Blow ups

1 Introduction

In this lecture, we give a gentle introduction to the concept of blowing up – performing a particular birational transformation to a scheme or manifold X which allows one to look more closely at some zero locus Z inside it. When Z is smooth, the procedure basically consists of cutting Z out and replacing it with the projectivisation of its normal bundle. The blow up however is defined extrinsically, which makes it applicable in the case when Z is singular. This is particularly useful, as it allows one to analyse precisely the way that"tangent spaces come together" at the singular locus of Z.

2 First Example: Blowing up $\mathbb{C}^2$ at the origin

Following the outline in the introduction, we consider the problem of finding an algebraic variety which is obtained from $\mathbb{C}^2$ by cutting out the origin $\mathbf{0}$ and replacing it by the projectivisation of its normal bundle, which is of course just $\mathbb{P}^1$. Moreover we would like to glue in this $\mathbb{P}^1$ in such a way that when we approach the origin along a fixed complex line in $\mathbb{C}^2$, we arrive at the point in $\mathbb{P}^1$ which represents precisely this line.

The key idea is to consider the graph of the quotient map:

$$\begin{aligned} \mathbb{C}^2 \setminus \{0\} &\longrightarrow \mathbb{P}^1 \\ (x, y) &\mapsto [x : y]. \end{aligned}$$

Writing X and Y for the homogeneous coordinates on $\mathbb{P}^1$, this graph is the space

$$\{((x, y), [X, Y]) \subseteq \mathbb{C}^2 \setminus \{0\} \times \mathbb{P}^1 \, : \, xY = Xy\}.$$

Now we take the closure of this graph in $\mathbb{C}^2 \times \mathbb{P}^1$. The equation defining the graph still holds, so we just get

$$\{xY = Xy\} \subseteq \mathbb{C}^2 \times \mathbb{P}^1.$$

This defines the blow-up of $\mathbb{C}^2$ at $\mathbf{0}$ and we denote it as $\mathrm{Bl}_{\mathbf{0}}\mathbb{C}^2$. Let us verify that it is indeed the space we were after.

First note that we have the obvious projection

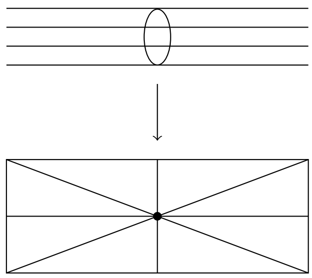
$$\pi : \mathrm{Bl}_{\mathbf{0}}\mathbb{C}^2 \subseteq \mathbb{C}^2 \times \mathbb{P}^1 \longrightarrow \mathbb{C}^2.$$

For any point $(x, y) \in \mathbb{C}^2 \setminus \{0\}$, there is a unique line through this point and the origin. Hence we get a unique point in $\mathbb{P}^1$ corresponding to this line, i.e. $\pi|_{\mathbb{C}^2 \setminus \{0\} \times \mathbb{P}^1}$ is a bijection. We also note that the preimage of $\mathbf{0} \in \mathbb{C}^2$ under

π is simply a copy of $\mathbb{P}^1$ since when $x = y = 0$, the variables X and Y are uncostrained.

Recall (see the lecture "Spec and Proj") that the tautological line bundle of $\mathbb{P}^1$ is defined as $\mathcal{O}_{\mathbb{P}^1}(-1) := \{(v, [v]) \in \mathbb{C}^2 \times \mathbb{P}^1\}$. So we see that the blow up of $\mathbb{C}^2$ at $\mathbf{0}$ is precisely $\mathcal{O}_{\mathbb{P}^1}(-1)$ but rather than viewing this space as a line bundle over $\mathbb{P}^1$, we now concentrate on the projection $\pi : \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathbb{C}^2$.

To visualise this maps, a good picture to keep in mind is the following.



In this picture the zero section of $\mathcal{O}_{\mathbb{P}^1}(-1)$ (i.e. the ellipse at the top of the picture) is contracted to the origin by the map π. This picture illustrates an important property: blowing up at the origin pulls apart directions at the origin. At the origin the blow-up remembers which line $\mathbf{0}$ is on, i.e. which line "I came in on." This is precisely the property we wanted the blow up to have.

3 Blowing up an affine variety along an ideal

After having seen this example, we now describe a general blow-up of an affine variety along an ideal.

We start by considering an affine variety $X \subseteq \mathbb{A}^n$ and an ideal I of the coordinate ring $\frac{\mathbb{C}[x_1,\ldots,x_n]}{\mathbb{I}(X)}$ of X. The zero locus

$$Z(I) = \{\mathbf{x} \in X \colon f(\mathbf{x}) = 0 \;\forall f \in I\}$$

defines a closed subvariety of X and the blow up of X along Z will be a variety isomorphic to X everywhere away from $Z(I)$. It is defined as follows:

Definition 3.1. Let $\{f_0, f_1, \ldots, f_r\}$ be a set of generators for I. We define the *blow-up* of X at I to be

$$\mathrm{Bl}_I X = \overline{\{(\mathbf{x}, [f_0(\mathbf{x}) : f_1(\mathbf{x}) : \ldots : f_r(\mathbf{x})]) \in \mathbb{A}^n \times \mathbb{P}^r \colon \mathbf{x} \in X \setminus Z(I)\}},$$

that is, it is the closure inside $\mathbb{A}^n \times \mathbb{P}^r$ of the graph of the morphism

$$F \colon X \setminus Z(I) \to \mathbb{P}^r, \quad \mathbf{x} \mapsto [f_0(\mathbf{x}) : f_1(\mathbf{x}) : \ldots : f_r(\mathbf{x})].$$

We see that this is well-defined, since at any point $\mathbf{x}$ in $X \setminus Z(I)$ at least one of the f_i does not vanish. Note that $\mathrm{Bl}_I X$ comes with a morphism $\pi \colon \mathrm{Bl}_I X \to X$ given by projection onto the first coordinate. We define

Definition 3.2. The *exceptional locus* of the blow up of X at I is defined to be $E = \pi^{-1}(Z(I))$.

Remark 3.3. You might be worried that the above notation doesn't make sense, at least not until we show that $\mathrm{Bl}_I X$ does not depend on the choice of generators for I. We do this in Proposition 3.4 below. Before that, we make the following observations:

1. Note that $\mathrm{Bl}_I X \setminus E$ is isomorphic to $X \setminus Z(I)$ since there the morphism π is invertible with inverse

$$\begin{aligned} \sigma\colon X \setminus Z(I) &\longrightarrow \mathrm{Bl}_I X \setminus E \\ \sigma\colon \mathbf{x} &\longmapsto (\mathbf{x}, [f_0(\mathbf{x}) : f_1(\mathbf{x}) : \ldots : f_r(\mathbf{x})]). \end{aligned}$$

2. Let us investigate more closely how the closure in Definition 3.1 is formed. For this purpose, we need to know how to express the Zariski closed subsets of $\mathbb{A}^n \times \mathbb{P}^r$. We can embed $\mathbb{A}^n \times \mathbb{P}^r$ into $\mathbb{P}^n \times \mathbb{P}^r$ and there we know from the Segre embedding that the closed subsets are intersections of vanishing sets of polynomials of the form $f(x_0, x_1, \ldots, x_n, X_0, X_1, \ldots, X_r)$ which are bihomogeneous in the x and X variables, that is, they satisfy

$$f(\lambda x_0, \lambda x_1, \ldots, \lambda x_n, X_0, X_1, \ldots, X_r) = \lambda^p f(x_0, x_1, \ldots, x_n, X_0, X_1, \ldots, X_r)$$
$$f(x_0, x_1, \ldots, x_n, \lambda X_0, \lambda X_1, \ldots, \lambda X_r) = \lambda^q f(x_0, x_1, \ldots, x_n, X_0, X_1, \ldots, X_r)$$

for some p, $q \in \mathbb{N}$ and all $\lambda \in \mathbb{C}$ (the pair (p, q) is then called the *bidegree* of f). Thus a basis for the closed subsets of $\mathbb{A}^n \times \mathbb{P}^r$ are precisely vanishing sets of polynomials $f(x_1, \ldots, x_n, X_0, X_1, \ldots, X_r)$, which are homogeneous only in the X-variables.

We now prove the independence of generators. This proof will become obsolete when we give a much more general definition of blow up, but it is useful for our current hands-on approach.

Proposition 3.4. Let $\{g_0, g_1, \ldots, g_s\}$ be another set of generators for I and let us temporarily write

$$\mathrm{Bl}_{(f_0,\ldots,f_r)} X = \overline{\{(\mathbf{x}, [f_0(\mathbf{x}) : f_1(\mathbf{x}) : \ldots : f_r(\mathbf{x})]) \in \mathbb{A}^n \times \mathbb{P}^r \colon \mathbf{x} \in X \setminus Z(I)\}}$$

and

$$\mathrm{Bl}_{(g_0,\ldots,g_s)} X = \overline{\{(\mathbf{x}, [g_0(\mathbf{x}) : g_1(\mathbf{x}) : \ldots : g_s(\mathbf{x})]) \in \mathbb{A}^n \times \mathbb{P}^s \colon \mathbf{x} \in X \setminus Z(I)\}}.$$

Then we have that

$$\mathrm{Bl}_{(f_0,\ldots,f_r)} X \cong \mathrm{Bl}_{(g_0,\ldots,g_s)} X$$

Proof. For this proof we may think of the ideal I as lying inside $\mathbb{C}[x_1, \ldots, x_n]$ and containing the vanishing ideal of X, so that $\{f_0, \ldots, f_r\} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ and $\{g_0, g_1, \ldots, g_s\} \subseteq \mathbb{C}[x_1, \ldots, x_n]$. Thus we have polynomial relations $f_i = \sum_j h_{ij} g_j$ and $g_l = \sum_m k_{lm} f_m$ for some $\{h_{ij}, k_{lm} \colon 0 \le i, m \le r,\ 0 \le j, l \le s\} \subseteq \mathbb{C}[x_1, \ldots, x_n]$. We define the morphism

$$\begin{aligned} \phi\colon\ \mathrm{Bl}_{(f_0,\ldots,f_r)} X &\longrightarrow \mathrm{Bl}_{(g_0,\ldots,g_s)} X \\ \phi\colon\ (\mathbf{x}, [X_0 : X_1 : \ldots : X_r]) &\longmapsto \left(\mathbf{x}, \left[\sum_{m=0}^{r} k_{0m}(\mathbf{x}) X_m : \ldots : \sum_{m=0}^{r} k_{sm}(\mathbf{x}) X_m\right]\right). \end{aligned}$$

We need to check that this is well-defined, i.e. that $\{\sum k_{0m}(\mathbf{x})X_m, \ldots, \sum k_{sm}(\mathbf{x})X_m\}$ cannot vanish simultaneously on $\mathrm{Bl}_{(f_0,\ldots,f_r)} X$. Consider the set of polynomials $\left\{\psi_i = X_i - \sum_{j=o}^{s} h_{ij} \left(\sum_{m=0}^{r} k_{jm} X_m\right) : 0 \le i \le r\right\}$. These are homogeneous in the X-variables and vanish on

$$\{(\mathbf{x}, [f_0(\mathbf{x}) : f_1(\mathbf{x}) : \ldots : f_r(\mathbf{x})]) \in \mathbb{A}^n \times \mathbb{P}^r : \mathbf{x} \in X \setminus Z(I)\}.$$

It follows from our discussion in Remark 3.3 that they must vanish on the whole of $\mathrm{Bl}_{(f_0,\ldots,f_r)} X$. Thus we can't have $\{\sum k_{0m}(\mathbf{x})X_m, \ldots, \sum k_{sm}(\mathbf{x})X_m\}$ all vanishing at a point in $\mathrm{Bl}_{(f_0,\ldots,f_r)} X$, since then vanishing of the ψ_i will imply $X_i = 0 \ \forall i$, a contradiction.

So F is well-defined and it is clearly invertible with an analogously constructed inverse morphism. Hence $\mathrm{Bl}_{(f_0,\ldots,f_r)} X \cong \mathrm{Bl}_{(g_0,\ldots,g_s)} X$. □

We now define one more notion, regarding subvarieties of the one we wish to blow up.

Definition 3.5. Let $Y \subseteq X$ be a closed subvariety different from Z. We define the *strict transform* of Y in $\pi\colon \mathrm{Bl}_I X \to X$ to be $\overline{\pi^{-1}(Y \setminus Z(I))}$.

Remark 3.6. Note that with this definition it is clear that the strict transform of Y is in fact $\mathrm{Bl}_{I \cap \mathbb{I}(Y)} Y$. This is the crucial **functoriality** property of blow ups - we can compute the blow up of a variety by first embedding it into a larger one, blowing that up and taking strict transform. The price we'll pay for writing down a general, coordinate free definition of blowing up is that this functoriality property will not be obvious (though, of course, still true).

Using the theory we've just developed, we will now revisit the example from the beginning.

Example 3.7. The blow up of $\mathbb{C}^2 = \mathrm{Spec}\,(\mathbb{C}[x_1, x_2])$ at the (reduced) origin. The ideal we wish to blow up in is then $(x_1, x_2) \triangleleft \mathbb{C}[x_1, x_2]$ and we will write $\mathrm{Bl}_{\mathbf{0}} \mathbb{C}^2$ as a shorthand for $\mathrm{Bl}_{(x_1,x_2)} \mathbb{C}^2$. So we have

$$\begin{aligned}
\mathrm{Bl}_{\mathbf{0}} \mathbb{C}^2 &= \overline{\{((x_1, x_2), [x_1 : x_2]) \in \mathbb{C}^2 \times \mathbb{P}^1 \colon (x_1, x_2) \neq (0, 0)\}} \\
&= \{((x_1, x_2), [X_1 : X_2]) \in \mathbb{C}^2 \times \mathbb{P}^1 \colon x_1 X_2 - X_1 x_2 = 0\} \\
&= \mathcal{O}_{\mathbb{P}^1}(-1)
\end{aligned}$$

Exercise 3.8. If we are working over $\mathbb{R}$, show that $\mathrm{Bl}_{\mathbf{0}} \mathbb{R}^2$ is topologically $\mathbb{R}^2 \# \mathbb{R}P^2$ (remove a small disc around the origin and identify antipodal points on the resulting boundary). Then we see that in fact $\mathcal{O}_{\mathbb{R}P^1}(-1)$ is the Möbius bundle and this justifies the common depiction of the blow up we give in Figure 1.

Exercise 3.9. Similarly, show that $\mathrm{Bl}_{\mathbf{0}} \mathbb{C}^2$ is topologically $\mathbb{C}^2 \# \overline{\mathbb{P}^2}$, where the bar denotes opposite orientation. To do this, show that the blow up removes a small D^4 centered at the origin and then identifies the Hopf fibres on the introduced boundary S^3.

Exercise 3.10. Let E denote the exceptional divisor in $\mathrm{Bl}_{\mathbf{0}} \mathbb{C}^n$. Show that the intersection product of E with itself equals -1.

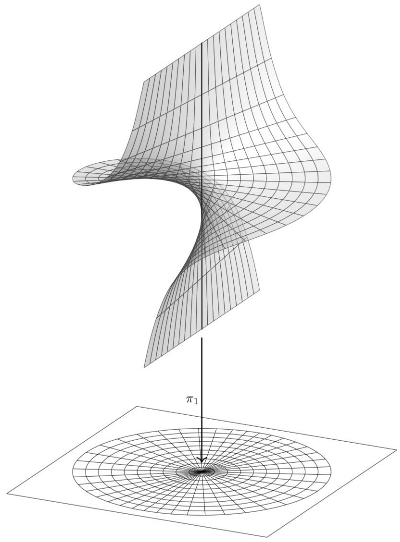


Figure 1: Common depiction of $\mathrm{Bl}_{\mathbf{0}}\mathbb{R}^2$

4 Blowing up a complex manifold

We have seen how to blow up $\mathbb{C}^n$ at the origin. We now extend this notion to blowing up a complex manifold at a point.

Definition 4.1. Let $P \subseteq \mathbb{C}^n$ be a polydisc centered around $\mathbf{0}$. We define the blow up of P at $\mathbf{0}$ to be $\tilde{P} = \pi^{-1}(P)$, where $\pi : \mathrm{Bl}_{\mathbf{0}}\, \mathbb{C}^n \to \mathbb{C}^n$ is the usual surjection.

Definition 4.2. Let M be a smooth complex manifold of complex dimension n and let $x \in M$. Let $U \subseteq M$ be a coordinate neighbourhood of x with chart $\varphi\colon U \to P$, where P is a polydisc in $\mathbb{C}^n$ and $\varphi(x) = \mathbf{0}$. Then we have a biholomorphism $\varphi^{-1} \circ \pi : \tilde{P} \setminus E \to U \setminus \{x\}$. We define the *blow up* $\mathrm{Bl}_x M$ of M at x to be the complex manifold obtained by gluing $\tilde{P}$ to $M \setminus \{x\}$ along $\varphi^{-1} \circ \pi$.

Here we implicitly use the charts on $\tilde{P}$, viewed as a submanifold of $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$, which now become charts on $\mathrm{Bl}_x M$. We also have, by abuse of notation, the surjection map $\pi\colon \mathrm{Bl}_x M \to M$ and the exceptional divisor is $E = \pi^{-1}(x)$. It is not immediately obvious however that the construction is independent of coordintes. So suppose $\varphi' : U' \to P'$ is another holomorphic chart, with $\varphi'(x) = \mathbf{0}$ and let $F = \varphi' \circ \varphi^{-1}$ be the transition map defined on some open neighbourhood of $\mathbf{0}$ in $\mathbb{C}^n$. Using this chart we obtain an a priori different blow up, which we denote $\mathrm{Bl}'_x M$. We wish to show that it is biholomorphic to $\mathrm{Bl}_x M$.

Exercise 4.3. Show that it suffices to give a biholomorphism between some open neighbourhoods of the exceptional divisors in $\tilde{P}$ and $\tilde{P}'$. Then note that away from the exceptional divisors we can use F (or, rather, its lift) and this can be extended over $E \cong \mathbb{P}^{n-1}$ by the map

$$[X_1 : X_2 : \ldots : X_n] \longmapsto \left[\begin{pmatrix} \frac{\partial F_1}{\partial x_1}\Big|_{\mathbf{0}} & \frac{\partial F_1}{\partial x_2}\Big|_{\mathbf{0}} & \cdots & \frac{\partial F_1}{\partial x_n}\Big|_{\mathbf{0}} \\ \frac{\partial F_2}{\partial x_1}\Big|_{\mathbf{0}} & \frac{\partial F_2}{\partial x_2}\Big|_{\mathbf{0}} & \cdots & \frac{\partial F_2}{\partial x_n}\Big|_{\mathbf{0}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1}\Big|_{\mathbf{0}} & \frac{\partial F_n}{\partial x_2}\Big|_{\mathbf{0}} & \cdots & \frac{\partial F_n}{\partial x_n}\Big|_{\mathbf{0}} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \right]$$

Remark 4.4. Note again that the exceptional divisor of the blow up of a variety (manifold) X at a smooth point x is $\mathbb{P}(T_x X)$. So blowing up at a point has the effect of separating the tangent directions at that point.

5 Key examples

We now give a recipe for computing the blow up of an affine variety at a point. Let $X = Z(g_1, g_2, \ldots, g_t) \subseteq \mathbb{A}^n$ be an affine variety such that $\mathbf{0} \in X$. Then by functoriality of blow ups we have $\mathrm{Bl}_{\mathbf{0}} X \subseteq \mathrm{Bl}_{\mathbf{0}} \mathbb{C}^n \subseteq \mathbb{C}^n \times \mathbb{P}^{n-1}$. For each $1 \leq i \leq t$ let us write g_i^{hom} for the lowest degree homogeneous part of g_i which is not identically 0 and set $d_i = \deg(g_i^{\mathrm{hom}})$. Consider a formal replacement procedure, where we obtain a polynomial $\tilde{g}_i \in \mathbb{C}[x_1, \ldots, x_n, X_1, \ldots, X_n]$ from g_i by replacing exactly d_i of the variables x_j in each monomial of g_i with the corresponding variables X_j (for monomials of degree greater than d_i the procedure involves a choice, but this is will not be important for our purposes). Note that the polynomial we obtain in this way is homogeneous of degree d_i in the X-variables. We now claim that

$$\mathrm{Bl}_{\mathbf{0}} X = \{((x_1, \ldots, x_n), [X_1 : \ldots : X_n]) \in \mathrm{Bl}_{\mathbf{0}} \mathbb{C}^n \colon \tilde{g}_i(\mathbf{x}, \mathbf{X}) = 0 \;\forall\, 1 \leq i \leq t\}.$$

In particular, the exceptional divisor of $\mathrm{Bl}_{\mathbf{0}} X$ is given by

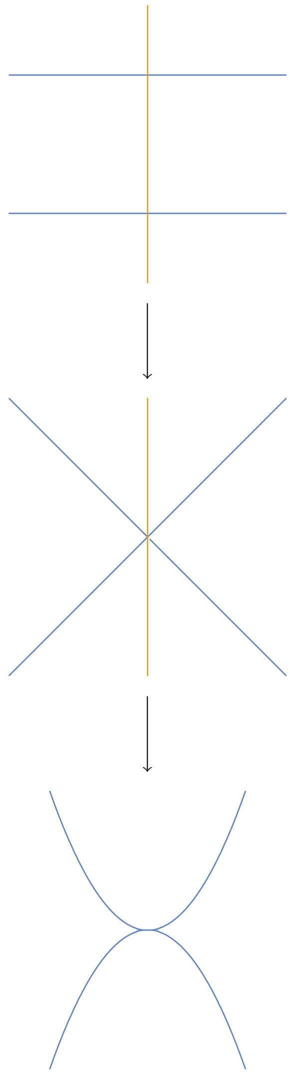
$$Z(\{g_i^{\mathrm{hom}}(X_1, \ldots, X_n) \colon 1 \leq i \leq t\}) \subseteq \mathbb{P}^{n-1}.$$

Exercise 5.1. Prove the above claim.

Exercise 5.2. Compute the blow ups at the origin of the following curves in $\mathbb{C}^2$: $\{x_1 x_2 = 0\}$, $\{x_2^2 - x_1^3 - x_1^2 = 0\}$, $\{x_2^2 - x_1^3 = 0\}$, $\{x_2^2 - x_1^4 = 0\}$. Draw their real cartoons and investigate their exceptional divisors.

Observe that the exceptional divisor in each of the above examples consists of precisely one point for each tangent direction to the curve at $\mathbf{0}$. In fact, by blowing up we have *resolved* the singularities in the first three curves, but not in the fourth one. We have however made the singularity milder – the two irreducible components no longer have a common tangent at their point of intersection. So we might hope that performing a second blow up will separate them.

Exercise 5.3. Desingularize the curve $\{x_2^2 - x_1^4 = 0\}$ by applying two repeated blow ups and obtain the following sequence of diagrams.



In fact, we could have accomplished the same effect in one go. Note that the problematic tangent direction is the one corresponding to the x_1-axis. So instead of blowing up the *reduced* origin we may consider blowing up the thickened point, which carries infinitesimal information in the x_1-direction.

Exercise 5.4. Calculate explicitly $\mathrm{Bl}_{(x_1^2,x_2)} X$, where X is the curve $Z(x_2^2 - x_1^4)$.

The last exercise indicates that it really is the *ideal* in which we are blowing up which is important and not just its reduced vanishing locus. In other words, we need the language of schemes to define the blow up in full generality and this is what we do next.

6 The general definition

We start by defining the blow up of an affine Noetherian scheme $X = \mathrm{Spec}\, A$ along a closed subscheme Z. Since Z is given as the vanishing set of some ideal

$I \triangleleft A$ we have $Z = \text{Spec } A/I$.

Definition 6.1. For a Noetherian affine scheme $X = \text{Spec } A$ and a closed subscheme $Z = \text{Spec } A/I$ we define the *blow up* of X along Z to be

$$\text{Bl}_Z X = \text{Proj }_A(A \oplus I \oplus I^2 \oplus \cdots),$$

where we view $A \oplus I \oplus I^2 \oplus \cdots$ as a graded A-algebra whose degree k piece is I^k (with $I^0 = A$).

Note first that this strange-looking algebra is in fact generated in degree 1 by construction and also, by the Noetherian hypothesis, it is generated by finitely many elements, say $\{x_1, x_2, \ldots, x_s\}$. using this, we can give a more explicit model for it. The key point is to view I not as a subset of A but rather as the quotient module

$$(A \cdot X_1 \oplus A \cdot X_2 \oplus \cdots \oplus A \cdot X_s)/\ker \phi,$$

where $X_1, X_2, \ldots, X_s$ are just formal variables and ϕ is the surjective $A-$module homomorphism

$$\begin{aligned} \phi\colon A \cdot X_1 \oplus A \cdot X_2 \oplus \cdots \oplus A \cdot X_s &\longrightarrow I \\ \phi\colon X_i &\longmapsto x_i. \end{aligned}$$

In fact we can extend ϕ to a surjective graded algebra homomorphism

$$\begin{aligned} \Phi\colon A[X_1, X_2, \ldots, X_s] &\longrightarrow A \oplus I \oplus I^2 \oplus \cdots \\ \Phi\colon X_0^{\alpha_0} X_1^{\alpha_1} \ldots X_n^{\alpha_n} &\longmapsto (0, 0, \ldots, 0, x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}, 0, \ldots), \end{aligned}$$

where on the right hand side the non-zero term lies in position $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ and the map is $A-$linear. In many concrete cases we have a much more concise way of writing down the kernel of this map and thus get a handle on the algebra $A \oplus I \oplus I^2 \oplus \cdots$. Let us see how we can compute our favourite example $\text{Bl}_{\mathbf{0}} \mathbb{C}^2$ with this new definition.

Example 6.2. We consider $X = \text{Spec } A$ and $Z = \text{Spec } A/I$ with $A = \mathbb{C}[x_1, x_2]$ and $I = (x_1, x_2)$. So from the discussion above we have that $A \oplus I \oplus I^2 \oplus \cdots$ is the quotient of the graded algebra $\mathbb{C}[x_1, x_2][X_1, X_2]$ by its ideal $(x_1 X_2 - X_1 x_2)$. So in fact we have:

$$\text{Bl}_{\mathbf{0}} \mathbb{C}^2 = \text{Proj } (\mathbb{C}[x_1, x_2, X_1, X_2]/(x_1 X_2 - X_1 x_2))$$

whose closed points are precisely $\{((x_1, x_2), [X_1 : X_2])\colon x_1 X_2 - X_1 x_2 = 0\}$.

We now move on to give the general definition of a blow up.

Let X be any Noetherian scheme and Z – a closed subscheme. Then on every affine open $U = \text{Spec } A$ in X, $Z \cap U$ is given as the (scheme-theoretic) vanishing set of some ideal $I \triangleleft A$ which defines an ideal sheaf $\tilde{I}$ on Spec A. These local ideal sheaves glue together to give a sheaf $\mathscr{I}_Z$. We can now consider the sheaf of graded $\mathcal{O}_X-$algebras $\mathcal{O}_X \oplus \mathscr{I}_Z \oplus \mathscr{I}_Z^2 \oplus \cdots$. On each affine open subset $U =$ Spec A of X we can then define a scheme $\text{Proj } \big(\mathcal{O}_X(U) \oplus \mathscr{I}_Z(U) \oplus \mathscr{I}_Z(U)^2 \oplus \cdots\big)$. In fact these schemes glue together to define a new scheme which we denote $\text{Proj }_X \big(\mathcal{O}_X \oplus \mathscr{I}_Z \oplus \mathscr{I}_Z^2 \oplus \cdots\big)$. Moreover there is a natural morphism

$$\pi\colon \text{Proj } \big(\mathcal{O}_X \oplus \mathscr{I}_Z \oplus \mathscr{I}_Z^2 \oplus \cdots\big) \to X.$$

Finally we state the most general definition of a blow up:

Definition 6.3. Let X be a Noetherian scheme and Z a closed subscheme. Then we define the *blow up* of X along Z to be

$$\mathrm{Bl}_Z X = \mathrm{Proj}\left(\mathcal{O}_X \oplus \mathscr{I}_Z \oplus \mathscr{I}_Z^2 \oplus \cdots\right).$$

The *exceptional divisor* is $\pi^{-1}(Z)$, where $\pi\colon \mathrm{Proj}_X\left(\mathcal{O}_X \oplus \mathscr{I}_Z \oplus \mathscr{I}_Z^2 \oplus \cdots\right) \to X$ is the natural morphism.