Line Bundles and Bend-and-Break

1 Line Bundles and the Kodaira Embedding

Let X be a compact complex manifold or a smooth algebraic variety, let say over $\mathbb{C}$. The content of this section is succinctly expressed by the mantra: 'line bundles of X give rational maps of X into projective space'.

In order to make precise this sentiment consider a line bundle $\mathscr{L}$ on X which admits a global section. To each $x \in X$, so long as the evaluation map ev_x on global sections of $\mathscr{L}$ is not identically zero, we can describe a hyperplane in $H^0(\mathscr{L})$ given by $\{s \in H^0(\mathscr{L}) \mid s(x) = 0\}$. This yields a rational map

$$X \stackrel{\Phi_{\mathscr{L}}}{\dashrightarrow} \mathbb{P}(H^0(\mathscr{L})^\vee).$$

The map $\Phi_{\mathscr{L}}$ is defined on the whole of X whenever ev_x is not, for all $x \in X$, identically zero. When this is the case we say $\mathscr{L}$ is *generated by sections* or base-point free.

Under suitable conditions the line bundle $\mathscr{L}$ will yield, via $\Phi_{\mathscr{L}}$, an embedding of X into projective space: in which case $\Phi_{\mathscr{L}}$ is known as the *Kodaira Embedding*. A line bundle $\mathscr{L}$ is called

- *very ample* $\Leftrightarrow$ $\Phi_{\mathscr{L}}$ is an embedding.
- *ample* $\Leftrightarrow$ $\Phi_{\mathscr{L}^N}$ is an embedding for some $N > 0$.
- *semiample* $\Leftrightarrow$ $\Phi_{\mathscr{L}^N}$ is a regular morphism for $N >> 0$.

Exercise 1. *Work with X an algebraic variety and $\mathscr{L}$ a very ample line bundle on X. Reconcile the image of $\Phi_{\mathscr{L}}$ with* Proj *of a suitable graded ring.*

Exercise 2. *Show that $\mathscr{L}$ is very ample if and only if $H^0(\mathscr{L})$ separates points and tangents of X,* i.e. *if and only if*

- *for all $x, y \in X$ there exists $s_x, s_y \in H^0(\mathscr{L})$ such that $s_x(x) \neq 0, s_y(y) \neq 0$ and $s_x(y) = s_y(x) = 0$,*
- *and for all $v \in T_xX$ there exists $s_x \in H^0(\mathscr{L})$ such that $s_x(x) = 0$ and $D_v s_x(x) \neq 0$.*

Exercise 3. *Rephrase the two conditions of the previous exercise algebraically, i.e., in terms of sections generating $L \otimes k(x)$. Here $k(x)$ is the residue field of x.*

Exercise 4. *Show that, so long as $\mathscr{L}$ is generated by sections, $\mathscr{L} \simeq \Phi^*_{\mathscr{L}}\mathscr{O}(1)$.*

Exercise 5. *Show that $\mathscr{L}$ is semi-ample if and only if $\mathscr{L}$ is the pull-back of an ample line bundle under a regular map.*

2 When is a Kähler manifold projective?

If $\mathscr{L}$ is an ample line bundle on a compact complex manifold X then X is projective and so restricting the Fubini-Study metric ω_{FS} on $\mathbb{P}^n$ endows X with the structure of a Kähler manifold. By Exercise 4 an integer multiple of the first chern class of $\mathscr{L}$ is the pullback along $\Phi_{\mathscr{L}}$ of $c_1(\mathscr{O}(1))$. Since $c_1(\mathscr{O}(1)) = [\omega_{FS}]$ we have that $c_1(\mathscr{L})$ is represented by a Kähler form. In fact:

Kodaira's Embedding Theorem. *If $\mathscr{L}$ is positive in the sense that $c_1(\mathscr{L}) \in H^2(X, \mathbb{R})$ is represented by a Kähler form then $\mathscr{L}$ is ample.*

Corollary. *Suppose X is a Kähler manifold. It is projective if and only if it has an integral Kähler form. We say a 2-form ω is integral if its class lies in the image of $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$.*

Exercise 6. *Prove this corollary.*

3 When is a line bundle ample?

In this section we describe a necessary and sufficient condition for a line bundle over a projective variety X to be ample in terms of intersection theory. If C is a curve in X and $\mathscr{L}$ a line bundle over X we define

$$\mathscr{L} \cdot C := \int_C c_1(\mathscr{L}).$$

For a curve $C \subset \mathbb{P}^n$, we can write $\mathscr{O}(1) \cdot C$ in a variety of ways:

- The number of zeroes of a section of $\mathscr{O}(1)$ restricted to C; recall that $c_1(\mathscr{O}(1))$ may be defined as the Poincare Dual to the locus of zeroes of a section of $\mathscr{O}(1)$.
- The volume of C with respect to the Fubini-Study metric; this is because $c_1(\mathscr{O}(1)) = \omega_{FS}$.
- The intersection pairing of C with H, the Poincare Dual to $c_1(\mathscr{O}(1))$. This H is a hyperplane since sections of $\mathscr{O}(1)$ are homogeneous linear polynomials in $x_0, \ldots, x_n$; their vanishing describes a hyperplane in $\mathbb{P}^n$.

In particular $\mathscr{O}(1) \cdot C > 0$. If $\mathscr{L}$ is ample over X then $\Phi_{\mathscr{L}^N}$ describes an embedding for some $N > 0$. If $C \subset X$ is a curve then

$$N(\mathscr{L} \cdot C) = \mathscr{L}^N \cdot C = \mathscr{O}(1) \cdot \Phi_{\mathscr{L}^N}(C) > 0.$$

Kleiman's Criterion. *Suppose X is a projective variety and $\mathscr{L}$ is a line bundle over X. Fix an ample line bundle $\mathscr{L}'$ on X and define, for each curve $C \subset X$, the degree $\deg C := \mathscr{L}' \cdot C$. A necessary and sufficient condition for $\mathscr{L}$ to be ample is that, for all curves $C \subset X$, $L \cdot C > \varepsilon \deg C$ for some positive constant ε (depending on $\mathscr{L}'$).*

Exercise 7. *Beware! There are line bundles with $L \cdot C > 0$ for all curves C which are not ample. Use google to find a counterexample.*

4 When $\Phi_{\mathscr{L}}$ is not an embedding

In this section take X a compact complex manifold over $\mathbb{C}$. If $\mathscr{L}$ is once more a line bundle over X what can we say about $\Phi_{\mathscr{L}}$ if $C \subset X$ is a curve with $\mathscr{L} \cdot C = 0$? If C has genus zero then

$\mathscr{L}\cdot C = 0 \Leftrightarrow \mathscr{L}|_C \simeq \mathscr{O}_C$. This is not true for curves of higher genus. For example if E is an elliptic curve and P_0, P_1 are distinct points on E then $\mathscr{O}_E(P_0 - P_1)$ is not a trivial bundle.

If $\mathscr{L}|_C$ is trivial then sections of $\mathscr{L}$ are constant along C (because sections are just functions $X \to \mathbb{C}$ and X is compact) and so $\Phi_{\mathscr{L}}$ is either ill-defined on all of C or maps each point of C onto a single point of projective space. If $\mathscr{L}$ is generated by sections then of course $\Phi_{\mathscr{L}}$ cannot be ill-defined. Also a general result says that for $Y \subset X$ a submanifold of codimension at least 2 sections of $\mathscr{L}|_{X\setminus Y}$ can be lifted to sections of $\mathscr{L}$; thus if $\dim X = 2$ then $\Phi_{\mathscr{L}}$ will always be defined on a neighbourhood of C.

Example. Let S be a surface and let $C \subset S$ be a genus zero curve with self-intersection -1. Let $\mathscr{L}$ be a very ample line bundle on S and set $d := \mathscr{L}\cdot C > 0$. The line bundle $\mathscr{L}(dC) := \mathscr{L}\otimes \mathscr{O}_S(dC)$ has first chern class $c_1(\mathscr{L}) + d[C]$ and so

$$\mathscr{L}\cdot C = d + \int_C d[C] = d + d[C]\cdot[C] = 0.$$

Thus $\mathscr{L}(dC)|_C$ is trivial and so $\Phi_{\mathscr{L}(dC)}$ contracts C to a point. If we interpret $\mathscr{O}_S(-C)$, as the subsheaf of $\mathscr{O}_S$ of functions which vanish along C then we see that $\mathscr{O}_S(-C)|_{S\setminus C} = \mathscr{O}_{S\setminus C}$. Hence $\mathscr{O}_S(dC)|_{S\setminus C}$ is trivial and so $\mathscr{L}(dC)|_{S\setminus C}$ is still very ample. In particular $\Phi_{\mathscr{L}(dC)}$ is an embedding of $S \setminus C$ into projective space which contracts C to a point.

Exercise 8. *Let $C \subset \mathbb{P}^2$ be a smooth cubic and let $Z = \{12 \text{ points on } C\}$.*

- *For generic Z show that there does not exist a non-trivial line bundle $\mathscr{L}$ on $\mathrm{Bl}_Z\,\mathbb{P}^2$ which is trivial (holomorphically or algebraically) on $\overline{C}$.*
- *However, show that if Z is the intersection of C with a quartic then $\mathscr{L}$ may be chosen trivial on $\overline{C}$. Further show that in this case $\Phi_{\mathscr{L}}$ contracts $\overline{C}$ to a singular surface.*

5 When a line bundle is not generated by sections

If $\mathscr{L}$ is a line bundle on an algebraic variety X, we write $\underline{H^0(\mathscr{L})}$ for $\mathcal{O}_X \otimes_{\mathbb{C}} H^0(\mathscr{L})$. We have a natural map

$$\underline{H^0(\mathscr{L})} \to \mathscr{L}$$

which acts via $(x, s) \mapsto (x, s(x))$. To ask that $\mathscr{L}$ be generated by sections amounts to asking that $\underline{H^0(\mathscr{L})} \to \mathscr{L}$ be surjective. What happens when this is not the case? TO DO!

Exercise 9. *Assume we are in the setting just described. Then $\Phi_{\mathscr{L}}$ is ill-defined along Z. Show that after blowing up along Z we get obtain regular map*

$$\mathrm{Bl}_Z\, X \xrightarrow{\Phi} \mathbb{P}(H^0(\mathscr{L})^\vee)$$

such that $\Phi^*\mathscr{O}(1) = \mathscr{L}(-E)$ where E is the exceptional divisor of the blow up.

Exercise 10. *Show that a degree one curve in $\mathbb{P}^3$ must be a line.*

Exercise 11. *Let Z be 6 distinct points in $\mathbb{P}^2$ and let $\mathscr{I}_Z$ be the ideal sheaf corresponding to Z. Exercise 9 provides us with a map*

$$\mathrm{Bl}_Z\,\mathbb{P}^2 \longrightarrow \mathbb{P}(H^0(\mathscr{O}_{\mathbb{P}^2}(3)\otimes \mathscr{I}_Z)^\vee)$$

- *Check the image is a surface of degree 3.*

- *Find* 27 *lines on it.*
- *Show any cubic surface has* 27 *lines and show that any configuration of* 6 *disjoint such lines can be blown down to give* $\mathbb{P}^2$*.*
- *Show that these lines have self-intersection* -1 *(use adjunction formula).*

6 The Minimal Model Program

In this final section we describe some results which make up the rudiments of what is known as the *minimal model program.* We wish to study K_X defined to be the line bundle $\det T_X^* = \bigwedge^{\text{top}} T_X^*$. One reason for this is because the global sections of K_X are a birational invariant of X (see Exercise 12). say somehing about positivity, kodaira dimension, rational curves, canonical models

Bend-and-Break Lemma. *Let* X *be a projective variety and* $C \subset X$ *a curve. Let* c *be a point on* C*. Suppose that hypothesis* $(*)$ *is verified (we'll tell you what* $(*)$ *is below). Then there exists a genus zero curve in* X *which contains* c*.*

We shall sketch a proof of this result. By blowing up we can assume that the curve C is smooth, of genus > 0 and that we have an embedding $f : C \to X$ with $f^*K_X \cdot C < 0$. Lets write p for the point in C with $f(p) = c$. The strategy is to look at a moduli space of deformations of f which all map p onto c. We assume the hypothesis:

$$\text{this moduli space has suitably large dimension (i.e., } > 0). \qquad (*)$$

Then we can choose a smooth curve D in this space of deformations. The upshot of this is that we obtain a morphism

$$\begin{aligned} \text{ev} :& D \times C \to X \quad \text{such that} \\ \text{ev}&(D \times \{p\}) = \{c\}. \end{aligned}$$

The maps $f_d := \text{ev}|_{\{d\}\times C} \to X$ are the deformations of f. We can compactify D to get a proper curve $\overline{D}$ and extend our evaluation map to a rational map

$$\overline{\text{ev}} : \overline{D} \times C \dashrightarrow X.$$

The key point is to observe that $\overline{\text{ev}}$ cannot possibly be defined everywhere on $\overline{D} \times \{p\}$. If it was then points $v \in C$ in a suitably small affine neighbourhood of p would have $\overline{\text{ev}}(\overline{D} \times \{v\})$ in a small affine neighbourhood of c. Since $\overline{D} \times \{v\}$ is proper this forces $\overline{\text{ev}}(\overline{D} \times \{v\})$ to be a point and so $f_d(v) = f(v)$ for all v in an affine neighbourhood of p and all $d \in \overline{D}$. This implies $f_d = f$ on a dense open subset and so $f_d = f$ on all of X. This is impossible.

Let Λ be the blow up of $\overline{D} \times C$ at all the points at which $\overline{\text{ev}}$ is not well-defined; we have regular maps $\varepsilon : \Lambda \to \overline{D} \times C$ and $e : \Lambda \to X$ such that $e = \overline{\text{ev}} \circ \varepsilon$. If $\overline{\text{ev}}$ is not defined at (d_0, p) then the fibre of d_0 under the projection $\Lambda \to \overline{D}$ consists of the strict transform of $\{d_0\} \times C$ and a copy of $\mathbb{P}^1$. Moreover e will not map all of this $\mathbb{P}^1$ to a point; the genus zero curve in X which is the image of this $\mathbb{P}^1$ is the curve we were looking for.

Theorem. *The hypothesis* $(*)$ *of the Bend-and-Break Lemma is verified if* $K_X \cdot C < 0$*.*

Again we shall only provide a very vague sketch of the proof of this result. It is possible to give the following lower bound on the space of deformations of our curve $f : C \to X$:

$$\underbrace{-K_X \cdot f_* C}_{>0} - g(C) \dim X.$$

If we could alter f so that it had arbitrarily high degree without changing the genus of $f(C)$ then we would be good. Unfortunately, in characteristic zero, this can only be done if C is an elliptic curve; compose f with the multiplication by n covering; if $g(C) > 1$ then we have a problem. If we were in characteristic > 0 then we might however not be such a bad position; powers of the Frobenius morphism have high degree but do not change the geometry of your curve. Thus in characteristic > 0 the theorem holds.

To get the theorem working in characteristic 0 we reduce mod p for all p and obtain curves mod p in X. Actually we don't necessarily reduce mod p because X need not have a model over $\mathbb{Z}$; instead we just adjoin to $\mathbb{Z}$ the coefficients of the polynomials defining X to obtain a ring R of finite type over $\mathbb{Z}$ and reduce mod $\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$. Some commutative algebra shows that $R/\mathfrak{m}$ is finite and that the maximal ideals are dense in $\operatorname{Spec} R$. This it turns out (by some theorems from algebraic geometry; see Hartshorne, Exercise 3.18-3.19) ensures that, since there exists rational curves in each $X \mod \mathfrak{m}$, there must be a rational curve in X.

Exercise 12. *Consider the natural map $\pi : \operatorname{Bl}_Z X \to X$. Show that π^* induces an isomorphism between $H^0(K_X)$ and $H^0(K_{\operatorname{Bl}_Z X})$.*