Lattices, Crystals & Tilings

M1R at Imperial

Yankı Lekili

20.03.2025

Overview

Lattices

Here are two equivalent definitions of a **lattice**.

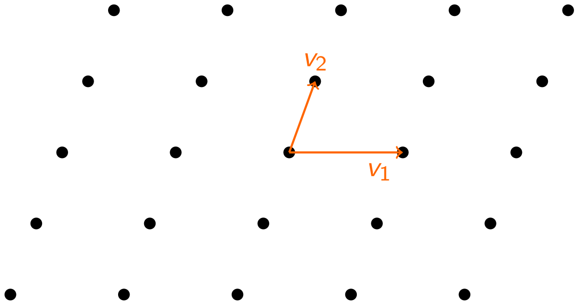
- A lattice in $\mathbb{R}^n$ is an infinite set of points with an arrangement and orientation that appears exactly the same, from whichever of the points of the set is viewed.
- A lattice consists of all points with position vectors of the form

$$m_1 v_1 + m_2 v_2 + \ldots + m_n v_n$$

  where $v_1, v_2, \ldots, v_n$ are any linearly independent vectors in $\mathbb{R}^n$, and $m_1, m_2, \ldots, m_n$ range through all integer values.

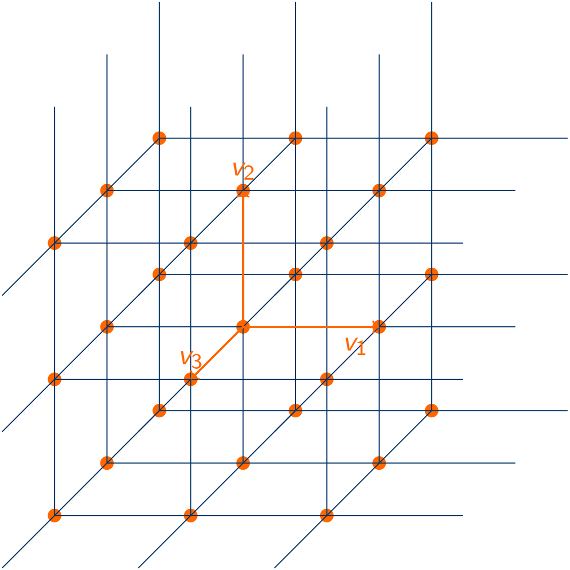
Think of a lattice as a grid of points in space, like the arrangement of atoms in a crystal or the pattern of tiles on a floor.

Example 2D Lattice

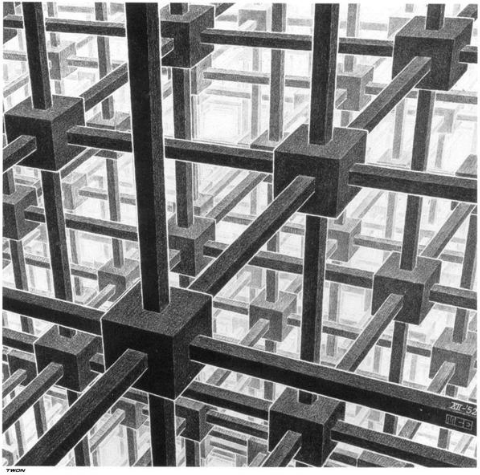


The vectors v_1 and v_2 define the basis of the lattice. All other points are generated by integer combinations of these vectors.

3D Cubic Lattice



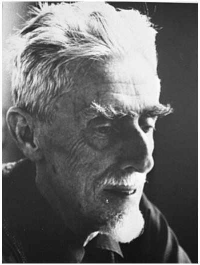
The vectors v_1, v_2, and v_3 define the basis of the 3D cubic lattice. Each point is generated by integer combinations of these vectors.

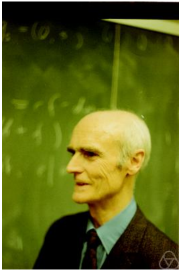


Cubic space division by M. C. Escher.

M.C. Escher's artwork often explores the concept of symmetry and tiling, making it a perfect visual representation of lattices.

A friendship



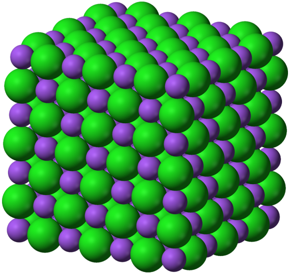


H. S. M. Coxeter (1907 – 2003) was a British-Canadian mathematician. He is regarded as one of the greatest geometers of the 20th century.

It is not surprising that Coxeter was a great friend and admirer of the Dutch graphic artist M.C. Escher. "Escher did it by instinct," Coxeter explained. "I did it by trigonometry."

Lattices in Chemistry

The atoms of a crystal are arranged in space in a discrete and extremely symmetrical way.



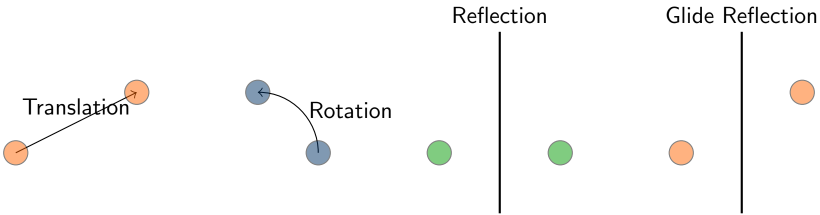
Microscopic structure of table salt. Purple is sodium ion Na^+, green is chlorine ion Cl^-.

The arrangement of sodium and chlorine ions in table salt forms a cubic lattice, which is a common structure in many crystals.

Symmetries

An **isometry** is a mapping of $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ onto itself preserving all distances.

$$\|\sigma(x) - \sigma(y)\| = \|x - y\| \quad \forall x, y \in \mathbb{R}^n$$



Theorem: Every isometry of $\mathbb{R}^2$ is of one of the above four types.

Proof: Elementary (cf. Martin - Transformation Geometry).

Group of Isometries of $\mathbb{R}^n$

- Every isometry of $\mathbb{R}^n$ can be described by a pair $\sigma = (A, v)$ consisting of a n-dimensional orthogonal matrix $A \in \mathrm{O}(n)$ and a n-dimensional vector v so that

$$\sigma(x) = Ax + v$$

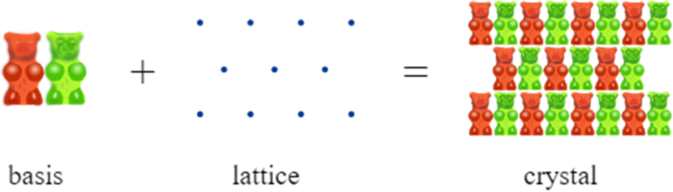
- We can compose isometries (A, v) and (B, w) as follows:

$$(A, v) \circ (B, w) = (AB, Aw + v).$$

- $\mathrm{Isom}(\mathbb{R}^n)$, the set of isometries of $\mathbb{R}^n$, is a group.

What are Crystallographic Groups?

A **crystal** is a lattice together with a "basis"[1] at each lattice point.



A crystallographic group $G \subset \mathrm{Isom}(\mathbb{R}^n)$ is the group of symmetries of a crystal, that is the set of all possible way you can move a crystal (rotate, translate, reflect) and still have it look the same.

[1]such as an atom or molecule or a motif

A mathematically precise definition

A n-dimensional **crystallographic group** G consists of *isometries* such that

1. any bounded region contains at most finitely many G-images of a given point,
2. the G-images of some bounded region cover all of $\mathbb{R}^n$.

Equivalently, we require that $G \subset \mathrm{Isom}(\mathbb{R}^n)$ is a subgroup such that each orbit $G \cdot x = \{gx \in \mathbb{R}^n : g \in G\}$ is a discrete subset of $\mathbb{R}^n$ and the **orbit space** $\mathbb{R}^n/G$ is a compact space.

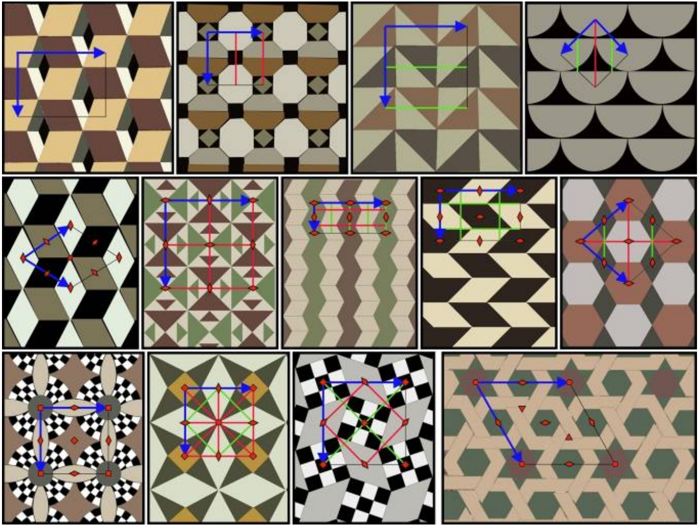
Exercise: Let $g_i = (A_i, v_i)$ for $i = 1, 2$ be given by

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Show that the subgroup $G = \langle g_1, g_2 \rangle \subset \mathrm{Isom}(\mathbb{R}^2)$ generated by g_1, g_2 is a crystallographic group, and describe the orbit space.

2-d Crystallographic Groups = Wallpaper Groups

Familiar as symmetry groups of tilings of the plane.



A collection of floor ornaments found in the San Marco Cathedral in Venice (Reference: Mehmet Erbudak)

Point groups and Translation groups

For a crystallographic group G, we have a group homomorphism

$$\begin{aligned} \rho : G &\to O(n) \\ (A, v) &\mapsto A \end{aligned}$$

- The subgroup of $\rho(G) \subset O(n)$ is called the **point group**.
- The kernel of ρ is called the the **translation group** $T(G)$.

Equivalently, there is an exact sequence of groups

$$0 \to T(G) \to G \to \rho(G) \to 0$$

Classification

Theorem (Bieberbach)

For a crystallographic group $G \subset \mathrm{Isom}(\mathbb{R}^n)$*, we have* $T(G) \simeq \mathbb{Z}^n$ *and* $\rho(G)$ *is finite.*

In particular, for any crystallographic group there exists a lattice (formed by a single orbit of $T(G)$) and G preserves this lattice. Thus, classification of crystallographic groups are understood in two steps:

- What is the underlying lattice, called the **Bravais lattice**, associated to G?
- What are the finite subgroups of the orthogonal group that preserve the given lattice?

Key Restriction

Only certain rotational symmetries are allowed in 2D and 3D lattices. Allowed orders: 1, 2, 3, 4, and 6.

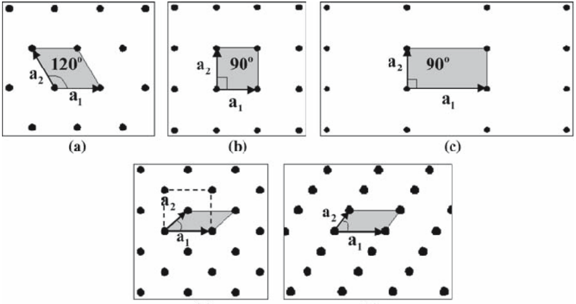
$$R^k = I \Rightarrow k = 1, 2, 3, 4, 6$$

Proof: If R is a rotational symmetry that preserves a lattice, then the matrix entries of R with respect to the basis of lattices vectors are integers, so $\mathrm{Tr}(R) \in \mathbb{Z}$. Because the trace is invariant under a change of basis

$$\mathrm{Tr} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 + 2\cos\theta \in \mathbb{Z}$$

Classification in 2d

In dimension two, there are exactly 5 types of Bravais lattices



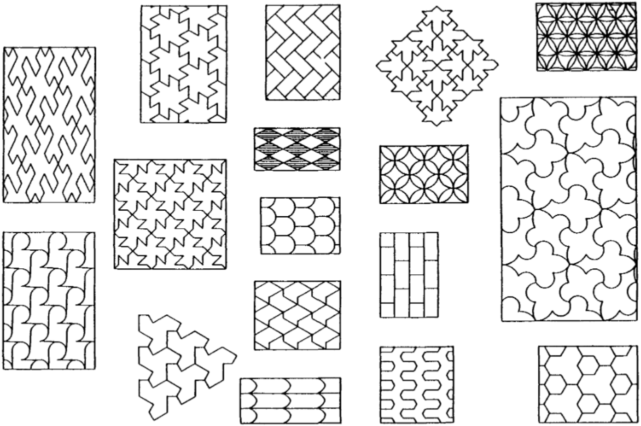
(a) (b) (c) (d) (e)

The corresponding point groups are:

$(a) C_1, C_2, C_3, C_6, D_1, D_2, D_3, D_6$	hexagonal,
$(b) C_1, C_2, C_4, D_1, D_2, D_4$	square,
$(c) C_1, C_2, D_1, D_2$	rectangular,
$(d) C_1, C_2, D_1, D_2$	rhombic,
$(e) C_1, C_2$	general.

Classification in 2d

There are exactly 17 two-dimensional wallpaper groups.



Picture from M. Artin - Algebra

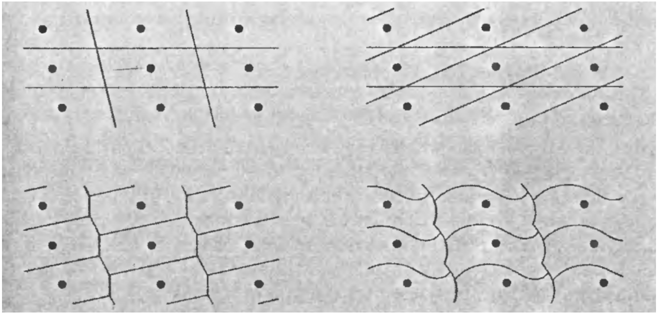
Higher dimensions

In dimension three, there are 230 three-dimensional crystallographic groups (14 Bravais lattices and 32 distinct point groups).
The exact number in higher dimensions is unknown but Bierbach also proved that in each dimension there are finitely many.

Dimension	**Number of Crystallographic Groups**
1D	2
2D	17
3D	230
4D	4894
5D	222097

Tilings

Given a lattice in $\mathbb{R}^2$, we can obtain a tiling of $\mathbb{R}^2$ by translating a "primitive unit cell" (Wigner-Seitz cell).



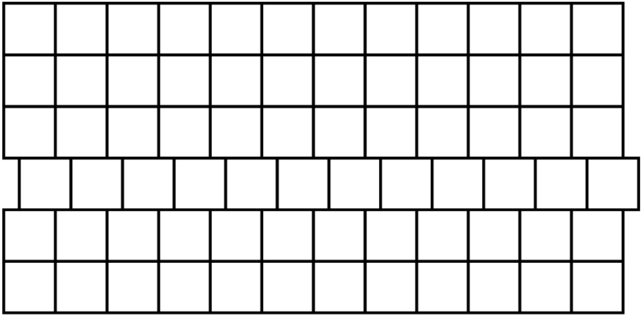
Several possible choices of primitive cell for a single two-dimensional lattice. (Reference: Ashcroft & Mermin)

These give us periodic tilings.

Tilings

A **plane tiling** is a countable family of closed set $\{T_1, T_2, \dots\}$ which cover the plane without gaps and tiles do not overlap except along boundaries.

A tiling is called periodic if it has translational symmetry. A tiling that cannot be constructed from a single primitive cell is called nonperiodic.

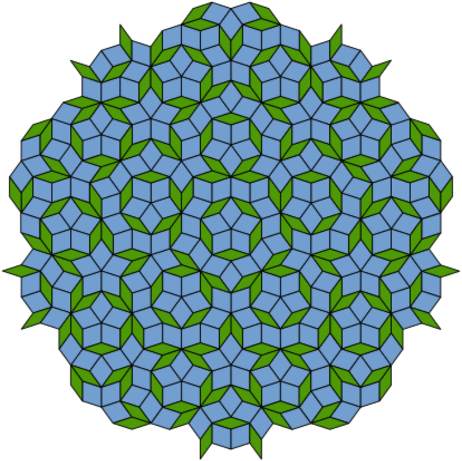


A nonperiodic tiling

Aperiodic tilings

If a given set of tiles allows **only** nonperiodic tilings, then this set of tiles is called **aperiodic**. The tilings obtained from an aperiodic set of tiles are often called aperiodic tilings, though strictly speaking it is the tiles themselves that are aperiodic.

Penrose tilings

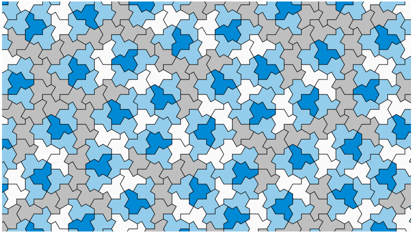


Penrose tilings are aperiodic. They never repeat exactly the same pattern, despite being highly ordered.

Aperiodic Monotiles: The Einstein

Breakthrough (2023): Mathematicians[2] discovered the first **aperiodic monotile**, nicknamed the **Einstein tile** (from the German "ein Stein" meaning "one stone").

- A single tile shape that can tile the plane **only non-periodically**. No repeating pattern exists, yet the tiling covers the plane without gaps or overlaps.
- The tile is a **polygon with 13 sides** (a "hat" shape).
- The tiling exhibits **quasicrystalline symmetry**, similar to Penrose tilings.



[2] Smith–Myers–Kaplan–Goodman-Strauss

Possible directions to explore for projects:

- Classification of crystallographic groups in dimension 2 (with proofs).
- Proof of Bieberbach theorem.
- Penrose's aperiodic tilings (expository).
- Aperiodic Einstein tilings (expository).
- The physics behind aperiodic tilings: Quasicrystals (expository)
- Classification of Frieze Groups. (with proofs)
- Describe the quotient spaces (orbifolds) $\mathbb{R}^2/G$ for the 17 crystallographic groups G.
- I'd be glad to hear from anyone (via email) who has another project idea in mind that relates to lattices and tilings. However, please check with me first to ensure that I consider the project to be at an appropriate level before you begin working on it seriously.

References:

- George E. Martin - Transformation Geometry (1982, Springer) (just the right level)
- Colin Adams - The tiling book (2023, AMS) (just the right level, pretty pictures!)
- Andrzej Szczepański - Geometry of Crystallographic Groups (2012, World Scientific)
- Francesco D'Andrea - A Guide to Penrose Tilings (2023, Springer)
- José M. Montesinos - Classical Tesselations and Three-Manifolds (1985, Springer) (somewhat advanced)
- B. Grünbaum & G. C. Shephard - Tilings and Patterns (1986, Freeman) (encyclopedic and includes many pictures and further references for inspiration)

Youtube videos (just for some distraction):

- Dr. Trefor Bazett - The Beauty of Symmetry: An Introduction to the Wallpaper Group
- minutephysics - Why Penrose Tiles Never Repeat
- Numberphile - A New Tile in Newtyle

A tiling database: `https://tilingsearch.mit.edu/`

17 wallpaper groups among traditional Japanese patterns:
`https://www.ms.u-tokyo.ac.jp/~tsuboi/urabe/public_html/pattrn/PatternE.html`

Thanks for listening!