates the tangent space $T_\alpha Y_w \cong \mathfrak{g}/\mathfrak{b}_1 \times \mathfrak{g}/\mathfrak{b}_2$. But, this latter condition is equivalent to

$$\langle x_1, u\rangle + \langle x_2, u\rangle = 0 \text{ for all } \quad u \in \mathfrak{g}$$

which implies $x_1 + x_2 = 0$. Hence the union of conormal bundles to Y_w get identifies with Z. Now, the closures of $T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$ are irreducible because the conormal bundles themselves are irreducible, Furthermore, they are of the same dimension since they are Lagrangians in $T^*(\mathcal{B} \times \mathcal{B})$. $\square$

The following is the key theorem of this section. Fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ be the nilradical.

Theorem 3.37. *Let $\mathbb{O}$ be a coadjoint orbit in $\mathfrak{g}^*$. Let $x \in \mathbb{O}$ be such that $x_{|\mathfrak{n}} = 0$. Then $\mathbb{O} \cap (x + \mathfrak{b}^\perp)$ is a (possibly singular) Lagrangian subvariety in $\mathbb{O}$ with respect to the natural symplectic structure on coadjoint orbits.*

Note that under an identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ via an invariant form, we have $b^\perp = \mathfrak{n} \subset \mathfrak{g}$. The theorem can then be stated as:

For any adjoint orbit $\mathbb{O} \subset \mathfrak{g}$ and any $x \in \mathbb{O} \cap \mathfrak{b}$, the set $\mathbb{O} \cap (x + \mathfrak{n})$ is a Lagrangian subvariety in $\mathbb{O}$ when it is equipped with the symplectic form coming from the identification with a coadjoint orbit.

We will prove the harder (and interesting) case, which corresponds to when $x \in \mathfrak{n}$, and leave the rest as an exercise for the reader.

Proof. Let $p : Z \to \mathcal{N}$ be the projection map. Given an adjoint orbit $\mathbb{O} \subset \mathcal{N}$, let us put $Z_o = \mu^{-1}(\mathbb{O})$. Thus, Z_o is the set of all triples $(x, \mathfrak{b}, \mathfrak{b}') \in Z$ such that $x \in \mathbb{O}$.

Claim: $\dim(\mathbb{O} \cap \mathfrak{n}) \leq 1/2 \dim \mathbb{O}$.

Let $n = \dim \mathfrak{n} = \dim \mathcal{B}$. Then we have $\dim T^*\mathcal{B} = \dim Z = 2n$.

We have a fibration $Z_o \to \mathbb{O}$ with fiber over $x \in \mathbb{O}$ given by $\mathcal{B}_x \times \mathcal{B}_x$. Note that this fibre is in general singular. Therefore, we have:

$$\dim \mathbb{O} + 2\dim \mathcal{B}_x \leq \dim Z_o$$

Since $\dim Z_o \leq \dim Z = 2n$, we conclude that:

$$\frac{1}{2}\dim \mathbb{O} + \dim \mathcal{B}_x \leq n$$

Now, given $x \in \mathbb{O}$ and $x \in \mathfrak{n} \subset \mathfrak{b}$, we let $S = \{g \in G : gxg^{-1} \in \mathfrak{b}\} = \{g \in G | g\mathfrak{b}g^{-1} \in \mathcal{B}_x\}$. Clearly, S is stable under the multiplication by B on the left and the map $Bg \to g^{-1}\mathfrak{b}g$ gives an isomorphism:

$$B \backslash S \cong \mathcal{B}_x$$

Therefore, we get the inequality:

$$\text{dim}S - \text{dim}B + \frac{1}{2}\text{dim}\mathbb{O} \leq n$$

On the other hand, $\mathbb{O} \cap \mathfrak{n}$ is equal to the set of all gxg^{-1} such that $g \in S$. Therefore, we have the isomorphism:

$$S/C_G(x) \cong \mathbb{O} \cap n$$

given by $g \cdot C_G(x) \to gxg^{-1}$, which, together with the above inequality, gives:

$$\text{dim}(\mathbb{O} \cap n) + \frac{1}{2}\text{dim}(\mathbb{O}) \leq n + \text{dim}B - \text{dim}C_G(x)$$

But $n + \text{dim}B - \text{dim}C_G(x) = \text{dim}G - \text{dim}C_G(x) = \text{dim}\mathbb{O}$, hence we get:

$$\text{dim}(\mathbb{O} \cap n) \leq \frac{1}{2}\text{dim}(\mathbb{O})$$

as required.

We will next show that $\mathbb{O} \cap \mathfrak{n}$ is a coisotropic subvariety in $\mathbb{O}$ and this will then imply the result.

To that end, view $\mathbb{O}$ as a symplectic manfiold with a Hamiltonian B-action. In particular, we have a moment map:

$$\mu : \mathbb{O} \to \mathfrak{b}^*$$

But this factors through $\mathbb{O} \to \mathfrak{g}^* \to \mathfrak{b}^*$, where the first map is the moment map obtianed by considering $\mathbb{O}$ as a Hamiltonian G-variety and the second map is the dual of the inclusion $\mathfrak{b} \to \mathfrak{g}$. Let us write this as $\iota : \mathfrak{g}^* \to \mathfrak{b}^*$.

Now, $\iota^*(0) = \mathfrak{b}^\perp = \mathfrak{n}$, hence we conclude that $\mu^{-1}(0) = \mathbb{O} \cap \mathfrak{n}$. Now 0 is a coadjoint B-orbit in $\mathfrak{b}^*$, hence by the main theoem of the first part of the course, we conclude that $\mu^{-1}(0)$ is coisotropic.

□

Example 3.38. *Let $G = SL_n(\mathbb{C})$ and $\mathbb{O} \subset \mathfrak{sl}_n$ the variety of rank 1 nilpotent matrices. Here is a concrete way of describing this variety. Let $v \in V \cong \mathbb{C}^n$ and $w \in V^*$, then consider the rank 1 map $x : \mathbb{C}^n \to \mathbb{C}^n$ given by:*

$$u \to (v \otimes w)(u) = w(u) \cdot v$$

Clearly, any rank 1 linear map can be written in this way. The nilpotency condition on x amounts to $w(v) = trace(x) = 0$. Thus, we have a surjection from the set

$$\{v \otimes w, v \in V, w \in V^*, w(v) = 0\}$$

to the variety $\mathbb{O}$. This surjection is not injective since for any $\lambda \in \mathbb{C}^*$, we have that $\lambda v \otimes \frac{1}{\lambda}w$ gives rise to the same rank 1 map as $v \otimes w$. But this is all the ambiguity, in particular:

$$dim\mathbb{O} = (2n-1) - 1 = 2n - 2$$

In coordinates, we cna write

$$\mathbb{O} = \{x = (a_{ij}) | a_{ij} = \alpha_i \beta_j, \sum \alpha_i \beta_i = 0$$

Now let $\mathfrak{n}$ be the Lie algebra of upper triangular nilpotent matrices. Consider $\mathbb{O} \cap \mathfrak{n}$. This has $n-1$ irreducible components. Indeed, in order for elements of $\mathbb{O}$ to be in $\mathfrak{n}$, one must have:

$$\begin{aligned} \alpha &= (\alpha_1, \alpha_2, \dots, \alpha_k,\ 0\ ,\ 0\ ,\ 0\ ,\ \dots\ 0) \\ \beta &= (0\ ,\ 0,\ \dots\ ,\ 0, \beta_{k+1}, \beta_{k+2}, \dots, \beta_n) \end{aligned}$$

The different irreducible components correspond to different values of $k = 1, \dots n-1$.

Notice that the dimension of each of these components is equal to $n-1$ which is half of dim($\mathbb{O}$ as it should be for Lagrangian.

We now consider the opposite case where we assume that $\mathbb{O}$ is the adjoint G orbit of a semisimpler regular elemetn $x \in \mathfrak{g}$. Let $\mathfrak{b}$ be a Borel subalgebra containing x and $\mathfrak{n}$ be its nilradical and B the Borel subgroup with Lie(B) $= \mathfrak{b}$. The Theorem 3.37 claim that $(x + \mathfrak{n}) \cap \mathbb{O}$ is a Lagrangian subvariety in $\mathbb{O}$. Indeed, we have seen before that in Prop. 3.27 that :

$$x + \mathfrak{n} = B \cdot x \subset \mathbb{O}$$

is a single B-orbit. Indeed, we can compute the dimension of $x + \mathfrak{n}$ as:

$$\dim(x + \mathfrak{n}) = \dim(\mathfrak{n}) = 1/2 \dim(G/T) = \frac{1}{2} \dim \mathbb{O}$$

So indeed $x + \mathfrak{n}$ is a B-stable half-dimensional variety of $\mathbb{O}$. Since we can also express it as the preimage of the corresponding coadjoint B-orbit in $\mathfrak{b}^*$, we conclude indeed that

Lemma 3.39. *For a regular semisimple element $x \in \mathfrak{b}$, the affine linear space $x + \mathfrak{n}$ is a B-stable Lagrangrian subvariety of adjoint orbit $\mathbb{O}$ through x.*

Let us consider the example of $\mathfrak{sl}_2(\mathbb{C})$ as usual.

Example 3.40. *As we have seen before, semisimple adjoint orbits are the quadrics determined by the equations:*

$$a^2 + bc = \lambda \quad \text{for } \lambda \neq 0$$

As is well known smooth quadrics have two rulings. We can see them here. Indeed let $x \in \mathfrak{sl}_2(\mathbb{C})$ be a semi-simple regular element, let $\mathfrak{h} = C_{\mathfrak{g}}(x)$ be the Cartan subalgebra containing x. Now, any Borel subalgebra containing x, contains $\mathfrak{h}$ and they are related by the action of the Weyl group. In our case, we get two Borel subalgebras $\mathfrak{b}$ and $\mathfrak{b}^{op}$ containing x. Then, the adjoint orbit through x contains the Lagrangians $x + \mathfrak{n}$ and $x + \mathfrak{n}^{op}$. The two rulings of the adjoint orbit through x can be given as the image under μ of $G \times_B (x + n)$ and $G \times_{B^{op}} (x + \mathfrak{n}^{op})$.

Let us generalize and at the same time explain the previous example in more detail. Let x be a regular semisimple element in $\mathfrak{g}$. The affine space $x + \mathfrak{n}$ is determined by the choice of an element $\tilde{x} = (x, \mathfrak{b})$ in the fibre $\mu^{-1}(x)$. Since x is regular semismiple, there are $\#W$ different Lagrangian affine linear spaces going through each point of the adjoint orbit $\mathbb{O}$ through x. In fact, each of these choices can be made compatibly in a way to exhibit $\mathbb{O}$ as a Lagrangian fibration in $\#W$ different ways. Indeed, let $\tilde{\mathbb{O}} = \mu^{-1}(\mathbb{O})$. Then $\mu : \tilde{\mathbb{O}} \to \mathbb{O}$ is a finite cover. On the other hande, $\mathbb{O}$ can be identified with G/T which has trivial fundamental group. Indeed, we may assume that G is simply connected as passing to a simply connected cover does not change its Lie algebra and the conjugacy classes in $\mathfrak{g}$. Then consider the long exact sequence of the fibration $T \to G \to G/T$. This gives:

$$\pi_1(G) \to \pi_1(G/T) \to \pi_0(T) \to \pi_0(G) \to \pi_0(G/T) \to 1$$

It follows that $\pi_1(G/T) = 0$.

Therefore, $\tilde{\mathbb{O}}$ is a disjoint union of $\#W$ connected components, each isomorphic to $\mathbb{O}$ by μ. On the other hand recall that we had the rprojection $p : \tilde{\mathfrak{g}} \to \mathbb{B}$ was a vector bundle which was identified with the orbit space $G \times_B \mathfrak{b}$ after choosing a Borel. Restriction of p to $\tilde{\mathbb{O}}$ makes each connected component of $\tilde{\mathbb{O}}$ into a G-equivariant fibration over $\mathbb{B}$ with fiber $x + \mathfrak{n}$ and the fibration takes the form

$$p : G \times_B (x + \mathfrak{n}) \to G/B \cong \mathbb{B}$$

These different fibrations can be transferred to $\mathbb{O}$ via the isomorphism μ.

We next turn our attention to nilpotent cone. Recall that the moment map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ and the Steinberg variety $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$. Given an adjoint orbit $\mathbb{O} \subset \mathcal{N}$, we let $Z_o = \mu^{-1}(\mathbb{O})$, the set of all triples $(x, \mathfrak{b}, \mathfrak{b}')$ such that $x \in \mathfrak{b} \cap \mathfrak{b}'$ and $x \in \mathbb{O}$.

Proposition 3.41. *For any orbit $\mathbb{O} \subset \mathcal{N}$, each irreducible component of Z_o has the same dimension equal to:*

$$2dim(G/B) = 2dim\mathfrak{n}$$

Proof. Indeed, letting $\tilde{\mathbb{O}} = \mu^{-1}(\mathbb{O})$, we have $Z_o = \tilde{\mathbb{O}} \times_{\mathbb{O}} \tilde{\mathbb{O}}$. The projection $p : \tilde{N} = G \times_B \mathfrak{n} \to G/B$ restricted to $\tilde{\mathbb{O}}$ gives an isomoprhism $\tilde{\mathbb{O}} = G \times_B (\mathbb{O} \cap \mathfrak{n})$ which fibers over G/B with fibre $\mathbb{O} \cap \mathfrak{n}$. But, we have seen that $\mathbb{O} \cap \mathfrak{n}$ is a Lagrangian subvariety of $\mathbb{O}$, in particular, each of its irreducible components have dimension equal to $\dim(\mathbb{O})/2$. Hence, Z_o has dimension equal to :

$$2\dim(\tilde{\mathbb{O}}) - \dim(\mathbb{O}) = 2\dim(G/B)$$

□

We have arrived at an interesting situation. Indeed, we have

$$Z = \sqcup_{\mathbb{O}} Z_o$$

where the union is over all nilpotent conjugacy classes. Furthermore, each Z_o is locally closed of pure dimension equal to $2\dim\mathfrak{n}$. Therefore, the closure of an irreducible component of Z_o is an irreducible component of Z. But, we have seen before that the irreducible components of Z are given by closures of the conormal bundles $T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$ where for $w \in W$, Y_w is an orbit of diagonal action of G on $\mathcal{B} \times \mathcal{B}$. Therefore, we conclude:

Corollary 3.42. *The number of nilpotent conjugacy classes in $\mathfrak{g}$ is finite.*

Finally, we have the following:

Corollary 3.43. *All irreducible compnents of a Springer fibre $\mathcal{B}_x$ have the same dimension, $dim\mathcal{B}_x$, and*

$$\frac{1}{2}dim\mathbb{O} + dim\mathcal{B}_x = dim\mathcal{B}$$

Proof. For $x \in \mathbb{O}$, we can identify the adjoint orbit through x with $G/C_G(x)$ where $C_G(x)$ is the centralizer of x. Now, $C_G(x)$ act on the borel subalgebras $\mathcal{B}_x$ containing x, and we have a G-equivariant isomoprhism: $\tilde{\mathbb{O}} = G \times_{C_G(x)} \mathcal{B}_x$. From this, we deduce that :

$$Z_o = \tilde{\mathbb{O}} \times_{\mathbb{O}} \tilde{\mathbb{O}} = G \times_{C_G(x)} (\mathcal{B}_x \times \mathcal{B}_x)$$

Therefore, irreducible components of Z_o are of the form $G \times_{C_G(x)} (\mathcal{B}_1 \times \mathcal{B}_2)$ where $\mathcal{B}_1$ and $\mathcal{B}_2$ are irreducible components of $\mathcal{B}_x$. Thus, for any such $\mathcal{B}_1$ and $\mathcal{B}_2$, we have:

$$\dim\mathbb{O} + \dim\mathcal{B}_1 + \dim\mathcal{B}_2 = 2\dim\mathcal{B}$$

Taking $\mathcal{B}_1 = \mathcal{B}_2$ gives the desired result. □

4 Borel-Moore homology

Below, we always work over the field $\mathbb{C}$.

Let M be a not necessarily compact, smooth (real) manifold. As for any topological space, we have the singular homology $H_*(M)$ and cohomology $H^*(M)$. We also have the compactly supported cohomology $H_c^*(M)$. For an oriented manifold M of dimension m, Poincaré duality statement is an isomorphism:

$$H_c^*(M) \cong H_{m-*}(M)$$

given by the cap product with the fundamental class $[M]$. The usual Poincaré duality statement for a *compact* manifold follows because of the isomorphism $H^*(M)$ and $H_c^*(M)$ if M is compact.

Staying in the non-compact setting, we can ask what is the Poincaré dual of $H^*(M)$? The answer is the Borel-Moore homology $H_*^{lf}(M)$ where the super-script lf refers to locally-finite chains.

For a topological space M, we define the complex $C_*^{lf}(M)$ of infinite singular chains:

$$\sum_{i=0}^{\infty} a_i\sigma_i$$

where the σ_i are singular chains of the same dimension and where any compact set $K \subset M$ intersects the support of only finitely many of the σ_i with a_i non-zero. The Borel-Moore homology $H_*^{lf}(M)$ is defined to be the homology of $C_*^{lf}(M)$ under the standard boundary map.

Of course, if M is compact, by definition, $H_*^{lf}(M)$ coincides with $H_*(M)$. But, this is no longer true for non-compact spaces.

For example, if $M = \mathbb{R}$, we can add up all the intervals $[i, i+1]_{i\in\mathbb{Z}}$ to get a locally finite chain with no boundary. This represents the fundamental class of M in $H_1^{lf}(M)$.

An equivalent definition of Borel-Moore homology is given by considering the one-point compactification $M \cup \{\infty\}$, then we have :

$$H_*^{lf}(M) = H_*(M \cup \infty, \infty)$$

From this, it is immediate that there is a proper push-forward map on Borel-Moore homology. That is, if $f : M \to N$ is a proper map (i.e. preimage of a compact set is compact), then there is an induced map $f_* : H_*^{lf}(M) \to H_*^{lf}(N)$. ($f$ being proper ensures that the map from $M \cup \{\infty\}$ to $N \cup \{\infty\}$ induced by f sending ∞ to ∞ is continuous.)

If M is a smooth but not necessarily compact m-dimensional manifold, then we have the Poincaré duality isomorphism (depending on an orientation):

$$H^*(M) \cong H_{m-*}^{lf}(M)$$

To state certain general properties of Borel-Moore homology (such as Poincaré duality), we need to restrict to a "reasonable" class of spaces. However, manifolds are too restrictive of a class, as most of Springer fibers are singular algebraic varieties.

We will insist that M is locally compact, has the homotopy type of a finite CW-complex, and admits a closed embedding to a C^∞ manifold. (I am not sure how much of these assumptions are necessary but they are certainly mild assumptions for us and all the real or complex algebraic varieties satisfy these assumptions).

Now if X is a closed subset of an m-dimensional manifold M, then we have the following version of Poincaré duality:

$$H_*^{lf}(X) = H^{m-*}(M, M\backslash X)$$

Most of the properties of Borel-Moore homology can be established by using this and appealing to the standard properties of cohomology.

In particular, if F is closed in X, then we have Mayer-Vietoris type long exact sequence:

$$\cdots \to H_p^{lf}(F) \to H_p^{lf}(X) \to H_p^{lf}(X\backslash F) \to H_{p-1}^{lf}(F) \to \cdots$$

But, for us, most importantly, every complex algebraic variety X (not necessarily smooth or compact) has a fundamental class $[X]$. Indeed, first we note that any smooth oriented manifold X has a well-defined fundamental class in Borel-Moore homology $[X] \in H_m(X)$ where $m = \dim_{\mathbb{R}}(X)$.

For a complex algebraic variety X, if X is irreducible of real dimension m, then $[X]$ is the unique class in $H_m^{lf}(X)$ that restricts to the fundamental class of the non-singular part X^{reg} of X. Being a smooth complex algebraic variety X^{reg} has a canonical orientation and hence a fundamental class $[X^{reg}] \in H_m^{lf}(X^{reg})$. Since the real codimension of $X\backslash X^{reg}$ is at least 2, one shows that

$H_k(X\backslash X^{reg}) = 0$ for any $k > m-2$ from which one concludes using the Mayer-Vietoris long exact sequence above that:

$$H_m^{lf}(X) \to H_m^{lf}(X^{reg})$$

is an isomorphism. If X is not irreducible, one defines $[X]$ as a non-homogenous class equal to $\sum [X_i]$ where X_i are irreducible components of $[X]$.

For a complex algebraic variety, the top Borel-Moore homology is particularly easy to compute. Namely, we have:

Proposition 4.1. *Let X be a complex variety of complex dimension n and let $X_1, \ldots, X_m$ be the n-dimensional irreducible components of X. Then the fundamental classes $[X_1], \ldots, [X_m]$ form a basis for the vector space $H_{top}^{lf}(X) = H_{2n}(X)$.*

Let M be a manifold and Z_1, Z_2 be closed subsets in M. Then, there is an intersection product on Borel-Moore homology that takes the form:

$$\cap : H_i^{lf}(Z_1) \times H_j^{lf}(Z_2) \to H_{i+j-m}^{lf}(Z_1 \cap Z_2)$$

We can define this using the product structure on cohomology:

$$H^{m-i}(M, M\backslash Z_1) \otimes H^{m-j}(M, M\backslash Z_2) \to H^{2m-i-j}(M, M\backslash (Z_1 \cup Z_2))$$

Note that if Z_1 and Z_2 are intersecting transversely, then the product satisfies:

$$[Z_1] \cap [Z_2] = [Z_1 \cap Z_2]$$

There is a Künneth isomorphism:

$$\boxtimes : H_i^{lf}(Z_1) \otimes H_j^{lf}(Z_2) \to H_{i+j}^{lf}(Z_1 \times Z_2)$$

The proof of this can be given just like the usual Kïnneth isomorphism for the (relative) singular homology.

Using this isomorphism, for a trivial fibration $p : B \times F \to B$ with F smooth oriented of real dimension d, we may define the smooth pullback $p^* : H_*^{lf}(B) \to H_{*+d}^{lf}(B)$ given by $p(c) = c \boxtimes [F]$, where $[F]$ is the fundamental class of F. It is possible to define this map more generally on any locally trivial fibration with oriented fibers, but we mention only that it restricts to the map we have described over any open set where the fibration is trivial.

Convolution product:

For us the most important property of the Borel-Moore homology is that it has a *convolution product.*

First recall the following toy example of a convolution. Let $\mathbb{C}(M)$ denote a finite-dimensional vector space of $\mathbb{C}$ valued functions on M. Given finite sets M_1, M_2, M_3. We define a convolution product:

$$\mathbb{C}(M_1 \times M_2) \otimes \mathbb{C}(M_2 \times M_3) \to \mathbb{C}(M_1 \times M_3)$$

via the formula:

$$f_{12} * f_{23}(m_1, m_3) = \sum_{m_2 \in M_2} f_{12}(m_1, m_2) f_{23}(m_2, m_3)$$

Writing d_i for the cardinality of M_i, we may naturally identify $\mathbb{C}(M_i \times M_j)$ with the vector space of $d_i \times d_j$ matrices. Then the formula given above is just the matrix multiplication.

Another famous example of convolution product appears in deRham cohomology. For a manifold M, recall that $\Omega^\bullet(M)$ denotes the graded vector space of C^∞ differential forms. Let M_1, M_2, M_3 be smooth compact manifolds and $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ be projection maps to (i, j)-factor. Let $d = \dim M_2$, then we have a convolution product:

$$\Omega^i(M_1 \times M_2) \otimes \Omega^j(M_2 \times M_3) \to \Omega^{i+j-d}(M_1 \times M_3)$$

given by

$$f_{12} * f_{23} = \int_{M_2} p_{12}^* f_{12} \wedge p_{23}^* f_{23}$$

One verifies the following Leibniz rule in a standard way:

$$d(f_{12} * f_{23}) = (df_{12}) * f_{23} + (-1)^j f_{12} * d(f_{23})$$

Hence, one gets an induced convolution product in deRham cohomology:

$$H^i(M_1 \times M_2) \otimes H^j(M_2 \times M_3) \to H^{i+j-d}(M_1 \times M_3)$$

One can transport this to homology via Poincaré duality.

After these examples, we now turn to the general formalism of correspondences, which is an abstract way of constructing convolution products.

Let M_1, M_2 be C^∞ manifolds, say. A correspondence is a closed subset $Z_{12} \subset M_1 \times M_2$. Given two correspondences $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$, we define their compositions as:

$$Z_{12} \circ Z_{23} = \{(m_1, m_3) : M_1 \times M_3 : \exists m_2 \in M_2 \text{ such that } (m_1, m_2) \in Z_{12}, (m_2, m_3) \in Z_{23}\}$$

An intuitive way to think about this operation is to treat Z_{12} (resp. Z_{23}) as multi-valued maps from $M_1 \to M_2$ (resp. $M_2 \to M_3$). Then, $Z_{12} \circ Z_{23}$ is the composition of these maps.

For example, if $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are maps. Let $Z_{12} = Graph(f)$ and $Z_{23} = Graph(g)$, then $Z_{12} \circ Z_{23} = Graph(g \circ f)$.

To define convolution product on Borel-Moore homology, we need a little bit more hypothesis which we now formulate. Let $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ be the projection to (i, j)-factor. Given correspondences $Z_{12} \in M_1 \times M_2$ and $Z_{23} \in M_2 \times M_3$ assume:

$$p_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times M_3$$

is *proper*.

Note that $p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) = (Z_{12} \times M_3) \cap (M_1 \times Z_{23})$, hence

$$Z_{12} \circ Z_{23} = p_{23}(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}))$$

Now, the properness assumption above lets us push-forward cycles in Borel-Moore homology. Let $d = \dim_{\mathbb{R}} M_2$, we define a convolution product:

$$H_i^{lf}(Z_{12}) \times H_j^{lf}(Z_{23}) \to H_{i+j-d}^{lf}(Z_{12} \circ Z_{23})$$

via the formula:

$$c_{12} * c_{23} = (p_{13})_*(p_{12}^* c_{12} \cap p_{23}^* c_{23})$$

Recall that since p_{12} and p_{23} are locally trivial fibrations, the pullback is well-defined and given by $p_{12} * c_{12} = c_{12} \boxtimes [M_3]$ and $p_{23}^* c_{23} = [M_1] \boxtimes c_{23}$.

Remark 4.2. *A similar convolution product can be defined for any generalized homology theory that has pullback morphisms for smooth maps, pushforward morphisms for proper maps and and intersection pairing with supports. At a later part of the book [CG] convolution product on K-homology theory is studied.*

There is also a relative version. Namely, let $f_i : M_i \to S$ be smooth locally trivial fibrations over S. Let $Z_{12} \subset M_1 \times_S M_2$ and $Z_{23} \subset M_2 \times_S M_3$, then assuming that $p_{13} : p_{12}^{-1}(Z_1) \cap p_{23}^{-1}(Z_3) \to M_1 \times_S M_3$ is proper, one can define a convolution product:

$$* : H_*^{lf}(Z_{12}) \times H_*^{lf}(Z_{23}) \to H_*^{lf}(Z_{12} \circ Z_{23})$$

Before we go on, we mention that the following associativity equation holds in Borel-Moore homology:

$$c_{12} * (c_{23} * c_{34}) = (c_{12} * c_{23}) * c_{34}$$

where $c_{12} \in H_*(Z_{12})$, $c_{23} \in H_*(Z_{23})$ and $c_{34} \in H_*(Z_{34})$. The verification of this formula is straightforward and given in [CG] if you have difficulty convincing yourself.

Suppose now that M is a smooth complex manifold and N a (possiblty) singular variety. Let $\pi : M \to N$ be a proper map. Put $M_1 = M_2 = M_3 = M$ and $Z = Z_{12} = Z_{23} = M \times_N M$. Expilicitly, we have:

$$Z = \{(m_1, m_2) \in M \times M : \pi(m_1) = \pi(m_2)\}$$

It is obvious that

$$Z \circ Z = Z$$

Therefore, we have the convolution map:

$$H_*(Z) \times H_*(Z) \to H_*(Z)$$

which makes $H_*(Z)$ into an associative algebra with unit given by the fundamental class of the diagonal $\Delta_M \subset Z$. Furthermore, let $M_x = \pi^{-1}(x)$, then we have $Z \circ M_x = M_x$. Hence, the convolution map gives:

$$H_*(Z) \times H_*(M_x) \to H_*(M_x)$$

which makes $H_*(M_x)$ into a left $H_*(Z)$ module.

We next discuss another important structural map on Borel-Moore homology. Let $(S, 0)$ be a smooth manifold with a base point 0. Let $S^* = S\backslash\{0\}$. Suppose $\pi : Z \to S$ is a map from a (possibly singular) space Z. Let $Z^* = \pi^{-1}(S^*)$ and $Z_0 = \pi^{-1}(0)$. Assume $\pi : Z^* \to S^*$ is a locally trivial fibration (with possibly singular fiber F). We will define a *specialization* map:

$$\lim_{s\to 0} : H_*^{lf}(Z^*) \to H_{*-d}^{lf}(Z_0), \text{ where } d = \dim_{\mathbb{R}} S$$

Let $(\mathbb{R}^d, 0) \to (S, 0)$ be a local chart for S around 0. We have the restriction map : $H_*^{lf}(Z) \to H_*^{lf}(\pi^{-1}(\mathbb{R}^d))$, hence we may assume that $(\mathbb{R}^d, 0) = (S, 0)$. In this case, write S_+ for the positive half plane in the first coordinate, I_+ for the positive first coordinate axis, and $I_{\geq 0}$ for the non-negative first coordinate axis. The specialization is then given by the composition:

$$H_*^{lf}(Z^*) \to H_*^{lf}(\pi^{-1}(S_+)) \cong H_{*-d}^{lf}(F) \otimes H_d^{lf}(S_+) \to H_{*-d}^{lf}(F) \otimes H_1^{lf}(I_+) \cong H_{*-d+1}^{lf}(\pi^{-1}(I_+)) \xrightarrow{\partial} H_{*-d}^{lf}(Z_0),$$

where the first map is given by restriction, the middle maps by the Kunneth theorem, and the last map by the exact sequence of the pair $(\pi^{-1}(I \geq 0), \pi^{-1}(I_+))$.

It is verified in [CG] Ch. 2 that the specialization is independent of the choices of the coordinates that we made above. We will not discuss this in class.

Another important property is that specialization commutes with convolution in Borel-Moore homology under natural hypothesis. We state this resutl as follows. The proof will be omitted.

Suppose $f_i : M_i \to S$ be smooth locally trivial fibrations over S. Assume $Z_{ij} \in M_i \times_S M_j$ for $i < j$, $i, j = 1, 2, 3$ be correspondences where we put $Z_{13} = Z_{12} \circ Z_{23}$ such that the natural projections $Z_{ij}^* \to S^*$ are locally trivial fibrations. Furthermore, assumat that $p_{13} : p_{12}^{-1}(Z_{12}^*) \times p_{23}^{-1}(Z_{23}^*) \to Z_{13}^*$ is a morphism of locally trivial fibrations, then we have the following commutative diagram:

$$\begin{array}{ccc}
H_*^{lf}(Z_{12}^*) \otimes H_*^{lf}(Z_{23}^*) & \xrightarrow{\lim_{s\to 0}} & H_*^{lf}((Z_{12})_0) \otimes H_*^{lf}((Z_{23})_0) \\
\downarrow * & & \downarrow * \\
H_*(Z_{13}^*) & \xrightarrow[\lim_{s\to 0}]{} & H_*((Z_{13})_0)
\end{array}$$

5 Weyl group representations

We now come to one of the main theorems of the course. Let $\mathbb{Q}[W]$ be the group algebra over $\mathbb{Q}$ of the abstract Weyl group. Recall that we use Z to denote the Steinberg variety and let $m = \dim_{\mathbb{R}} Z = \dim_{\mathbb{R}} \tilde{N}$. Recall also that $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ is a fibre product with respect to the Springer map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$. In particular, $Z \circ Z = Z$, hence we have a natural convolution algebra structure on $H_*^{lf}(Z)$.

Theorem 5.1. *There is a canonical algebra isomorphism*

$$H_m^{lf}(Z;\mathbb{Q}) \cong \mathbb{Q}[W]$$

Proof. Recall that we write Y_w for the G-orbit corresponding to $w \in W$ for the diagonal G action on $\mathcal{B}\times\mathcal{B}$, and we have that $Z = \bigsqcup_w T^*_{Y_w}(\mathcal{B}\times\mathcal{B})$. The irreducible components $\overline{T^*_{Y_w}(\mathcal{B}\times\mathcal{B})}$ of Z are all of dimension m. We have seen that the top-dimensional Borel-Moore homology group $H_m^{lf}(Z;\mathbb{Q})$ has a basis given by the fundamental classes of these irreducible components. In particular, it is clear that

$$\dim_{\mathbb{Q}} H_m^{lf}(Z;\mathbb{Q}) = \dim_{\mathbb{Q}}\mathbb{Q}[W]$$

The more difficult part of the theorem is to show that the algebra structures match on both sides. This is far from obvious. The problem is that these irreducible components are singular in general and intersect in a complicated way. In fact, rather surprisingly, in order to prove this, we will need to pass to a different basis of $H_m^{lf}(Z;\mathbb{Q})$.

We would like construct a basis of $H_m^{lf}(Z)$ by exhibiting a map from $W \to H^{lf}(Z)$ assigning $w \to [\Lambda_w^0]$ so as to satisfy the equation:

$$[\Lambda_{yw}^0] = [\Lambda_y^0] * [\Lambda_w^0]$$

The superscript 0 signifies specialization, and it will be made clear later on.

Recall that we have seen that restriction of μ on the semisimple regular locus is a finite Galois cover $\mu : \tilde{\mathfrak{g}}^{sr} \to \mathfrak{g}^{sr}$. Given $w \in W$, consider the $Graph(w) \subset \tilde{\mathfrak{g}}^{sr} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}^{sr}$ of the fibrewise action of w. Recall that we had the adjoint quotient map:

$$\tilde{\chi} : \tilde{\mathfrak{g}} \to \mathfrak{H} = \mathfrak{h}$$

sending $(x, \mathfrak{b}) \to [x] \in \mathfrak{b}/[\mathfrak{b},\mathfrak{b}] = \mathfrak{H}$. We let $\tilde{\mathfrak{g}}_h = \tilde{\chi}^{-1}(h)$ to be the fibre above $h \in \mathfrak{h}$. Note that $\tilde{\chi} : \tilde{g} \to \mathfrak{H}$ is a locally trivial fibration. Indeed, recall that $\tilde{\mathfrak{g}} = G \times_B \mathfrak{b}$ and for any subset $S \in \mathfrak{H} = \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$, we have

$$\tilde{\chi}^{-1}(S) = G \times_B (S + \mathfrak{n})$$

Let $pr_2 : \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ denote the projection to the second component. For $h \in \mathfrak{h}^{sr}$, let us write

$$\Lambda_w^h = Graph(w) \cap (pr_2 \circ \tilde{\chi})^{-1}(h) = \{(x, \mathfrak{b}, x, \mathfrak{b}') : x \in \mathfrak{b} \cap \mathfrak{b}', (\mathfrak{b}, \mathfrak{b}') \in Y_w \ , \ \tilde{\chi}(x, \mathfrak{b}') = h\}$$

where Y_w is the G-orbit corresponding to $w \in W$ for the diagonal action of G on $\mathcal{B} \times \mathcal{B}$.

In other words, for $h \in \mathfrak{h}^{sr}$, Λ_w^h is the graph of the map $w : \tilde{\mathfrak{g}}_h \to \tilde{\mathfrak{g}}_{w(h)}$ induced by the W-action on $\tilde{\mathfrak{g}}^{rs}$. Therefore, in Borel-Moore homology of $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$, for $h \in \mathfrak{h}^{sr}$, we have

$$[\Lambda_{yw}^h] = [\Lambda_y^h] * [\Lambda_w^h]$$

The idea of the argument is to analyze what happens as $h \to 0$.

Let us define:

$$\Lambda_w := \{(x, \mathfrak{b}, x, \mathfrak{b}') : x \in \mathfrak{b} \cap \mathfrak{b}', (\mathfrak{b}, \mathfrak{b}') \in Y_w\}$$

Hence, the projection $\Lambda_w \to Y_w$ is a vector bundle over Y_w and in particular, has dimension $\dim\mathfrak{g}$. So, we have a decomposition of $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ into irreducible components :

$$\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} = \bigcup_w \overline{\Lambda_w}$$

In fact, it is easy to conclude from what we have seen before that:

$$\overline{\Lambda_w} := \overline{Graph(w)} \subset \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$$

is the closure of of the $Graph(w)$.

Moreover, over the special point $0 \in \mathfrak{h}$, we have :

$$\Lambda_w^0 = \Lambda_w \cap \tilde{\chi}^{-1}(0) \subset (\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) \cap (\tilde{\mathcal{N}} \times \tilde{\mathcal{N}}) = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} = Z$$

(In fact, Λ_w^0 turns out to be all of Z).

Therefore, we would like to construct the cycles $[\Lambda_w^0]$ by specialization. We may work by restricting $\tilde{\chi}$ to one Λ_w at a time. Let $\mathfrak{H}_w = Graph(\mathfrak{H} \xrightarrow{w} \mathfrak{H}) = \{(w \cdot h, h) \in \mathfrak{H} \times \mathfrak{H}\}$. We have a locally trivial fibration:

$$\nu : \tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}} = \{(y, x) \in \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} : \tilde{\chi}(y) = w \cdot \tilde{\chi}(x)\} \to \mathfrak{H}_w$$

given by $(y, x) \to (\tilde{\chi}(y), \tilde{\chi}(x))$.

By definition, we can identify Λ_w in $\tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}}$ as the preimage of the diagonal under the projection $\mu \boxtimes \mu : \tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}} \to \mathfrak{g} \times \mathfrak{g}$. Note that $\nu^{-1}(0) = \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$.

Then the restriction of ν to Λ gives a family of varieties $\nu_w : \Lambda_w \to \mathfrak{H}_w$, with the desired properties that

$$\nu_w^{-1}(h) = Graph(\tilde{\mathfrak{g}}^{sr} \xrightarrow{w} \tilde{\mathfrak{g}}^{sr}) \;\text{ for }\; h \in \mathfrak{H}^{sr}, \;\text{ and }\; \nu_w^{-1}(0) \subset Z$$

The problem is that this restriction is no longer locally trivial.

But, the specialization set-up required that a locally trivial fibration outside of the special point 0.

To overcome this difficuly, we replace $\mathfrak{H}$ with a smaller set $\mathfrak{l} \subset \mathfrak{H}$. For a fixed $w \in W$, choose a 2-dimensional real subspace $\mathfrak{l}$ such that $\mathfrak{l}^* = \mathfrak{l} - \{0\}$ lies in $\mathfrak{H}^{rs}$. Consider $\mathfrak{l}_w = Graph(\mathfrak{l} \xrightarrow{w} w \cdot \mathfrak{l})$

Now, over $\mathfrak{l}_w$ we have the restriction $\nu_w : \Lambda_w^{\mathfrak{l}} := \nu_w^{-1}(\mathfrak{l}_w) \to \mathfrak{l}_w$ which is locally trivial over $\mathfrak{l}_w^* = \mathfrak{l}_w - \{0\}$.

We can therefore define a specialization map:

$$H_*^{lf}(\Lambda_w^{\mathfrak{l}}) \to H_{*-2}^{lf}(\Lambda_w^0) = H_{*-2}(Z)$$

Define the homology class $[\Lambda_w^0] = \lim_{h\to 0}[\Lambda_w^{\mathfrak{l}^*}] \in H_m(Z)$.

That this specialization does not depend on the choice of $\mathfrak{l}^*$ is an easy argument using the transitivity of specialization and the fact that any two elements $h, h' \in \mathfrak{H}^{rs}$ can be connected by a piecewise linear (real) path. The desired equation :

$$[\Lambda_{yw}^0] = [\Lambda_y^0] * [\Lambda_w^0]$$

follows from the fact that "specialization commutes with convolution".

Finally, we need to prove that $\{[\Lambda_w^0].w \in W\}$ form a basis. Firstly, note that, the projection of each $\Lambda_w^h \to \mathcal{B}\times\mathcal{B}$ is supported in Y_w by definition. Therefore, $[\Lambda_w^0]$ is supported at most on $\overline{Y_w}$. Hence, this implies that:

$$[\Lambda_w^0] = \sum_{v\le w} c_{w,v}\overline{[T^*_{Y_v}(\mathcal{B}\times\mathcal{B})]}$$

for some $c_{w,v}$ and where $v \le w$ is in the Bruhat ordering.

Claim: $c_{w,w} = 1$ for all $w \in W$.

We give a sketch. First observe that $\Lambda_w^h \to Y_w$ can be identified with the flat family of affine bundles over Y_w:

$$G \times_{B\cap w\cdot B} (h + \mathfrak{n} \cap w \cdot n) \to G/(B \cap w \cdot B)$$

As $h \to 0$, $[\Lambda_w^h]$ degenerates to the fundamental class $G \times_{B\cap w\cdot B} (\mathfrak{n} \cap w \cdot \mathfrak{n})$ which is exactly $[T^*_{Y_w}(\mathcal{B}\times\mathcal{B})]$.

Therefore, the matrix (c_{vw}) is upper triangular with diagonal entries 1, hence is invertible. This completes the proof. □

Example 5.2. *Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ Then, we have $W = 1, s$. $c_{11} = c_{ss} = 1$, $c_{1s} = 1$. (Explain this in class via graphs of Dehn twists...)*

Notation: From now on, we will write $H(Z)$ for $H_m^{lf}(Z)$ where $m = \dim_{\mathbb{R}} Z$.

Recall that we have the Springer resolution $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ whose fibre over $x\mathcal{N}$ is denoted by $\mathcal{B}_x$. $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ is the Steinberg variety that maps to $\mathcal{N}$ via μ. For $x \in \mathcal{N}$, put $Z_x = \mathcal{B}_x \times B_x$.

Clearly, we have :

$$Z \circ Z_x = Z_x = Z_x \circ Z$$

Therefore, if $d = \dim_{\mathbb{R}}\mathcal{B}_x$, then we have that $H(Z_x) := H_{2d}^{lf}(Z_x)$ is an $H(Z)$-bimodule.

Note that Künnerth formula gives $H_d^{lf}(\mathcal{B}_x)\otimes H_d^{lf}(\mathcal{B}_x) = H_{2d}^{lf}(Z_x)$. Furthermore, since $\mathcal{B}_x \circ Z = \mathcal{B}_x$ and $Z \circ \mathcal{B}_x = \mathcal{B}_x$, we can make $H(\mathcal{B}_x) := H_d^{lf}(\mathcal{B}_x)$ into a $H(Z)$-bimodule. We will write ${}_LH(\mathcal{B}_x)$ and $H(\mathcal{B}_x)_R$ when we view $H(Z)$ as a left and right $H(Z)$-modules respectively.

Künneth isomorphism yiels an isomorphism of $H(Z)$-bimodules:

$$H(Z_x) \cong {}_L H(\mathcal{B}_x) \otimes_{\mathbb{Q}} H(\mathcal{B}_x)_R$$

Claim: (This is not obvious and we leave its justification for later) There is an isomorphism of right $H(Z)$-modules

$${}_L H(\mathcal{B}_x)^\vee \cong H(\mathcal{B}_x)_R$$

Here recall that if A is a $\mathbb{Q}$-algebra and V is a left A-module, then $V^\vee = Hom_{\mathbb{Q}}(V, \mathbb{Q})$ is a right A-module, where the multiplicatioon of $f \in Hom_{\mathbb{Q}}(V, \mathbb{Q})$ is given by $(f \cdot a)(v) = f(a \cdot v)$.

Next, since G acts on $\mathcal{N}$ by adjoint action, we get an induced map $\mathcal{B}_x \to \mathcal{B}_{g(x)}$ for all $g \in G$. This induces a morphism on Borel-Moore homology:

$$g : H_*(\mathcal{B}_x) \to H_*(\mathcal{B}_{g(x)})$$

We have the following compatibility:

Lemma 5.3. *The left (resp. right) $H(Z)$-action on Borel-Moore homology $H(\mathcal{B}_x)$ is compatible with the G-action, in the sense that for $z \in H(Z)$, $g \in G$ and $c \in H_*(\mathcal{B}_x)$, we have:*

$$z \cdot g(c) = g(z \cdot c)$$

Proof. Since G acts on Z, and maps $\mathcal{B}_x \to \mathcal{B}_{g(x)}$, it follows easily from the definition of convolution product that:

$$g(z) * g(c) = g(z * c)$$

Therefore, we need to show that $g(z) = z$ for $z \in H(Z)$. Note however that G is assumed to be connected throughout, therefore its action on homology is trivial (since the identity element in G acts as identity.) □

Now, recall that $C_G(x)$, the centralizer of $x \in \mathcal{N}$ in G acts on the variety $\mathcal{B}_x$ by conjugation. Hence, we get an induced action on the homology of $\mathcal{B}_x$. The identity component $C_G^\circ(x)$ acts trivially on homology, so that the action factors through the finite component group $C(x) = C_G(x)/C_G^\circ(x)$

It follows the we have the following decomposition into $C(x)$-isotypical components:

$$\mathbb{C} \otimes_{\mathbb{Q}} {}_L H(\mathcal{B}_x) = \bigoplus_{\chi \in C(x)^\wedge} \chi \otimes H(\mathcal{B}_x)_\chi$$

where $C(x)^\wedge$ denotes the set of (equivalence classes of) irreducible complex representations of the (finite) group $C(x)$ that occur in $\mathbb{C} \otimes_{\mathbb{Q}} {}_L H(\mathcal{B}_x)$ with non-zero multiplicity.

Here is how to understand this, consider the action of $C(x)$ on $\mathbb{C} \otimes_{\mathbb{Q}} {}_L H(\mathcal{B}_x)$ and decompose it into irreducible representations parametrized by characters χ but note that same character might appear many times. Now, the action of $H(Z)$ will not necessarily preserve these irreducible representations but will permute them. However, this permutation has to send an irreducible representaiton of $C(x)$ to an isomorphic one because of the compatibility between $C(x)$ and $H(Z)$ actions. So, we collect all the irreducible representations of $C(x)$ with character χ that appear and call their direct sum $H(\mathcal{B}_x)_\chi$. The latter is then a representation fo $H(Z)$.

Recall that by Maschke's theorem group algebra $\mathbb{Q}[G]$ of any finite group G is a semisimple algebra. In particular, $H(Z) \cong \mathbb{Q}[W]$ is a semisimple algebra. Therefore, we must have that $H(\mathcal{B}_x)$ decompose into a direct sum of irreducible representations.

We are now in a position to state the Springer's main theorem:

Theorem 5.4. *(Springer correspondence)*

- *For any $x \in \mathcal{N}$, and $\chi \in C(x)^\wedge$, $H_d(\mathcal{B}_x)_\chi$ is an irreducible $H(Z)$-module where $d = dim_{\mathbb{R}}\mathcal{B}_x$.*
- *The modules $H_d(\mathcal{B}_x)_\chi$ and $H_d(\mathcal{B}_y)_\psi$ are isomorphic if and only if the pairs (x, χ) and (y, ψ) are G-conjugate.*
- *The set $\{H_(\mathcal{B}_x)_\chi | x \in \mathcal{N}, \chi \in C(x)^\wedge\}$ is a complete collection of isomorphism classes of simple complex $H(Z)$-modules.*

Remark 5.5. *Note that in the case $\mathfrak{g} = sl_n(\mathbb{C})$, it turns out that $H_d(\mathcal{B}_x)$ is irreducible as an $H(Z)$-module for any $x \in \mathcal{N}$. This can be seen by carrying out all the constructions using the reductive algebra $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. Indeed, $sl_n(\mathbb{C})$ is the derived subalgebra of $\mathfrak{gl}_n(\mathbb{C})$. All the constructions that we did can be carried out using a reducitve Lie algebra, and the varieties $\mathcal{B}, \mathcal{N}, \mathcal{Z}$ remain unchanged if we replace a reductive Lie algebra $\mathfrak{g}$ with its derived (semisimple) subalgebra $[\mathfrak{g}, \mathfrak{g}]$. The advantage of considering the $\mathfrak{gl}_n$ is that we may now assume $G = GL_n(\mathbb{C})$. But then for any $x \in \mathfrak{g}$, $C_G(x)$ is connected. Hence, $C(x) = C_G(x)/C_G^\circ(x) = 1$. To see that $C_G(x)$ is connected, observe that $C_G(x) = \{y \in M_n(\mathbb{C}) : xy = yx, dety \neq 0\}$. This is a complement of a complex hypersurface in a vector space, which is connected. (Note that it is not true in general that the component groups of a centralizer in $SL_n(\mathbb{C})$ is trivial.)*

Before, we go on with the proof, we need the following result (which we have claimed before):

Lemma 5.6. *There is an isomorphism of right $H(Z)$-modules*

$$_L H(\mathcal{B}_x)^\vee \cong H(\mathcal{B}_x)_R$$

Proof. There is an involution on Z given by switching the factors in $\tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$. This gives rise to an involution $c \to c^t$ on $H(Z)$. With respect to the convolution algebra structure on $H(Z)$, this gives us an algebra anti-involution, that is:

$$(c_1 * c_2)^t = c_2^t * c_1^t$$

Now given a right $H(Z)$-module V, define a left $H(Z)$-module structure on V by:

$$c \cdot v = v \cdot c^t \text{ for } c \in H(Z), \quad v \in V$$

Call this left $H(Z)$-module V^t. Then, by definition we have:

$$(H(\mathcal{B}_x)_R)^t = {_L H(\mathcal{B}_x)^\vee}$$

Let $V = {}_L H(\mathcal{B}_x)^\vee$. Thus, the claim is to show that there is an isomorphism of $H(Z)$-modules:

$$V \cong (V^\vee)^t$$

Now, we claim that under the isomorphism $\mathbb{Q}[W] \cong H(Z)$, the anti-involution $c \to c^t$ on $H(Z)$ corresponds to the anti-involution $w \to w^{-1}$ on $\mathbb{Q}[W]$. To see this, obser ver that the involution on Z applied to a graph Λ_w^h at a semisimple regular h gives

$$(\Lambda_w^h)^t = \Lambda_{w^{-1}}^{wh}$$

But, wh is also regular semisimple, therefore, we can obtain consider the limiting cycles as $h \to 0$ or $wh \to 0$. This implies that $[\Lambda_w^0]^t = [\Lambda_{w^{-1}}^0]$.

It follows that, the left $H(Z)$-module $(V^\vee)^t$ is the contragredient module of V. Recall that this just means that

$$c \cdot f(v) = f(c^{-1}v), \text{ for } f \in Hom(V, \mathbb{Q}), \ c \in H(Z), \ v \in V$$

But now, recall that $H(Z) = \mathbb{Q}[W]$. So, it suffices to show that V is isomorphic to its contragradient as a W-module. But, W is a finite group, there any finite-dimensional W-module admits a W-invariant positive definite bilinear form. Hence, such modules are isomorphic to their contragredients. □

We now turn back to the proof of Springer's correspondence.

Partially order the nilpotent orbits of $\mathcal{N}$ according to containment in closure, and for such an orbit $\mathbb{O}$, let $Z_{<\mathbb{O}}$, $Z_{\mathbb{O}}$, and $Z_{\leq\mathbb{O}}$ be the corresponding preimages in Z. We notice that $H(Z_{<\mathbb{O}})$ and $H(Z_{\leq\mathbb{O}})$ are both 2-sided ideals in $H(Z)$. We put

$$H_{\mathbb{O}} = H(Z_{\leq\mathbb{O}})/H(Z_{<\mathbb{O}})$$

The partial order on the set of orbits gives us a filtration on the algebra $H(Z)$ by two-sided ideal $H(Z_{\leq\mathbb{O}})$. Let $grH(Z)$ denote the associated graded bimodule. On the other hand since $H(Z) = \mathbb{Q}[W]$ is the group algebra of a finite group, it is a semisimple algebra. Therefore, we have an isomorphism of $H(Z)$-bimodules $H(Z) \cong grH(Z)$. Thus, we have the bimodule isomorphism:

$$H(Z) \cong \bigoplus_{\mathbb{O} \subset \mathcal{N}} H_{\mathbb{O}}$$

Now, recall that $H(Z_{\leq\mathbb{O}})$ and $H(Z_{<\mathbb{O}})$ have bases given by the fundamental classes of the irreducible components of their respective spaces. Therefore, $H_{\mathbb{O}}$ has a basis given by fundamental classes of the irreducible component of $Z_{\mathbb{O}}$.

Recall that $Z_{\mathbb{O}} = \tilde{\mathbb{O}} \times_{\mathbb{O}} \tilde{\mathbb{O}} = G \times_{C_G(x)} (\mathcal{B}_x \times \mathcal{B}_x)$ after picking any $x \in \mathbb{O}$. Hence, its irreducible components are the in one-to-one correspondence with $C(x) = C_G(x)/C_G^\circ(x)$-orbits of pairs of irreducible components of $\mathcal{B}_x$. Thus, we can write

$$H_{\mathbb{O}} = H(Z_x)^{C(x)} = H(B_x \times B_x)^{C(x)} = ({}_L H(\mathcal{B}_x) \otimes H(\mathcal{B}_x)_R)^{C(x)}$$

where we used the Künneth isomorphism for the last equality.

(Strictly speaking, we have only given a justification for the first equality as addittive vector spaces but this identification is in fact an identification of $H(Z)$-bimodules. This is not hard just a bit more tedious than I would like to discuss. It is based on the fact that the modules $H(\mathcal{B}_x)$ do not vary as x varies in an orbit $\mathbb{O}$. See Sec. 3.5 of [CG] for details.)

Now, we can use the Lemma that we proved above to conclude that:

$$H_{\mathbb{O}} = ({}_LH(\mathcal{B}_x) \otimes {}_LH(\mathcal{B}_x)^{\vee})^{C(x)} = End_{C(x)}({}_LH(\mathcal{B}_x))$$

Next, recall that we have the decomposition, tensoring with $\mathbb{C}$ over $\mathbb{Q}$, we have the decomposition:

$$\mathbb{C} \otimes_{\mathbb{Q}} {}_LH(\mathcal{B}_x) = \bigoplus_{\chi \in C(x)^{\wedge}} \chi \otimes H(\mathcal{B}_x)_{\chi}$$

Summing over all orbits $\mathbb{O} \subset \mathcal{N}$, we obtain:

$$\mathbb{C}[W] = \mathbb{C} \otimes_{\mathbb{Q}} H(Z) = \bigoplus_{\mathbb{O}} H_{\mathbb{O}} = \bigoplus_{[x,\chi]} End_{\mathbb{C}}({}_LH(\mathcal{B}_x)_{\chi})$$

where the last sum is over G-conjugay classes of pairs (x, χ) where $\chi \in C(x)^{\wedge}$. (We used the fact that $Hom_{C(x)}(\chi, \psi) = \mathbb{C}$ if $\chi = \psi$ and 0 otherwise).

The statement of Springer correspondence follows from this by using the semisimplicity of $\mathbb{C}[W]$. Namely, let $\{E_{\alpha}\}$ be a complete collection of irreducible complex left $H(Z)$-modules. Since $\mathbb{C} \otimes_{\mathbb{Q}} H(Z)$ is semisimple, we have:

$$\mathbb{C} \otimes_{\mathbb{Q}} H(Z) = \bigoplus_{\alpha} End(E_{\alpha})$$

On the other hand ${}_LH(\mathcal{B}_x)_{\chi}$ is a left $H(Z)$-module, hence it decomposes into irreducible components with multiplicities:

$${}_LH(\mathcal{B}_x)_{\chi} = \sum n^{\alpha}_{x,\chi} E_{\alpha}$$

But then we get:

$$\mathbb{C} \otimes_{\mathbb{Q}} H(Z) = \bigoplus_{[x,\chi]} \bigoplus_{\alpha,\beta} n^{\alpha}_{x,\chi} n^{\beta}_{x,\chi} Hom_{\mathbb{C}}(E_{\alpha}, E_{\beta})$$

From which, we conclude that $\sum_{[x,\chi]} n^{\alpha}_{x,\chi} n^{\beta}_{x,\chi} = \delta_{\alpha,\beta}$. Hence, it follows that ${}_LH(\mathcal{B}_x)_{\chi} = E_{\alpha}$ for some α. □

Here is some fun consequence. Recall that for any finite group G, we have the elementary result:

$$|G| = \sum_{\alpha} (dim E_{\alpha})^2$$

where E_α runs through irreducible reprensentations of G. For example, this follows from what we said above. Namely,

$$\mathbb{C}[G] = \sum_\alpha End(E_\alpha)$$

Here is an alternative way to see it for Weyl groups. Say, for simplicity of notation we restrict to $\mathfrak{g} = \mathfrak{sl}_n$. Then the component group $C(x)^\wedge$ is trivial.

Now, we have $Z = \bigcup_{\mathbb{O}} Z_{\mathbb{O}}$ and the irreducible components of $Z_{\mathbb{O}} = G \times_{C_G(x)} (\mathcal{B}_x \times \mathcal{B}_x)$ is parametrized by pairs of irreducible components of $\mathcal{B}_x$. Therefore :

$$|S_n| = \#\{ \text{ components of } Z\} = \sum_{\mathbb{O}} (\#\text{components of } \mathcal{B}_x)^2 = \sum_{\mathbb{O}} (\dim H(\mathcal{B}_x))^2$$

Since $H(\mathcal{B}_x)$ gives us all the irreducible representations of S_n, this reproduces the elementary result that we mentioned above.

Here is another interesting combinatorial identity.
Corollary 5.7. *We have*

$$\sum_\alpha dim(E_\alpha) = \#\{involutions\ in\ S_n\}$$

where the sum runs through all the irreducible representations of S_n.

Proof. An element $w \in S_n$ corresponds to the irreducible component of Z given by the conormal bundle $T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$. As we saw such a component is sent to $T^*_{Y_{w^{-1}}}(\mathcal{B} \times \mathcal{B})$ under the involution on Z. Therefore, the number of irreducible components of Z fixed by the involution on Z is equal to the number of involutions in S_n.

On the other hand, we can also describe irreducible components of Z via the decomposition $Z = \bigsqcup_{\mathbb{O}} Z_{\mathbb{O}}$. A component of $Z_{\mathbb{O}}$ given by $G \times_{C(x)} (\mathcal{B}_x^\alpha \times \mathcal{B}_x^\beta)$, where $x \in \mathbb{O}$, is fixed by the involution if and only if $\alpha = \beta$. □

Example 5.8. *A distinguished Springer fiber is $\mathcal{B}_0 = \mathcal{B}$. This is a smooth irreducible variety. Hence $H_{top}(\mathcal{B}) = \mathbb{C}$. It is not too hard to see that W action on this gives the sign representation of W since a simple reflection changes the orientation of G/B.*

If x is regular nilpotent, then we know that $\mathcal{B}_x$ is a single point. Hence $H_{top}(\mathcal{B}_x)$ corresponds to the trivial representation.

Say $\mathfrak{g} = \mathfrak{sl}_n$, and x has Jordan type $(n-1, 1)$, then B_x consists of $(n-1)$ copies of $\mathbb{P}^1$ connected linearly according to the Dynkin diagram of type A_{n-1}. The action of W yields the $(n-1)$-dimensional irreducible subrepresentation of the permutation representation of S_n, where each reflection acts by exchanging the corresponding $\mathbb{P}^1$'s.

6 Springer theory for $\mathcal{U}(\mathfrak{sl}_n)$

In this section, we give a survey of the results in Ch. 4 of [CG].

We now have a twist in the story. Namely, until now, we have focused on representations of the Weyl groups. These are finite groups. Admittedly, their representation theory is accessible via more elementary tools. Nonetheless, geometrically minded reader would surely appreciate the beauty of the constructions given in the previous sections. On the other hand, our main motivation to study the geometry of the Springer fibration was to go beyond the representation theory of Weyl groups. The machinery of constructing representations of convolution algebras like $H(Z)$ applies in a much more general and the geometry enables us to study representation theory of much more complicated algebras than $\mathbb{C}[W]$ whose representation theory is not accessible otherwise.

We fix an integer $n \geq 1$ corresponding to $\mathfrak{sl}_n(\mathbb{C})$. We also fix another integer $d \geq 1$ bearing no relation to n whatsoever. Let us pose

$$\mathcal{F} = \{0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^d\}$$

be the set of all n-step flags in $\mathbb{C}^d$. In the current situation, $\mathcal{F}$ will play the role that the flag variety $\mathcal{B}$ played when we studied Springer representations for the Weyl group.

The space $\mathcal{F}$ is a compact smooth manifold with connected components parametrized by partitions:

$$\mathbf{d} = (d = d_1 + d_2 + \ldots + d_n, \quad d_i \in \mathbb{Z}_{\geq 0})$$

To the partition $\mathbf{d} = (d_1, \ldots, d_n)$, we associate the connected component of $\mathcal{F}$ consisting of flags:

$$\{\mathcal{F}_{\mathbf{d}} = \{0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^d : \dim F_i/F_{i-1} = d_i\}$$

Note that d_i are allowed to be zero, hence flags F_i and F_j are allowed to coincide for $i \neq j$. The dimension of the component $\mathcal{F}_{\mathbf{d}}$ is given by:

$$\dim_{\mathbb{C}} \mathcal{F}_{\mathbf{d}} = \frac{d!}{d_1! d_2! \ldots d_n!}$$

We next introduce the nilpotent variety in this setting to be:

$$N = \{x \in M_{d \times d}(\mathbb{C}) : x^n = 0\}$$

We are going to define an analogue of the Springer fibration. Let us define:

$$M = \{(x, F) \in N \times \mathcal{F} : x(F_i) \subset F_{i-1}, i = 1, \ldots, n\}$$

The projection to the first component is denoted by $\mu : M \to N$ and is analogous to the Springer fibration. We denote $\mu^{-1}(x) = \mathcal{F}_x$ for "Springer fibers".

On the other hand, the projection fo M to the second component exhibits it as:

$$M = T^*\mathcal{F}$$

The above decomposition of $\mathcal{F}$ into its connected components yields a decomposition of $M = \bigsqcup_{\mathbf{d}} M_{\mathbf{d}}$, where $M_{\mathbf{d}} = T^*\mathcal{F}_{\mathbf{d}}$. We write

$$\mathcal{F}_{\mathbf{d},x} = \mathcal{F}_x \cap M_{\mathbf{d}}$$

for the connected components of $\mathcal{F}_x$.

Observe that we have a natural $GL_d(\mathbb{C})$ action on M, N (by conjugation) and $\mathcal{F}$ and the projections commute with this action.

We define

$$Z = M \times_N M = \{(m_1, m_2) \in M \times M | \mu(m_1) = \mu(m_2)\} \subset M \times M$$

in an analogous way to the Steinberg variety.

We have the following lemma of Spaltenstein:

Lemma 6.1. *For any $x \in N$ and any partition $\mathbf{d}$, the variety $\mathcal{F}_{\mathbf{d},x}$ is connected of pure dimension (each irreducible component is of the same dimension) and*

$$\dim \mathbb{O}_x + 2\dim \mathcal{F}_{\mathbf{d},x} = 2\dim \mathcal{F}_{\mathbf{d}}$$

where $\mathbb{O}_x$ is the is the $GL_d(\mathbb{C})$ orbit of $x \in \mathfrak{sl}_n(\mathbb{C})$.

The proof of this result is by an explicity computation, which we omit. Furthermore, the equidimensionality assertion fails for simple groups of types other than SL_n.

Lemma 6.2. *The number of $GL_d(\mathbb{C})$-diagonal orbits in $\mathcal{F} \times \mathcal{F}$ is finite.*

Proof. Write $\mathcal{B}$ for the flag variety of $GL_d(\mathbb{C})$. There is a surjective $GL_d(\mathbb{C})$-equivariant map $\mathcal{B} \to \mathcal{F}_{\mathbf{d}}$ from the set of complete flags in $\mathbb{C}^d$ to any $\mathcal{F}_{\mathbf{d}}$. Hence, there is an $GL_d(\mathbb{C})$-equivariant surjection from $\mathcal{B} \times \mathcal{B} \to \mathcal{F}_{\mathbf{d}_1} \times \mathcal{F}_{\mathbf{d}_2}$. □

In a completely analogous way (to the Weyl group case), we have that

$$Z = \bigsqcup_{\alpha} T^*_{Y_\alpha}(\mathcal{F} \times \mathcal{F})$$

as a union of conormal bundles, where Y_α are the $GL_d(\mathbb{C})$-diagonal orbits in $\mathcal{F} \times \mathcal{F}$ (where as before the symplectic form on $T^*(\mathcal{F} \times \mathcal{F})$ differs from the one on $T^*\mathcal{F} \times T^*\mathcal{F}$) by a switch on the sign on the second component.)

Corollary 6.3. *We have $Z \circ Z = Z$, in particular, $H(Z)$ is an associative algebra with unit and $H(\mathcal{F}_x)$ is an $H(Z)$-module, for any $x \in N$.*

Note that Z is not connected and it has irreducible components of different dimensions, however if Z^α is an irreducible component of Z contained in $T^*\mathcal{F}_{\mathbf{d}_1} \times T^*\mathcal{F}_{\mathbf{d}_2}$ then it has half the dimension (as a Lagrangian should have).

Above, we write $H(Z)$ for the vector subspace of $H_*^{lf}(Z)$ spanned by the fundamental classes of the irreducible components of Z, and similarly $H(\mathcal{F}_x)$ for the vector subspace of $H_*^{lf}(\mathcal{F}_x)$ spanned by the fundamental classes of the irreducible components of $\mathcal{F}_x$.

The following theorem is analogous to what we have seen in the case of the Springer representations for the Weyl group:
Theorem 6.4.

- *$H(Z)$ is a finite-dimensional, semisimple associative algebra with unit.*
- *The representations $H(\mathcal{F}_x)$ and $H(\mathcal{F}_y)$ are isomorphic as $H(Z)$-modules if and only if x and y are conjugate by $GL_d(\mathbb{C})$.*
- *The collection $\{H(\mathcal{F}_x)\}$ as x runs through representations of the $GL_d(\mathbb{C})$ conjugacy classes in N is a complete collection of the isomorphism classes of irreducible $H(Z)$-modules.*

The following is the main theorem of Chapter 4 of [CG], we will omit the proof.
Theorem 6.5. *There is a natural surjective algebra homomorphism:*

$$\mathcal{U}(\mathfrak{sl}_n(\mathbb{C})) \to H(Z)$$

with kernel I_d is the two-sided ideal given by the annihilator of $(\mathbb{C}^n)^{\otimes d}$, the d^{th} tensor product of the fundamental representation $\mathbb{C}^n$ of $\mathfrak{sl}_n(\mathbb{C})$.

Note that we cannot expect to have an isomorphism because $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$ is infinite dimensional whereas $H(Z)$ is finite-dimensional.

Recall that any finite-dimensional irreducible $sl_n(\mathbb{C})$ representation has a highest weight $w_1\omega_1 + \ldots + w_{n-1}\omega_{n-1}$ where ω_i are fundamental weights.
Corollary 6.6. *Under the above isomorphism $H(\mathcal{F}_x)$ is the irreducible highest weight representation of $\mathfrak{sl}_n$ with highest weight $w_1\omega_1 + \ldots + w_{n-1}\omega_{n-1}$ and w_i is the number of $(i \times i)$-Jordan blocks in the Jordan normal form of x.*

Note that the sum of the weights $w_1 + \ldots + w_{n-1}$ gives a partition of d. It is known that the irreducible $sl_n(\mathbb{C})$ modules corresponding to partitions of d are precisely those that occur with non-zero multiplicity in the decomposition of $(\mathbb{C}^n)^{\otimes d}$. Clearly, by increasing d if necessary, we can exhibit any finite-dimensional irreducible representation of $\mathfrak{sl}_n(\mathbb{C})$ as $H(\mathcal{F}_x)$ for some x.

7 Modern approach to Springer theory

Let X be a topological space and $\mathbb{K}$ be a field. A sheaf of $\mathbb{K}$-vector spaces, F , on X is a contravariant functor:

$$\mathcal{F} : \{\text{open sets in } X \text{ and inclusions}\} \to \{\mathbb{K}-\text{vector spaces and linear maps}\}$$

obeying the sheaf axiom : For any collection of open sets $\{V_i\}_{i\in I}$ in X and $s_i \in \mathcal{F}(V_i)$ that are compatible with one another in the sense that $s_{i|V_i\cap V_j} = s_{j|Vi\cap Vj}$, for all element $i, j \in I$ there exists a unique $s \in \mathcal{F}(\bigcup_{i\in I} V_i)$ such that $s_{|V_i} = s_i$ for all $i \in I$. We write $\mathcal{F}(U)$ for the image of U, and $r_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for the restriction map corresponding to the inclusion $V \to U$. The elements of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over U.

Definition 7.1. *A sheaf $\mathcal{F}$ on X is called locally constant if every $x \in X$ has a neighborhood U such that for all $y \in U$, the canonical map $\mathcal{F}(U) \to \mathcal{F}_y$ is an isomorphism. A local system is a locally constant sheaf with finite dimen- sional stalks. If X is connected then all these stalks automatically have the same dimension. This dimension is called the rank of the local system.*

The category of sheaves of $\mathbb{C}$-vector spaces is denoted by $Sh(X)$. It is an abelian category, i.e. the notions of injections, surjections, kernels, cokernels, exact sequences, adding maps betwee the same pair of sheaves, and direct sums all make sense and have the usual properties.

Define the category $\mathcal{K}^b(Sh(X))$ as the category whose objects are finite complexes of sheaves on X

$$A^\bullet = (0 \to A^{-m} \to A^{-m+1} \to \ldots \to A^{n-1} \to A^n \to 0) \quad m, n \geq 0$$

where each A^i is a sheaf on X. The morphisms are morphisms of complexes of $A^\bullet \to B^\bullet$ commuting with the differentials. Given a complex of sheaves $A^\bullet$, we let

$$\mathcal{H}^i(A) = Ket(A^i \to A^{i+1})/Im(A^{i-1} \to A^i)$$

denote the i-th cohomology sheaf. A morphim of complexes is called a quasi-isomorphism provided it induces isomorphisms between cohomology sheaves.

The derived category $D^b(Sh(X)$ is by definition the category with the same objects as $\mathcal{K}^b(Sh(X))$ and with morphisms which are obtained from those in $\mathcal{K}^b(Sh(X))$ by formally inverting all the quasi-isomorphisms (each hom space is obtained by localisation at quasi-isomorphisms); thus quasi-isomorphisms become isomorphisms in the derived category.

The notion of "isomorphism" in $D^b(Sh(X))$ is defined so as to insure that any object in $D^b(Sh(X))$ can be represented by a complex of injective sheaves.

From now on assume that X is a complex algebra variety. A sheaf $\mathcal{F}$ on X is said to be **constructible**, if there is an algebraic stratification $X = \sqcup_\alpha X_\alpha$ such that for each α, the restriction of $\mathcal{F}$ to the stratum X_α, is a locally constant sheaf of finite dimensional vector spaces. An object $A \in D^b(Sh(X))$ is said to be constructible complex if all the cohomlogy sheaves $\mathcal{H}^i(A)$ are constructible.

Definition 7.2. *Let $D^b_c(X)$ to be the full subcategory of $D^b(Sh(X))$ formed by constructible complexes.*

The category $D^b_c(X)$ is called the bounded derived category of constructible sheaves in spite of the fact that it is not the derived category of construcrible sheaves.

Recall that if $f : X \to Y$ is a map and $\mathcal{F}$ is a sheaf of $\mathbb{C}$-vectorr spaces on X, we can push it forward by

$$f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)$$

The derived functor of f_* is denoted by $Rf_* : D^b_c(X) \to D^b_c(Y)$. (Recall that derived functor is defined by replacing an object with a complex of injective sheaves and then applying the functor to that complex). After sufficient familiarity, one usually writes f_* instead of Rf_* for the derived functor of pushforward.

While at it, let me mention that there is also $f_! : D^b_c(X) \to D^b_c(Y)$ called the push-forward with proper supports defined by for a sheaf $\mathcal{F}$ on X by:

$$f_!\mathcal{F}(U) = \{s \in \mathcal{F}(f^{-1}(U)) | f : Supp(s) \to U \text{ is proper .}\}$$

There are also adjoint functors to these, given by pullbacks f^* and $f^!$. Namely, there are canonical isomorphisms:

$$Hom(f^*A_2, A_1) = Hom(A_2, f_*A_1) \text{ and } Hom(f_!A_2, A_1) = Hom(f^!A_2, A_1)$$

The cohomology of the fibers of a map are precisely the local invariants of the derived pushforward of the constant sheaf along the map. Global results about the pushforward will imply local results about the cohomology of the fibers. For example, much of Springer's theory of Weyl group representations is encoded in the following statement

Theorem 7.3. *(Springer correspondence) The restriction of the derived pushforward $R\mu_*\mathbb{C}_{\tilde{\mathfrak{g}}}$ of the constant sheaf on $\tilde{\mathfrak{g}}$ to the open locus of regular semisimple elements $\mathfrak{g}^{rs} \subset \mathfrak{g}$ is a local system with monodromies given by the regular representation of the Weyl group of $\mathfrak{g}$. The entire pushforward is the canonical intersection cohomology extension of this local system.*

The above statement is an application of the following very deep result of Beilinson-Bernstein-Deligne:

Theorem 7.4. *(Decomposition Theorem). Let $\mu : M \to S$ be a projective map of varieties with M smooth. The pushforward $R\mu_*\mathbb{C}_M$ is a direct sum of (shifted) intersection cohomology sheaves of local systems on subvarieties of S. Furthermore, the local systems giving the twisted coefficients are semisimple.*

We haven't defined intersection cohomology yet, but let us discuss a very special case of the above decomposition theorem.

Suppose $f : X \to Y$ is a C^∞ fiber bundle with smooth compact fiber F , let $\mathcal{H}^j(F)$ denote the local system on Y whose fiber at the point $y \in Y$ is $H(f^{-1}(y))$. There are the associated Leray-Serre spectral sequence

$$E_2^{i,j} = H^i(Y; \mathcal{H}^j(F)) \to H^{i+j}(X)$$

and the monodoromy representation $\rho_i : \pi_1(Y, y_0) \to GL(H^i(F))$.

Note that even if Y is simply connected, the spectral sequence can be non-trivial, as can be seen from the example of the Hopf fibration $f : S^3 \to S^2$.

We define a family of projective manifolds to be a proper holomorphic submersion $f : X \to Y$ of nonsingular varieties that factors through some product $Y \times \mathbb{P}^N$ and for which the fibers are connected projective manifolds. The nonsingular hypersurfaces of a fixed degree in some projective space give an interesting example. By a classical result of Ehresmann, such a map is also a C^∞ fiber bundle. The results that follow are due to Deligne.

Theorem 7.5. *(Decomposition and semisimplicity for families of projective manifolds) Suppose $f : X \to Y$ is a family of projective manifolds. Then*

- *The Leray spectral sequence degenerates at the E^2 -page and induces an isomor- phism*

$$H^i(X) = \bigoplus_{a+b=i} H^a(Y; \mathcal{H}^b(F))$$

- *The monodromy representation is semisimple: it is a direct sum of irreducible representations.*

The second part of the theorem is remarkable because the fundamental group of Y can be infinite.

This is a special case of the following result on intersection homology:
Theorem 7.6. *Let $f : X \to Y$ be a proper map of varieties. There exist finitely many triples (Y_a, L_a, d_a) made of locally closed, smooth and irreducible algebraic subvarieties $Y_a \subset Y$, semisimple local systems L_a on Y_a and integer numbers d_a , such that for every open set $U \subset Y$ there is an isomorphism*

$$IH^r(f^{-1}U) = \bigoplus_a IH^{r-d_a}(U \cap Y_a, L_a)$$

In particular, if X is smooth, taking $U = Y$, we get a decomposition of $H^*(X)$.

So, what are these intersection cohomology complexes? They are the simple (in the sense of irreducible) objects of an abelian heart of the triangulated category $D^b_c(X)$. The heart is not the full subcategory of constructible sheaves, rather it is the full subcategory of *perverse sheaves.*

7.1 Intersection homology and Perverse sheaves

(We follow the beautiful exposition given in [KW]).

We begin with a topological treatment of intersection homology. For L compact Hausdorff topological, let $C(L)$ denote the open cone on L, i.e. the result of identifying the subset $L \times \{0\}$ of $L \times [0, 1)$ to a single point.
Definition 7.7. *A topological stratified space is defined inductively on dimension. A 0-dimensional stratified space X is a countable set with the discreet topology. For $n > 0$, an n-dimensional topological stratification of X is a filtration*

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \ldots \subset X_{n-1} \subset X_n = X$$

of X by closed subspaces such that for each i and for each point x of $X_i \setminus X_{i-1}$, there exists a neighborhood $U \subset X$ of x in X, a compact $n - i - 1$-dimensional stratified space L, and a filtration-preserving homeomorphism

$$U \cong \mathbb{R}^i \times C(L)$$

If X is a topologically stratified space, the i-dimensional stratum of X is the space $X_i \setminus X_{i-1}$. Connected components of $X_i \setminus X_{i-1}$ are called strata and L is called the link of the stratum on which x lies.

A topological pseudomanifold of dimension n is a paracompact Hausdorff topological space X which possesses a topological stratification such that

$$X_{n-1} = X_{n-2}$$

and $X \setminus X_{n-1}$ is dense in X.

An important class of pseudomanifolds is the class of complex quasi-projective varieties. In fact, this is a subclass of the class of piecewise-linear pseudomanifolds, i.e. X has a triangulation such that each X_j is a union of simplices.

Geomertric realization of a simplicial complex is an n-dimensional piecewise-linear pseudomanifold if and only if every simplex is a face of an n-simplex, and every $(n-1)$-simplex is a face of precisely two n-simplices.

Let $T : |K| \to X$ be a triangulation of X compatible with the stratification. Write $C_i^T(X)$ for the space of all (finite) simplicial i-chains of X with respect to T.
Definition 7.8. *The support $|\xi|$ of a simplicial i-chain $\xi = \Sigma_{\sigma \in K^{(i)}} \xi_\sigma \sigma$ is given by*

$$|\xi| = \bigsqcup_{\xi_\sigma \neq 0} T(\sigma)$$

The intersection complex $IC_i^T(X)$ is a subcomplex of $C_i^T(X)$ whose elements are those i-chains ξ such that there are restrictions on the intersections of $|\xi|$ and X_j for each j. These restriction are imposed by a "perversity function".
Definition 7.9. *A perversity is a function $p : \{2, \ldots, n\} \to \mathbb{N}$ such that $p(2) = 0$ and $p(i+1) = p(i)$ or $p(i) + 1$.*

The most important examples of perversity are:

1. Zero-perversity: $p(i) = 0$ for all i. $(0, 0, \ldots, 0)$.
2. Top perversity: $p(i) = i - 2$. $(0, 1, 2, \ldots, n-2)$.
3. The (lower) middle perversity p is defined by $p(i) =$ integer part of $(i-2)/2$. $(0, 0, 1, 1, 2, 2, \ldots)$.
4. The (upper) middle perversity has values $p(i) =$ the integer part of $(i-1)/2$. $(0, 1, 1, 2, 2, \ldots)$.

Fix a perversity p. The key definition is the following:
Definition 7.10. *We say that an i-chain $\xi \in C_i^T(X)$ is p-allowable if*

$$\dim_{\mathbb{R}} |\xi| \cap X_{n-k} \leq i - k + p(k)$$

Note that since the triangulation T is compatible with the stratification, the intersection $|\xi| \cap X_{n-k}$ is a union of simplices and hence has a well-defined dimension (the largest dimension of any of the faces of the constituent simplices).

Since X_{n-k} has codimension k an i-chain is dimensionally transverse to it if

$$\dim_{\mathbb{R}} |\xi| \cap X_{n-k} \leq i - k$$

so, turning on the perversity value $p(k)$ tells us how much non-transversity is allowed.

We then define:
Definition 7.11. *Let $I^pC_i^T(X)$ to be the subspace of $C_i^T(X)$ consisting of all i-chains $\xi \in C_i^T(X)$ such that*

1. *ξ is p-allowable i-chain,*
2. *$\partial\xi$ is a p-allowable $(i-1)$-chain.*

We define $I^pC_i(TX)$ to be the colimit of $I^pC_i^T(X)$ over all triangulations of T compatible with the stratification (note that refinements of triangulation induce inclusion maps).

The intersection homology group of X with perversity p is defined to be

$$I^pH_i(X) = \frac{\ker \partial : I^pC_i(X) \to I^pC_{i-1}(X)}{im\partial : I^pC_{i+1}(X) \to I^pC_i(X)}$$

Remark 7.12. *In simplicial homology, one has that $H_*(X) = H_*^T(X)$ for any triangulation T. This is not true in intersection homology (one really needs the colimit over all triangulations). However, if the triangulation T is flag-like, i.e for each i the intersection of any simplex σ with the closure $\overline{X}_i$ is a single face of σ then $I^pH_*(X) = I^pH_*^T(X)$.*

We have the following fundamental result:
Theorem 7.13. *(Goresky-MacPherson) $I^pH_i(X)$ does not depend on the choice of stratification. (Indeed, any homemomorphism between topological pseudomanifolds induces isomorphism.)*

One defines singular intersection homology groups in a similar way. We say that a singular i-simplex $\sigma : \Delta_i \to X$ is p-allowable if

$$\sigma^{-1}(X_{n-k} - X_{n-k-1}) \subset (i - k + p(k))\text{-skeleton of } \Delta_i \quad \text{for} \quad k \geq 2.$$

Let's get to work and compute this in an example. Let X be the topological space given by suspension $\Sigma(A \sqcup B)$ where $A \cong B \cong S^1$. (i.e. twice pinched torus).

First, let's compute the usual homology. This is not too hard, let us pick two points a, b on the circles A and B respectively. These are homologous in because there is a 1-chain in X that connectc them. Thus, in fact, it is clear that $H_0(X) = \mathbb{K}$. Furthermore, the union of suspensions of a and b, $\Sigma(a) - \Sigma(b)$ gives us a 1-cycle that generates $H_1(X) = \mathbb{K}$ and finally, we have $H_2(X) = \mathbb{K} \oplus \mathbb{K}$ generated by $\Sigma(A)$ and $\Sigma(B)$.

Next, let us turn to intersection homology.

Let us choose the stratification to be $X_2 = X$, $X_1 = X_0$ to be the two singular points. By definition, the only perversity function has $p(X_0) = 0$.

Now, the points a and b are valid 0-cycles in $I^pH_0(X)$, however the 1-chain in X, call it ξ, that connects a and b is no longer allowed. Because it is no $(p, 1)$-allowable : We have $0 = \dim_{\mathbb{R}}|\xi| \cap X_0 > 1-2+0 = -1$. Therefore, in fact, $I^pH_0(X) = \mathbb{K}+\mathbb{K}$. Similary, the 1-cycle that $\Sigma(a) - \Sigma(b)$ is not allowed, and one can see easily that $I^pH_1(X) = 0$. Finally, the generators $\Sigma(A)$ and $\Sigma(B)$ are $(p, 2)$-allowable, as we have, for ex. :

$$0 = \dim_{\mathbb{R}}|\Sigma(A)| \cap X_0 \leq 2-2+0 = 0$$

Hence, we have $IH_2^p(X) = \mathbb{K} \oplus \mathbb{K}$.

We observe here a remarkable feature of intersection homology: While the usual homology $H_*(X)$ did not satisfy Poincaré duality, $IH_*^p(X)$ satisfies Poincaré duality! (Of course, this is not a coincidence.)

One extends the definition of (p, i)-allowable to locally finite chains in a straightforward way and the corresponding Borel-Moore type intesection homology group is denoted by $I^pH^{lf}(X)$.

We say an m-dimensional pseudomanifold is irreducible if $X_m \setminus X_{n-2}$ is connected, in which case $H_m(X;\mathbb{Z})$ is either $\mathbb{Z}$ or 0. If it is $\mathbb{Z}$ then we say that X is orientable and a choice of a generator for $H_m(X)$ is an orientation.

We call p and q to be complimentary perversities if $p(i) + q(i) = i - 2$ for all $i \geq 2$.

Theorem 7.14. *(Poincaré duality, Goresky and MacPherson) Suppose that X is oriented topological pseudomanifold of dimension d, and p and q are complementary perversities. Then, there is a non-degenerate bilinear form:*

$$I^pH_i(X) \times I^qH_{d-i}^{lf}(X) \to \mathbb{Q}$$

7.2 Sheaf-theoretic approach to intersection homology

(We continue following [KW]).

Let us fix a topological pseudomanifold X with a toplogical stratification defined by $X_0 \subset X_1 \subset \ldots X_{n-2} = X_{n-1} \subset X_n = X$.

We define the Borel-Moore complex of sheaves $\mathcal{S}_X^\bullet$ by

$$\mathcal{S}_X^{-i}(U) = C_i^{lf}(U)$$

The restriction map for $V \subset U$ is defined as follows: Given an i-simplex σ with $\text{im}(\sigma) \subset U$, define a set J of i-simplices in V as follows: If $\text{im}(\sigma) \subset V$, then $J = \{\sigma\}$; otherwise performa barycentric subdivision of σ. If τ is an i-simplex in the subdivision with $\text{im}(\tau) \subset V$ then add τ to J. Further subdivide those i-simplices in the subdivision with $\text{im}(\tau) \not\subset V$ and repeat.

It is a straightforward exercise to check that this gives a well-defined restriction map from locally finite chains in U to locally finite chains in V. Furthermore, the restriction map is compatible with the boundary map. Finally, one can check that we get a complex of sheves $\mathcal{S}_X^\bullet$ (i.e. gluing axiom is satisfied.)

The same construction works verbatim for intersection chains. A locally finite singular i-chain ξ is in the subspace $I^p\mathcal{S}_X^{-i}(U)$ if it is p-allowable.

Note that we worked with Borel-Moore chains to have a well-defined restriction map. Given U, we can identify Borel-Moore intersection chains, $I^pC_i^{lf}(U)$, with the global sections $\Gamma(U; I^p\mathcal{S}_X^{-i})$. On the other hand, the sections with compact supports $\Gamma_c(UlI^p\mathcal{S}_X^{-i})$ recovers the intersection chains $I^pC_i(U)$ on U.

We next will show that both $\mathcal{S}_X^\bullet$ and $I^pS_X^\bullet$ have vanishing higher cohomology. For this recall the following definition:

Definition 7.15. *A sheaf $\mathcal{S}$ on X is soft if for every closed subset $A \subset X$, the restriction map $\mathcal{S}(X) \to \mathcal{S}(A)$ is surjective, where $\mathcal{S}(A)$ is, by definition, $colim_{A\subset U}\mathcal{S}(U)$.*

A sheaf $\mathcal{S}$ on X is c-soft if for every compact subset $K \subset X$, the restriction map $\mathcal{S}(X) \to \mathcal{S}(K)$ is surjective.

For locally compact and countable at infinity topological spaces, for ex. a topological pseudomanifold, the notion of soft and c-soft sheaves agree. Furthermore, it is well-known from homological algebra that soft sheaves have vanishing higher cohomology.

Lemma 7.16. *For any $i \leq 0$, the sheaves $\mathcal{S}_X^i$ and $I^p\mathcal{S}_X^i$ are soft. Hence, in particular, we have $H^k(X; \mathcal{S}_X^i) = 0 = H^k(X; I^p\mathcal{S}_X^i)$ for $k > 0$.*

Proof. Let $K \subset X$ compact and suppose $\xi \in \mathcal{S}_X^i(K)$. We can represent ξ by a locally finite singular chain $\eta \in \mathcal{S}_X^i(U)$ for some open neighborhood U of K. Note that η does not have to be a locally finite singular chain in X (its simplices may accumulate at a boundary point of U). Now, every point $x \in K$, has a neighborhood V_x which meets only finitely many singular simplices of η. Take a finite subcover of the cover of K formed by V_x. There are finitely many simplices of η meeting this open cover. So, we pose $\tilde{\eta} = \sum a_k\sigma_k \in \mathcal{S}_X^i(U)$ where σ_k are the simplices of η that intersect K. Note that this is a *finite* sum. Hence, we can view $\tilde{\eta} \in \mathcal{S}_X^i(X)$ and this clearly restricts to $\xi \in \mathcal{S}^) i_X(K)$ as required.

The same proof applies verbatim to $I^p\mathcal{S}_X^i$ when intersection conditions are imposed.

□

For a complex of sheaves $\mathcal{S}^\bullet$ on X with vanishing higher cohomology, a completely standard spectral sequence argument gives that there is a quasi-isomorphism between the hypercohomology complex of $\mathcal{S}^\bullet$ and the complex $\Gamma(X; \mathcal{S}^\bullet)$ of global sections. Hence, we conclude that:

$$H^k(Rf_*(\mathcal{S}^\bullet)) = H^k(X; \mathcal{S}^\bullet)$$

where $f : X \to \{pt.\}$ is the constant map, whose pushforward is the glocal sections functor Γ. In particular, the hypercohomology of $\mathcal{S}_X^*$ computes the Borel-Moore homology of X, and the hypercohomology of $I^p\mathcal{S}_X^\bullet$ computes the Borel-Moore intersection homology.

The same argument applies for the compactly supported push-forward $f_!$. Then, we get

$$H_i(X) = H^{-i}(Rf_!\mathcal{S}_X^\bullet) \quad \text{and} \quad I^pH_i(X) = H^{-i}(Rf_!I^p\mathcal{S}_X^\bullet)$$

Therefore, the complex of sheaves $I^p\mathcal{S}^\bullet$ is now the main concentration. It turns that these sheaves have a very nice characterization inside the the derived category of constructible sheaves on X, which we wrote as $D_c^b(X)$ before (where the contructibility here is with respect to the given stratification on X).

Theorem 7.17. *(Goresky-MacPherson) Up to canonical isomoprhism in the contructible bounded derived category $D_c^b(X)$ (w.r.t to the given stratification on X), there is a unique complex of sheaves $\mathcal{F}^\bullet$ which satisfies:*

- *For any $x \in X$, the stalk cohomology $H^i(j_x^*\mathcal{F}^\bullet) = 0$ for $i < -n$, (where $j_x : \{x\} \to X$ is the inclusion.)*
- *For any $x \in X \setminus X_{n-2}$, the stalk cohomology $H^i(j_x^*\mathcal{F}^\bullet) = \mathbb{K}$ for $i = -n$ and 0 otherwise. Furthermore, the $-n$-th stalk cohomology forms a local system.*
- *For any $x \in X_{n-k} \setminus X_{n-k-1}$, the i-th stalk $H^i(j_x^*\mathcal{F}) = 0$ for $i > p(k) - n$.*
- *For any $x \in X_{n-k} \setminus X_{n-k-1}$, the i-th costalk $H^i(j_x^!\mathcal{F}) = 0$ for $i < -q(k)$. where $p(k) + q(k) = k - 2$.*

  Furthermore, the intersection homology sheaves $I^p\mathcal{S}_X^\bullet$ satisfies these axioms.

7.3 Deligne's construction

Suppose we have a fixed filtration $X = X_n \supset X_{n-2} \supset \ldots \supset X_0$ by closed subset of X. Let

$$i_k : X - X_{n-k} \to X - X_{n-k-1}$$

denote the inclusions.

If $\mathcal{F}^\bullet$ is a complex of sheaves on X and $r \in \mathbb{Z}$, we define the truncated complex $\tau_{\leq r}\mathcal{F}^\bullet$ to be the complex such that: $\tau_{\leq r}\mathcal{F}^i = \mathcal{F}^i$ for $i < r$,

$$\tau_{\leq r}\mathcal{F}^r = \ker(d : \mathcal{F}^r \to \mathcal{F}^{r+1})$$

and $\tau_{\leq r}\mathcal{F}^i = 0$ for $i > r$.

Theorem 7.18. *Suppose the filtration on X comes from a topological stratification of a pseudomanifold, then the complex of sheaves:*

$$\mathcal{P}^\bullet = \tau_{\leq p(n)-n}R(i_n)_* \ldots \tau_{\leq p(2)-n}R(i_2)_*\mathbb{C}_{X-X_{n-2}}[n]$$

satisfies the properties listed above. In particular, it is isomorphic to the intersection homology sheaves $I^p\mathcal{S}_X^\bullet$.

Teh above definition also gives us an easy way of construction intersection homology sheaves on X which extend a local system on $X \smallsetminus X_{n-2}$. Given a local system $\mathcal{L}$ on $X \smallsetminus X_{n-2}$, one poses:

$$I^p\mathcal{S}^\bullet{}_{(X,\mathcal{L})} = \tau_{\leq p(n)-n}R(i_n)_* \dots \tau_{\leq p(2)-n}R(i_2)_*\mathbb{C}_{X-X_{n-2}}[n]$$

Of course, one can also give an interpretation of this sheaf in terms of singular chain complex with twisted coefficients given in the local system. One just has to note that it suffices to give the local system only on $X \smallsetminus X_{n-2}$, since the allowability conditions on singular chains forces that interior of any singular chain maps to $X \smallsetminus X_{n-2}$.

7.4 Verdier duality

Recall that $Sh(X)$ stands for the category of sheaves of $\mathbb{C}$-vector spaces, and $\mathcal{S}_X^\bullet$ is a complex of sheaves of singular chains with compact support. We define the Verdier duality (contravariant) functor:

$$D_X : D^b(Sh(X)) \to D^b(Sh(X))$$

given by

$$\mathcal{F} \to R\mathcal{H}om_{Sh(X)}(\mathcal{F}, \mathcal{S}_X^\bullet)$$

where $R\mathcal{H}om$ is the right derived functor of the left exact functor $\mathcal{F} \to \mathcal{H}om_{Sh(X)}(\mathcal{F}, \mathcal{S}_X^\bullet)$.

Duality functor satisfies the following properties:

- D_X takes distinguished triangles to distinguished triangles and $D_X(\mathcal{F}^\bullet[1]) = (D_X\mathcal{F}^\bullet)[-1]$
- When $X = pt.$, then $D_{pt}V^\bullet$ is the complex with $(V^{-i})^\vee$ in degree i, and the differentials are duals of the differentials of $V^\bullet$.
- There is a natural map $\mathcal{F}^\bullet \to D^2\mathcal{F}^\bullet$ which is an isomorphism if $\mathcal{F}^\bullet$ has constructible cohomology.
- For U open in X, $D_U(\mathcal{F}^\bullet_{|U}) = (D_X\mathcal{F}^\bullet)_{|U}$.
- For any continuous map $f : X \to Y$, and $\mathcal{F}^\bullet$ in $D^b(Sh(X))$, there is a natural isomorpshim:

  $$D_Y Rf_*\mathcal{F}^\bullet \cong Rf_! D_X\mathcal{F}^\bullet$$

- Verdier duality preserves cohomological constructibility. Given a topological stratification of X, we have a duality functor $D_X : D^b_c(X) \to D^b_c(X)$

Exercise: Let $f : X \to \{pt.\}$ be the constant map $U \subset X$ open and $\mathcal{F}^\bullet$ a complex of sheaves. Then, we can compute:

$$H^i(U; D_X\mathcal{F}^\bullet) = H^i(R(f_{|U})_*\mathcal{F}^\bullet) \cong H^i(D_{pt}R(f_{|U})_!\mathcal{F}^\bullet) = H^{-i}(R(f_U)_!\mathcal{F}^\bullet)^\vee \cong H_c^{-i}(U; \mathcal{F}^\bullet)^\vee$$

An important consequence is that, if p and q are complementary perversities, then, we have a canonical isomorphism:

$$I^q\mathcal{S}_X^\bullet \cong D_X I^p\mathcal{S}_X^\bullet[n]$$

In particular, taking cohomology, we get:

$$I^q H_i(X) \cong I^p H^{lf}_{n-i}(X)^\vee$$

Hence, Poincaré duality in interection homology is a consequence of Verdier duality.

We would also like to note that Verdier duality relates the derived pull-back and push-forward maps associated to a map $f : X \to Y$ by the following formulae:

$$f_! = D_Y f_* D_X \quad , \quad f^! = D_X f^* D_Y$$

7.5 Perverse sheaves

Let X be a topologically stratified pseudomanifold. For simplicity, let us assume that all strata of X are even-dimensional (for ex. take any complex quasi-projective variety). Furthermore, let us concentrate in the case of the middle perversity, which has $p(2k) = k - 1$. Then, the last two conditions in the characterizasion theorem of Goresk-Macpherson say become:

$$H^i(j_x^* I^p \mathcal{S}_X^\bullet) = 0 \text{ for } \; i > (-\text{dim}S - \text{dim}X)/2$$
$$H^i(j_x^! I^p \mathcal{S}_X^\bullet) = 0 \text{ for } \; i < (\text{dim}S - \text{dim}X)/2$$

where x is a point in a stratum S of codimension > 0.

Relaxing the strict inequality to allow equality leads to the notion of perverse sheaf:
Definition 7.19. *A complex of constructible sheaves $\mathcal{F}^\bullet \in D_c^b(X)$ is a perverse sheaf if*

$$H^i(j_x^* \mathcal{F}^\bullet) = 0 \textit{ for } \; i \geq (-dimS)/2$$
$$H^i(j_x^! \mathcal{F}^\bullet) = 0 \textit{ for } \; i \leq dimS/2$$

for all $x \in X$ (including $S = X \setminus X_{n-2}$).

In particular, we have that $I^p \mathcal{S}_X^\bullet[-dimX/2]$ is a perverse sheaf.

The following fundamental results are due to Beilinson-Bernstein-Deligne:
Theorem 7.20. *The category of perverse sheaves $P(X)$ is a Abelian. The Verdier duality functor D_X preserves the perverse sheaves. In fact, it is an exact contravariant functor.*

The category $P(X)$ is Artinian and Noetherian. Every perverse sheaf $\mathcal{F}^\bullet$, has a finite length composition series:

$$0 \hookrightarrow \mathcal{F}_0^\bullet \hookrightarrow \mathcal{F}_1^\bullet \cdots \hookrightarrow \mathcal{F}_n^\bullet = \mathcal{F}^\bullet$$

for which the quotients $\mathcal{F}_i^\bullet / \mathcal{F}_{i-1}^\bullet$ are simple perverse sheaves, and the maximal length of any such composition series for $\mathcal{F}^\bullet$ is finite.

Furthermore, any simple perverse sheaf has the form $I^p \mathcal{S}_{Y,\mathcal{L}}^\bullet[-s]$ where Y is the closure of a connected stratum S of dimension $2s$ and $\mathcal{L}$ is an irreducibled local system on S.

We next discuss the notion of small and semi-small maps. Suppose $f : X \to Y$ is a proper surjective map of varieties with X nonsingular. We say that a topological stratification $Y = \bigsqcup_k S_k$ into locally closed subvarities is a stratification relative to f if $f : f^{-1}(S_k) \to S_k$ is a topologically trivial fibration.

It turns out that the following conditions are equivalent.

- $Rf_*\mathbb{C}_X[dimX]$ is a perverse sheaf on Y ,
- $\dim X \times_Y X \leq \dim X$,
- $\dim S_k + 2(\dim f^{-1}(S_k) - \dim S_k) \leq \dim X$ for all k.

We say that f is semismall if any of the above conditions hold and small if the inequality in the last condition is strict for all strata S_k which are not dense in Y . Call a stratum relevant if equality holds.

If f is small, then $Rf\mathbb{C}_X[n] = I^mH(Y, \mathcal{L})$, where $\mathcal{K} = (Rf_*\mathbb{C}_X)_{|Y_0}$ is the restriction of the (derived) push-forward to the open stratum.

For small and semismall maps, the Decomposition theorem takes on the following especially nice form.
Theorem 7.21. *(Decomposition theorem). If $f : X \to Y$ is semismall, then*

$$Rf_*\mathbb{C}_X[dimX] = \bigoplus_\alpha I^m\mathcal{S}^\bullet_{(S_k,\mathcal{L}_k)}$$

where the sum is over relevant strata S_k and local systems $\mathcal{L}_k$ on S_k (and m signifies the middle perversity.)

Finally, let us mention that it is easy to see from what we have proved that the Springer map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is a semi-small resolution. This gives us the Springer decomposition :

$$R\mu_*\mathbb{C}_{\tilde{N}}[\dim N] = \bigoplus_{\chi \in \pi_1(\mathbb{O}(x))} I^m\mathcal{S}^\bullet(\mathbb{O}_x, \mathcal{L}_\chi) \otimes V_{(x,\chi)}$$

To put this into context, we note that it can be shown by a diagram chase and Verdier duality that

$$End(R\mu_*\mathbb{C}_{\tilde{N}}, R\mu_*\mathbb{C}_{\tilde{N}}) \cong H^{lf}_{top}(Z)$$

where Z is the Steinberg variety. Namely, since $I^m\mathcal{S}^\bullet(\mathcal{O}_x, \mathcal{L}_\chi)$ are simple objects in the abelian category of perverse sheaves,

$$Hom(I^m\mathcal{S}^\bullet(\mathcal{O}_x, \mathcal{L}_\chi), I^m\mathcal{S}^\bullet(\mathcal{O}_{x'}, \mathcal{L}_{\chi'})) = 0$$

unless $x = x'$ and $\chi = \chi'$ in which case it is $\mathbb{C}$.

On the other hand, for a fixed $\mathbb{O}$ (equivalently $x \in \mathbb{O}$), we can see that $H^{lf}_{top}(\mathcal{B}_x) = \bigoplus_\chi \mathbb{C}_\chi \otimes V_{(x,\chi)}$, by computing the stalks:

$$H^{-\dim \mathbb{O}}(\mathcal{B}_x, \mathbb{C}_{\mathcal{B}_x}[dim \mathcal{N}]) = H^{-\dim \mathbb{O}}(R\mu \mathbb{C}_{\tilde{\mathcal{N}}}[dimN]) = \bigoplus (I^m \mathcal{S}^\bullet(\overline{\mathbb{O}}, \mathcal{L}_\chi \otimes V_{(x,\chi)}))_x = \bigoplus \mathbb{C}_\chi \otimes V_{\xi,\chi}$$

On the other hand,

$$H^{lf}_{top}(\mathcal{B}_x) = H^{-2\dim_{\mathbb{C}} \mathcal{B}_x}(B_x, i^!_x \mathbb{C}_{\tilde{\mathcal{N}}}[2\dim \mathcal{N}])) = H^{-\dim \mathbb{O}}(\mathcal{B}_x, \mathbb{C}_{\mathcal{B}_x}[dim \mathcal{N}])$$

where $i_x : \mathcal{B}_x \to \tilde{\mathcal{N}}$ is the inclusion, and we used the fact that $i^!_x = i^*_x[-2\dim \mathbb{O}]$ because $\mathcal{B}_x \to \mathbb{O}$ is a fibration.

In other words, taking the self-endomorphism of the perverse sheaf $\mu_* \mathbb{C}_{\tilde{N}}$ reproduces for us the isomorphism:

$$\mathbb{C}[W] = \bigoplus_{x,\chi} End_{\mathbb{C}}(H^{lf}_{top}(\mathcal{B}_x)_\chi)$$

that we saw earlier in the course.

Finally, let us end with mentioning that the geometric W-action can also be reconstructed this way on the Springer sheaf $R\mu_* \mathbb{C}_{\tilde{N}}$, by applying the specialization construction but we shall stop here. Time to take a break and digest what we have covered...