Classifying spaces, equivariant cohomology and localisation

1 Classifying spaces

The goal of this section is to introduce the notion of *classifying space* BG of a (Hausdorff) topological group G. This space has the property that isomorphism classes of principal G-bundles over a paracompact Hausdorff base space correspond bijectively to homotopy classes of maps from the base to BG, hence the name. More explicitly this correspondence is given by pullback of a *universal principal G-bundle* $EG \to BG$.

1.1 The whole world in an example

We begin by considering the special case of $G = GL(r, \mathbb{C})$. But first we need some preliminaries on Grassmannians. Denote by $Gr(k, \mathbb{C}^n)$ the Grassmannian of k-dimensional linear subspaces of $\mathbb{C}^n$. We then have a tautological short exact sequence of vector bundles on $Gr(k, \mathbb{C}^n)$:

$$0 \longrightarrow S(k, \mathbb{C}^n) \longrightarrow \underline{\mathbb{C}^n} \longrightarrow U(k, \mathbb{C}^n) \longrightarrow 0.$$

Here $S(k, \mathbb{C}^n)$ denotes the tautological subspace bundle on $Gr(k, \mathbb{C}^n)$, $\underline{\mathbb{C}^n}$ is the trivial bundle with fibre $\mathbb{C}^n$ over $Gr(k, \mathbb{C}^n)$ and $U(k, \mathbb{C}^n)$ is the obvious quotient bundle. If we define $Gr(\mathbb{C}^\infty, k) \coloneqq \varinjlim Gr(n-k, \mathbb{C}^n)$ and $U \coloneqq \varinjlim U(n-k, \mathbb{C}^n)$ we obtain a vector bundle $U \to Gr(\mathbb{C}^\infty, k)$ which we can think of as the tautological quotient bundle on the infinite quotient Grassmannian $Gr(\mathbb{C}^\infty, k)$.

Now let $E \to M$ be a complex vector bundle of rank r over a manifold M; for the present discussion it is irrelevant whether this is a smooth vector bundle or just a topological one. Denote by $\underline{\Gamma(E)}$ the trivial vector bundle with fibre $\Gamma(E)$ over M, where $\Gamma(E)$ is the space of sections of E. We then have a short exact sequence of vector bundles

$$0 \longrightarrow \ker \phi \longrightarrow \underline{\Gamma(E)} \xrightarrow{\phi} E \longrightarrow 0,$$

where the map ϕ is the obvious one given by evaluation of sections. Thinking of $\Gamma(E)$ as the vector space $\mathbb{C}^\infty$, this expresses E as a quotient of $M \times \mathbb{C}^\infty$, i.e. every fibre of E is a r-dimensional quotient of $\mathbb{C}^\infty$. This corresponds to having a map (continuous or smooth, depending on the chosen setting) $f \colon M \to Gr(\mathbb{C}^\infty, r)$ such that E is the pullback f^*U of the tautological quotient bundle U under f. When E is the tangent bundle of M we can think of this map f as a generalised Gauss map for the manifold. Denote by $\mathrm{Vect}^r_{\mathbb{C}}(M)$ the set of isomorphism classes of complex vector bundles of rank r over M and let $[M, Gr(\mathbb{C}^\infty, r)]$ be the set of homotopy classes of maps $M \to Gr(\mathbb{C}^\infty, r)$. Since pullbacks under homotopic maps are isomorphic we have sketched the proof of a bijective correspondende

$$\mathrm{Vect}^r_{\mathbb{C}}(M) \longleftrightarrow [M, Gr(\mathbb{C}^\infty, r)]. \tag{1}$$

For the present case of $G = GL(r, \mathbb{C})$ we write $BG \coloneqq Gr(\mathbb{C}^\infty, r)$.

Example 1.1. Consider the case $r = 1$. By choosing inner products on the $\mathbb{C}^n$ (e.g. the obvious ones) we obtain an isomorphism $Gr(\mathbb{C}^\infty, 1) \cong \mathbb{C}P^\infty$. Combining this with what we have seen so far we obtain a bijective correspondence

$$\mathrm{Vect}^1_{\mathbb{C}}(M) \longleftrightarrow [M, \mathbb{C}P^\infty].$$

Example 1.2. Now let $M = S^n$ be the unit n-sphere. One can show by means of clutching functions that $\mathrm{Vect}^r_{\mathbb{C}}(S^n)$ corresponds bijectively to $[S^{n-1}, G]$. But by what we have seen now we also know that $\mathrm{Vect}^r_{\mathbb{C}}(S^n)$ corresponds to $[S^n, BG]$. Thus we conclude that there is an isomorphism

$$\pi_n(BG) \cong \pi_{n-1}(G) \tag{2}$$

for every n (actually we only know that it is a bijection, we will however see shortly that it is indeed an isomorphism). Note that we are explicitly working with $G = GL(r, \mathbb{C})$ and $BG = Gr(\mathbb{C}^\infty, r)$ here, but the same argument does work for general classifying spaces once we have defined them.

Note that (1) doesn't quite fit into the general statement given in the introduction: there are no principal bundles mentioned here. It turns out however that there is a somewhat canonical correspondence between complex vector bundles of rank r and principal $GL(r, \mathbb{C})$-bundles (this works for other fields too). Namely we define the *frame bundle* of a given vector bundle $E \to M$ of rank r to be the principal $GL(r, \mathbb{C})$-bundle $F(E) \to M$ whose fibre over a point is the space of isomorphisms from $\mathbb{C}^r$ to the fibre of E over that point. Note that there is an obvious right action of $GL(r, \mathbb{C})$ on $F(E)$ which preserves the fibres and acts freely and transitively on them. Noting that pulling back commutes with the frame bundle construction we can translate what we have seen so far into a correspondence

$$\{\text{principal } GL(r, \mathbb{C})\text{-bundles over } M\}/\text{isomorphism} \longleftrightarrow [M, Gr(\mathbb{C}^\infty, r)].$$

The correspondence here is given by pulling back the frame bundle of $U \to BG$. We will denote it by $EG \to BG$ and say that it is the *universal principal G-bundle* (precisely because of the above correspondence).

We now want to see that the space EG is weakly contractible. This will be true in the general setting and is part of the allure of the universal principal bundle. Knowing that $BG = Gr(\mathbb{C}^\infty, r)$ is paracompact we obtain a long exact sequence of homotopy groups:

$$\cdots \longrightarrow \pi_i(G) \longrightarrow \pi_i(EG) \longrightarrow \pi_i(BG) \xrightarrow{\partial_*} \pi_{i-1}(G) \longrightarrow \cdots .$$

All the maps except ∂_* are the obvious ones. To construct ∂_* start by picking an element $[a] \in \pi_i(BG)$ and choosing a representative $a \colon S^n \to BG$. We can think of this map as a map $a \colon D^n \to BG$ on the closed unit n-disk that is constant on its boundary $\partial D^n = S^{n-1}$. From the fact that $EG \to BG$ is a fibre bundle with paracompact base space we know that there is a lift $\tilde{a}\colon D^n \to EG$ of a. Since the image of S^{n-1} under a is a single point we know that its image under $\tilde{a}$ must be contained in a fibre. Thus we have $\tilde{a}|_{S^{n-1}}\colon S^{n-1} \to G$ and we can define

$$\partial_*([a]) \coloneqq [\tilde{a}|_{S^{n-1}}] \in \pi_{n-1}(G).$$

The fact that this is well-defined follows from the homotopy lifting property for fibre bundles over a paracompact base. If we now knew that ∂_* is an isomorphism we'd be done. This is indeed true as the following exercise shows:

Exercise 1.3. Meditate on the fact that ∂_* is the same map as the one in (2).

Exercise 1.4. Think about what the above construction of ∂_* looks like explicitly in the case of the Hopf fibration $S^1 \hookrightarrow S^3 \longrightarrow S^2$.

Exercise 1.5. Let $\mathbb{Z}$ be the group of integers under addition. Show that $B\mathbb{Z} = S^1$ and that $E\mathbb{Z} \to B\mathbb{Z}$ is $\mathbb{R} \to S^1$ with the map being the complex exponential. Do the above by noting that principal $\mathbb{Z}$-bundles on M are in correspondence with $H^1(M;\mathbb{Z})$ and that the latter is in correspondence with $[M, S^1]$.

Exercise 1.6. Think about what the total space of the pullback of $E\mathbb{Z} \to B\mathbb{Z}$ under the double cover $S^1 \xrightarrow{z^2} S^1$ looks like.

1.2 General groups

To define the classifying space and universal bundle for a general (Hausdorff) group we can start by constructing a principal G-bundle $EG \to BG$ with EG contractible. To do this we use the topological join construction:

Definition 1.7. Let X, Y be topological spaces. The *join* $X \star Y$ is then defined to be $(X \times Y \times [0,1])/\sim$, where the equivalence relation $\sim$ is generated by $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$ for $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$.

Note that the join of two spaces is just the space obtained by connecting every point of one space by the unit interval to every point of the other space. Moreover the construction is associative and commutative. We now define the universal bundle by setting $EG \coloneqq G \star G \star G \star \cdots$ and $BG \coloneqq EG/G$. One has to verify that this is indeed a principal G-bundle (in particular that it has local trivialisations), but that follows by thinking in an appropriately elegant way about EG.

Example 1.8. We have seen how to construct $EGL(r,\mathbb{C})$. If G has a faithful linear representation $G \hookrightarrow GL(r,\mathbb{C})$ then note that G acts freely on $EGL(r,\mathbb{C})$ and so we can just define $EG \coloneqq EGL(r,\mathbb{C})$ and $BG \coloneqq EGL(r,\mathbb{C})/G$.

We now have the following correspondence in general:

Theorem 1.9. *Let M be a paracompact Hausdorff space and let G be a Hausdorff topological group. There is a bijective correspondence*

$$\{\textit{principal } G\textit{-bundles on } M\}/\textit{isomorphism} \longleftrightarrow [M, BG],$$

where the map from right to left is given by pulling back $EG \to BG$ along a representative.

For Example 1.8 to fit into this general picture we have to mention another result:

Theorem 1.10. *Let G be a Hausdorff topological group. Then every principal G-bundle with contractible total space is a universal bundle in the sense of Theorem 1.9. In particular its total and base spaces are unique up to homotopy.*

Although a rigorous proof of Theorem 1.9 in the general topological setting requires some care regarding technicalities, we can easily argue why this should be true for the case of M a smooth manifold and G a compact Lie group. Namely let $P \to M$ be a principal G-bundle and consider the following diagram:

$$\begin{tikzcd} P \arrow[d] & EG \times P \arrow[l] \arrow[r] \arrow[d] & EG \arrow[d] \\ M & EG \times_G P \arrow[l, "p"'] \arrow[r, "\pi"] & BG \end{tikzcd}$$

Here $EG \times_G P \coloneqq EG \times P/\sim$ where $(eg, x) \sim (e, gx)$ for $e \in EG$, $x \in P$ and $g \in G$. Now note that the bundle $EG \times P \to EG \times_G P$ is the pullback of $EG \to BG$ under the obvious map π, and also that it is the pullback of $P \to M$ under the projection p. But we also have that $EG \times_G P \to M$ is a fibre bundle with fibre EG which is weakly contractible and so it has a global section $s\colon M \to EG \times_G P$. Finally it follows from what we have said so far that $P \to M$ is the pullback of $EG \to BG$ under $\pi \circ s$.

2 Equivariant cohomology

The goal now is to introduce a cohomology theory which is in some sense tailored to spaces that have a (smooth or continuous) group action on them. More specifically we note that if we have a topological group G acting on a space X, then the orbit space X/G generally won't be well-behaved unless the action is free. If it were free we'd be happy with looking at $H^*(X/G)$, but generally this won't cut it since in the case of a transitive action we'd loose all information. Thus we try to alter the space X in such a way that the action becomes free but the homotopy type of X doesn't change. Using the universal bundle construction from the previous section we can do precisely that by defining $X_G \coloneqq EG \times X/\sim$, where $(eg, x) \sim (e, gx)$ for every $e \in EG$, $x \in X$ and $g \in G$. This suits us because $EG \times X$ has a free G-action (since EG has one) and it has the same homotopy type as X. Finally we define the *equivariant cohomology of X with respect to G* to be

$$H^*_G(X) \coloneqq H^*(X_G).$$

Note that if G acts freely on X then we obtain what we wanted: $H^*_G(X) = H^*(X/G)$. If G is the trivial group then we just recover the usual cohomology of X. More generally if G acts trivially on X then we have $H^*_G(X) = H^*(BG \times X)$. In particular we note that $H^*_G(\text{point}) = H^*(BG)$ and so the theory is already much richer than ordinary cohomology.

There are two natural maps on X_G that are worth understanding. One is the induced projection $X_G \to X/G$. In general it is not a fibre bundle since the fibre over a free G-orbit is EG, the fibre over a fixed point is BG and the fibre over a general orbit Gx is $EG/G_x = BG_x$ where G_x is the stabiliser subgroup of $x \in X$. The other obvious map is $X_G \to BG$. This is indeed a G-bundle with fibre X. This is precisely because of how we constructed X_G: it's the associated bundle to the principal G-bundle $EG \to BG$ with respect to the G-space X. In particular we have a homomorphism $H^*(BG) \to H^*_G(X)$ which tells us that $H^*_G(X)$ is actually a $H^*(BG)$-module. We also have an obvious homomorphism $i^*\colon H^*_G(X) \to H^*(X)$ induced by the inclusion of X as fibre of X_G (all such inclusions are homotopic and thus induce the same map in homology).

Example 2.1. If $G = \mathbb{C}^*$ and we work with coefficients in $\mathbb{C}$ then $H^*(BG) = H^*(\mathbb{C}P^\infty) = \mathbb{C}[t]$ and so $H^*_G(X)$ carries a $\mathbb{C}[t]$ action.

Exercise 2.2. Note that $BS^1 = B\mathbb{C}^*$. Now assume that X has a free S^1-action. Show that the action of $\mathbb{C}[t]$ on $H^*_{S^1}(X)$ is given by t being the first Chern class of the S^1-bundle $X \to X/S^1$.

3 Localisation

We have seen that we can define a sensible cohomology theory for group actions that aren't necessarily free. It would be nice however if there were also some computational advantages. We might hope for that to be true since we have seen that the equivariant cohomology of a point is usually non-trivial and can be in fact infinite-dimensional. If for example the action of G on X has finitely many fixed points we might hope to get an inclusion $H^*_G(X) \hookrightarrow H^*_G(X^G)$, where X^G is the fixed point set in X under G. This isn't true in general, but under fairly weak assumptions and modulo torsion (in a very general sense) this is actually an isomorphism. Thus we can in some sense *localise* to the fixed point locus.

For simplicity of exposition assume that $G = S^1$ acts on a compact n-dimensional manifold X in a way such that every orbit is either free or fixed. We don't loose generality here since we work with $\mathbb{C}$ coefficients from now on and so any torsion caused by stabilisers in S^1 is killed anyways. Denote by $F \subseteq X$ the subset of fixed points.

Exercise 3.1. $F \subseteq X$ is a submanifold.

Proof. Let g be a Riemannian metric on X. Averaging the metric g by G ($G = S^1$ or any compact Lie group), we can suppose that G acts by isometry, i.e. $\tau_g : X \to X$, $x \to g \cdot x$, is an isometry of (X, g).

Suppose that the point $x \in F$ is not isolated and define the space of invariant vectors

$$H = \{v \in T_x X \,|\, d\tau_g(v) = v \quad \forall g \in G\}.$$

Note that $\exp_x(H) \subseteq F$. Indeed, since τ_g is an isometry, we have

$$\tau_g \cdot \exp_x(h) = \exp_{\tau_g x}(d\tau_g h) = \exp_x(h).$$

Let $y \in F \cap U$, $y \neq x$, where U is a neighbourhood of x such that $\exp_x : \exp^{-1}(U) \subseteq T_x X \to U$ is a diffeomorphism. Since $\exp_x$ is surjective onto U, there exists $v \in T_x X$ such that $y = \exp_x(v)$. Being y a fixed point,

$$\exp_x(v) = \tau_g \cdot \exp_x(v) = \exp_{\tau_g x}(d\tau_g v) = \exp_x(d\tau_g v).$$

By the injectivity of $\exp_x|_{\exp^{-1}(U)}$, we conclude that $v \in H$. Hence,

$$F \cap U = \exp_x(H) \cap U$$

and $\exp_x|_H$ is a local chart for F. $\square$

Let c be the codimension of F in X. Now consider the long exact sequence for cohomology of a pair:

$$\cdots \longrightarrow H^*(X, X \setminus F) \longrightarrow H^*(X) \longrightarrow H^*(X \setminus F) \longrightarrow H^{*+1}(X, X \setminus F) \longrightarrow \cdots.$$

By excision and the Thom isomorphism we have

$$\begin{aligned} H^*(X, X \setminus F) &\cong H^*(\nu_F, \partial\nu_F) \\ &\cong H^*_{\text{compact}}(\nu_F) \\ &\cong H^{*-c}(F), \end{aligned}$$

where ν_F is the normal bundle of F in X. By assumption $X \setminus F$ has a free G-action. Thus if we pass to equivariant cohomology we obtain

$$\cdots \longrightarrow H^{*-c}_G(F) \longrightarrow H^*_G(X) \longrightarrow H^*((X \setminus F)/G) \longrightarrow H^{*-c+1}_G(F) \longrightarrow \cdots.$$

This should be understood as a long exact sequence of $\mathbb{C}[t]$-modules. Note that $H^*((X \setminus F)/G)$ is finite dimensional as a $\mathbb{C}$-vector space. Since it is also a $\mathbb{C}[t]$-module it must be of the form $\mathbb{C}[t]/(t^k)$ for some $k \in \mathbb{N}$. We introduce the notation

$$\hat{H}^*_G(X) \coloneqq H^*_G(X) \otimes_{\mathbb{C}[t]} \mathbb{C}(t),$$

and call this *localisation*. Note that $\mathbb{C}(t)$ is a flat $\mathbb{C}[t]$-module and so if we localise the above exact sequence we obtain another exact sequence where $\hat{H}^*_G(X \setminus F) = 0$ and thus we obtain an isomorphism

$$\hat{H}^{*-c}_G(F) \xrightarrow{\cong} \hat{H}^*_G(X).$$

Similarly by considering the cohomology long exact sequence for the pair (X, F) we obtain another isomorphism

$$\hat{H}^*_G(X) \xrightarrow{\cong} \hat{H}^*_G(F).$$

We can now ask what the composition of the above two isomorphisms is. In the case of usual cohomology we have the following commutative diagram explaining the maps:

$$\begin{tikzcd} H^{*-c}(F) \arrow[r] \arrow[d, "\cong"] & H^*(X) \arrow[r] \arrow[d, "\cong"] & H^*(F) \arrow[d, "\cong"] \\ H_{n-*}(F) \arrow[r, "i_*"] & H_{n-*}(X) \arrow[r, "\cap F"] & H_{n-c-*}(F) \end{tikzcd}$$

The vertical isomorphisms are given by Poincaré duality. If we start at the top left with $1 \in H^{*-c}(F)$ and go around anticlockwise we end up getting mapped to the Euler class $e(\nu_F) \in H^*(F)$. Passing to equivariant cohomology again we have that the composition

$$H^{*-c}_G(F) \xrightarrow{i_*} H^*_G(X) \xrightarrow{i^*} H^*_G(F)$$

is given by cup product with the equivariant Euler class $e_G(\nu_F) \in H_G^*(F)$ (since ν_F carries an obvious G-action it induces a bundle on F_G and we understand its Euler class to be the equivariant Euler class of ν_F). By the above we now know that after localisation $e_G(\nu_F)$ is invertible in $\hat{H}_G^*(F)$. Combining everything so far we obtain the formula

$$\sigma = i_* \left(\frac{i^* \sigma}{e_G(\nu_F)} \right)$$

for $\sigma \in \hat{H}_G^*(X)$. Finally note that

$$\begin{aligned} \hat{H}_G^*(F) &= H_G^*(F) \otimes_{\mathbb{C}[t]} \mathbb{C}(t) \\ &\cong (H^*(F) \otimes_{\mathbb{C}} H^*(BG)) \otimes_{\mathbb{C}[t]} \mathbb{C}(t) \\ &\cong H^*(F) \otimes_{\mathbb{C}} (\mathbb{C}[t] \otimes_{\mathbb{C}[t]} \mathbb{C}(t)) \\ &\cong H^*(F) \otimes_{\mathbb{C}} \mathbb{C}(t). \end{aligned}$$

We have thus sketched the proof of

Theorem 3.2 (Atiyah–Bott–Berline–Vergne)**.** *Let $\sigma \in H^*(X)$ and assume that it comes from a $\tilde{\sigma} \in H_G^*(X)$ under the map $H_G^*(X) \to H^*(X)$. Then*

$$\int_X \sigma = \int_F \left[\frac{i^* \tilde{\sigma}}{e_G(\nu_F)} \right],$$

where the square brackets mean taking the constant term.

Exercise 3.3. Take S^1 to act on S^2 by rotation around the z-axis. Lift the volume form $\mathrm{vol} \in H^*(S^2)$ to $H_{S^1}^*(S^2)$, localise and calculate $\int_{S^2} \mathrm{vol}$ using Theorem 3.2.

Exercise 3.4. Show using the localisation theorem that a generic cubic surface in $\mathbb{P}^3$ contains 27 lines.