Atiyah-Singer Index theorem

1 K-theory

1.1 Topological K-theory

Let X be a compact smooth manifold. The set of (isomorpism classes of) smooth $\mathbb{C}$-vector bundles over X is a monoid with respect to the direct sum operation $\oplus$, which we shall refer to as **Vec(X)**. $Vec(X)$ also has the tensor product operation $\otimes$. In **topological K-theory** we add formal inverses to form an abelian group **K(X)**. We will start with some of the algebra underpinning this.

Theorem 1.1. *(Adding Formal Inverses)*

Let A be a semi-group, there is a canonically defined abelian group B, such that $A \hookrightarrow B$ as a semi-group. Define $B = A \times A/\sim$, where $(E_1, V_1) \sim (E_2, V_2)$ means that there exists G, G' such that $(E_1 \oplus G, V_1 \oplus G) = (E_2 \oplus G', V_2 \oplus G')$.

We usually write $(E, F) = E - F$. For $A = Vec(X)$ adding formal inverses gives a ring **K(X)** (see exercise).

Exercise 1.2.

1. *Check that if the vector bundles $E_1 \oplus F_2$ and $E_2 \oplus F_1$ are isomorphic, then indeed $E_1 - F_1 = E_2 - F_2$ in $K(X)$.*
2. *Show that tensor $\otimes$ passes to the group $K(X)$ making it into a ring.*

Example 1.3. *A point and the circle. $K(Pt) \cong \mathbb{Z}$. This isomorphism is given by the rank of a trivial vector bundle. Similarly $K(S^1) = \mathbb{Z}$ since every $\mathbb{C}$-vector bundle on S^1 is trivial ($Gl_n(\mathbb{C})$ is connected).*

Example 1.4. *The sphere. $K(\mathbb{CP}^1) = \mathbb{Z}[H]/(1 - H)^2$. Where H is the K-theory class of the vector bundle $\mathcal{O}(1)$.*

- The relation $(1-H)^2 = 0$ can be seen from the Euler sequence :

$$0 \to \mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathcal{O}(2) \to 0$$

  [A geometric interpretation of this exact sequence is to consider the middle term as 1-homogeneous vector fields on $\mathbb{C}^2 \setminus 0$, which turn out to be in bijection with sections of $\mathcal{O}(1) \otimes \mathbb{C}^2 = \mathcal{O}(1) \oplus \mathcal{O}(1)$ (The key observation is that (smooth) sections of $\mathcal{O}(1)$ are in bijection with 1-homogeneous functions on $\mathbb{C}^2 \setminus 0$).The final term $\mathcal{O}(2)$ is isomorphic to the tangent bundle of $\mathbb{CP}^1$, the second map is push-forward of vector fields along the map $\mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$]. From this we see $K(\mathcal{O}(1) \oplus \mathcal{O}(1)) = K(\mathcal{O}) + K(\mathcal{O}(2))$.

- To see the above relation from an algebraic geometry perspective, for $p \in \mathbb{CP}^1$ consider the short exact sequence of sheaves:

$$0 \to \mathcal{O} \to \mathcal{O}(1) \to \mathcal{O}_p \to 0$$

  [Here $\mathcal{O}_p$ is the (skyscraper) sheaf defined to assign $\mathbb{C}$ to an open set containing p and 0 otherwise. Thinking of $\mathcal{O}(1)$ as the sheaf of holomorphic functions with a pole of order 1 at p, The quotient $\mathcal{O}(1)/\mathcal{O}$ will vanish away from p and at p we will be the sheaf given by the co-efficient of the order -1 term, which is $\mathcal{O}_P$. For $p \neq q$ the stalk of $\mathcal{O}_p$ at q is O and vice versa, we have that $\mathcal{O}_p \otimes \mathcal{O}_q = 0$ showing that $(H-1)^2$].

Exercise 1.5. *Using the Whitney sum formula show that we can extend Chern classes to $K(X)$, by $c.(E-F) = c.(E)/c.(F)$*

In addition to Chern classes of a vector bundle we may also define a **Chern Character**, which is closely related and gives a ring homomorphism from the K-theory ring to the cohomology ring of X. [The definition for line bundles is easy to state, let L be a line bundle and let $x = c_1 \in H^2(X, \mathbb{Z})$ its first Chern class. Then the chern character of L is defined by the formal sum:

$$ch(L) = e^x = 1 + \frac{1}{2!}x + \frac{1}{3!}x^3 + ... \in H^{2*}(X, \mathbb{Q})$$

Define $ch(L_1 \oplus ... \oplus L_N) = \sum ch(L_i)$, now it is clear how to extend the chern character to general vector bundles by appealing to the splitting principle]. There is a ring homomorphism $ch\colon K(X) \otimes \mathbb{Q} \to H^{2*}(X, \mathbb{Q})$.

The following definition is needed to talk about the Bott periodicity theorem. The restriction homomorphism $R : K(X) \to K(pt)$ simply counts the dimension of vector bundle classes and extends linearly to $K(X)$. Define the **Reduced K-theory** of X to be $\tilde{\mathbf{K}}(\mathbf{X}) = ker(R)$ i.e. the K-theory classes generated by $E - F$ where $Rank(E) = Rank(F)$. We may apply this to our previous examples:

The map $K(S^1) \to \mathbb{Z}$ is clearly an isomorphism hence $\tilde{K}(S^1) = 0$. One can also see that $\tilde{K}(S^2) = \mathbb{Z}(H-1)$, the generator $\mathbf{b} = H - 1 = [\mathcal{O}(1)] - [\mathcal{O}]$ is called the **Bott Class** and appears in the statement of the periodicity theorem.

1.2 Compactly supported K-theory and the Periodicity theorem

Suppose now that X is a locally compact space. Let $X^+ = X \cup \{\infty\}$ be the one point compactification of X. We have again a restriction $i_\star : K(X^+) \to K(\{\infty\})$.

Definition 1.6. ***The K-theory with compact support*** *of X is the reduced K-theory of the one point compactification X^+.* $\tilde{\mathbf{K}}(\mathbf{X}) = \tilde{K}(X^+, \{\infty\}) = Ker(i_\star)$.

[Before reading the statement of the p-theorem it is helpful to note that in K-theory (with compact support) there is an **outer product** $\boxtimes : K(X) \times K(Y) \to K(X \times Y)$. At the level of vector bundles this amounts to forming the vector bundle with fibre $E_x \otimes F_y$ over the point (x, y)]

With this in mind recall the generator $b = H - 1$ of $K(S^2) = K(\mathbb{R}^2)$.

Theorem 1.7. *The Periodicity Theorem. For a locally compact space the map $\alpha : \tilde{K}(X) \to \tilde{K}(X \times \mathbb{R}^2)$, $A \mapsto A \boxtimes b$ is an isomorphism.*

This reduces the problem of computing the reduced k-theory of spheres of any dimension to just S^1 and S^2. We compute: $\tilde{K}(S^{2k+i}) = \tilde{K}(\mathbb{R}^{2k+i}) = \tilde{K}(\mathbb{R}^i) = \tilde{K}(S^i) = 0, \mathbb{Z}$ for $i = 1, 2$.

2 Fredholm Operators

One way of obtaining K-theory classes is to take a 2-term complex of vector bundles

$$E_1 \xrightarrow{d} E_2$$

we get get a K-theory class $[E_1] - [E_2] = [ker(d)] - [Im(d)]$. In fact all K-theory classes occur like this if we allow E_i to be infinite dimensional. This motivates the study of Fredholm operators.

Fix V to be a separable Hilbert space over $\mathbb{C}$. [A vector space V with an inner product that induces a complete metric on V].

Definition 2.1. *Let $T : V \to V$ be a bounded linear map (the image of the unit ball is bounded).*

1. *We say that T is* ***Fredholm*** *if both $Ker(T)$ and $Coker(T)$ are finite dimensional.*
2. *The* ***index*** *of a Fredolm operator T is defined $index(T) = dim(Ker(T)) - dim(Coker(T))$.*
3. $\mathcal{F} := \{$ *Fredholm operators* $: V \to V\}$.

Example 2.2. *Consider the Hilbert space $L^2(\mathbb{Z}_+) = \{(a_0, a_1, ...) \in \mathbb{C}^\infty : \sum |a_i|^2 < \infty\}$. The maps*

$$T : (a_0, a_1, ...) = (0, a_0, a_1, ...)$$

$$T^{-1} : (a_0, a_1, ...) = (a_1, a_2, ...)$$

have index $-1, 1$ respectively.

In fact $Index(T^n) = n \ \forall n \in \mathbb{Z} \ T^{-1}$ is a left inverse of T. The operator $T^nT^{-n} - T^{-n}T^n$ is the finite rank operator:

$$(a_0, a_1, ...) \mapsto (a_0, a_1, ..., a_{n-1}, 0, ...)$$

2.1 Constructing Index bundles

The theory of Fredholm operators is closely related to K-theory: Suppose that we had a smoothly varying family of Fredholm operators T_x over a compact, connected space X. Then such a family gives a K-theory class in X, and all K-theory classes arise in this way.

Definition/Theorem 2.3. *Let $f : X \to \mathcal{F}$ be a family of Fredholm operators $\{T_x\}$. Then we may associate a canonically defined K-theory class $Ind(T) \in K(X)$.*

Proof. The map $Index : \mathcal{F} \to \mathbb{Z}$ is continuous so $Ind(T_x) = dim(Ker(T_x)) - dim(Coker(T_x))$ is constant. In the simplified situation where $dim(Ker(T_x))$ and hence $dim(Coker(T_x))$ are also constant we can give an explicit description of the index bundles.

$$E = \bigsqcup_{x \in X} Ker(T_x), F = \bigsqcup_{x \in X} Coker(T_x)$$

inherit the structure of vector bundles from the family. Define $Ind(T) = E - F$.

Now for the general situation consider the map $H_n : (a_0, a_1, ...) \mapsto (0, ..., a_n, ...)$ (assuming for simplicity that $V = L^2(\mathbb{Z}_+)$). Fredholm operators are invertible up to compact operators, so for each $x \in X$ $\exists N : Im(H_N T_x) = Im(H_N)$. By compactness of X there is a global choice of N that works for all x. Now the family $\{H_N T_x\}$ has Cokernel of constant dimension so apply the above construction. [HARD BIT: prove this is independent of n]. □

Exercise 2.4. *Show that all K(X) classes arise this way by considering the sequence*

$$0 \to Ker \to L^2(E) \to E \to 0.$$

[Here $L^2(E)$ is the trivial vector bundle of L^2 sections of E. The second map is $(x, s) \mapsto s(x)$]

Example 2.5. *Fredholm operators and winding numbers.*

Let $V = L^2(S^1)$ be the space of square integrable (complex) functions on S^1. All such functions have a Fourier series

$$f(z) = \sum_{k \in \mathbb{Z}} a_k z^k$$

Define the Hardy space $\mathcal{H}^2$ to be the subspace of functions with $a_k = 0$ for all $k < 0$.

Let $g : S^1 \to \mathbb{C}$ be a continuous function. Define an operator $T_g : \mathcal{H}^2 \to \mathcal{H}^2$. Note that for $f \in L^2$ $f * g \in L^2$. Define $T_f(g)$ to be the projection of $f * g$ to $\mathcal{H}^2$ by killing the negative Fourier coefficients.

T_g is Fredholm $\Leftrightarrow$ $Im(g) \subset \mathbb{C}^\star$. The nice fact about these Fredholm operators is that the index is the winding number of g around 0.

3 The Atiyah-Singer Index formula (INCOMPLETE)

[Let E,F be vector bundles of rank m and n respectively over X a manifold of dimension k. A **a rank k differential operator** is a map $D : \Gamma(E) \to \Gamma(F)$ satisfying the following: About each point choose an open set such that E and F are trivial. $s \in \Gamma(E)$ and $Ds \in \Gamma(F)$ may be expressed locally as $s : U \to \mathbb{R}^m$, $Ds : U \to \mathbb{R}^n$. We say that D is a differential operator of order k if locally $Ds = As$ where A is a matrix with coefficients of the form:

$$\sum f_{i_1,\ldots,i_j} \frac{\partial^j}{\partial x_{i_1} \ldots x_{i_j}}$$

and $j \leq k$.

Example 3.1.
- *The Laplace operator.* $\Delta = \sum \frac{\partial^2}{\partial x_i^2}$ *acting on complex functions is a differential operator of degree* 2.
- *The exterior derivative.* $d : \Omega^p \to \Omega^{p+1}$ *is a differential operator of degree* 1 *(exercise).]*

Suppose that D is a differential operator of degree k. Now for each 1-form $\omega \neq 0$ we define $\sigma(D)(\omega) \in Hom(E, F)$ called the **symbol** of D. This measures the top-order behaviour of the operator and may be described locally:

We form a new matrix A^P from A above by forgetting all differentials of order $< p$, now write the 1-form ω in local coordinates $\omega(p) = (v_1, \ldots, v_k)$. Replace each differential $\frac{\partial}{\partial x_i}$ in the matrix A^p with $v_i \in \mathbb{R}$. This matrix gives a linear map $E_p \to V_p$ and globally we have a bundle homomorphism $\sigma(D)(\omega) \in Hom(E, F)$ which is called the symbol.

Example 3.2. *Consider the cotangent bundle* $\pi : T_X^\star \setminus 0 \to X$*, we may pull back the bundles to obtain* $\pi^\star(E), \pi^\star(F)$*. Rearrange the above to show that the symbol can be expressed as a bundle morphism* $\sigma(D) : \pi^\star(E) \to \pi^\star(F)$*.*

Definition 3.3. *D is called an Elliptic operator if $\sigma(D)(\omega)$ is a bundle isomorphism for all $\omega \in \Omega_X^1 \setminus 0$.*

Example 3.4. *For the Exterior derivative we may check that* $\sigma(d)(\alpha) : \Omega^k \to \Omega^{k+1} = \cdot \mapsto \alpha \wedge \cdot$.

By computing the rank of Ω^k we see that this can't be an isomorphism in all but the most trivial cases, therefore d is not an elliptic operator. Define $d^\star : \Omega^{k+1} \to \Omega^k$ by

$$f dx_1 ... dx_p \mapsto \sum_i \frac{\partial f}{\partial x_i} dx_1 ... \hat{dx_i} ... dx_p$$

. Then $d^\star$ is the formal adjoint of d. $Ker(d) = coker(d^\star)$ and $coker(d) = ker(d^\star)$, and $d + d^\star : \Omega^\star \to \Omega^\star$ is an elliptic operator (this is seen by computing the symbol: $\sigma(d + d^\star)(a)(\cdot) = a \wedge \cdot + a \lrcorner \cdot$). Note that $(d + d^\star)(\Omega^{2\star}) \subset \Omega^{2*+1}$, restricting the operator in this way $Ker(d + d^\star) = \oplus H^{2\star}(X)$, $Coker(d + d^\star) = \oplus H^{2\star+1}(X)$.

It turns out that an elliptic complex is always a Fredholm operator ($\Gamma(E), \Gamma(F)$ considered as trivial bundles over X) and we see in this case $Index(D) = \sum dim(H^{2i}) - \sum dim(H^{2i+1}) = e(X)$ (exercise).

Elliptic operators are invertible up to lower order operators. Using the compact Rellich lemma (which states that $L^2_k \to L^2_{k-1}$ is a compact embedding) shows that

$$L^2(E) \xrightarrow{D} L^2(F)$$

is Fredholm. Index(D) depends only on the homotopy class of the map $T^*_X \setminus 0 \to Iso(E, F)$ given $\omega \mapsto \sigma(D)(\omega)$ and can provide information about the topology of X.

Consider $\pi^\star(E) - \pi^\star(F) \in K(T^\star(X))$. Subtracting $\rho^\star(F)$ to get a class $\tilde{\sigma} \in \tilde{K}(X)$.

We may embed $X \hookrightarrow N_X \hookrightarrow \mathbb{R}^n$. Suppose that N_X was trivial $N_X = X \times \mathbb{R}^m$ ($m = codim(X)$).

Now there exists an operator on the trivial bundle $\mathbb{R} \times \mathbb{C}$ over $\mathbb{R}$. Whose symbol over $T^\star\mathbb{R} \cong \mathbb{R}^2$, gives the Bott class $b \in \tilde{K}(\mathbb{R}^2)$. We want to form $D \boxplus B^{\boxplus m} \in \tilde{K}(T^\star X \times R^{2m}) = \tilde{K}(T^\star(N_X))$.