Chern Classes

1 Introduction and Basic Definitions

We introduce C^∞ vector bundles, sections and other basic definitions before building up to the Chern classes of line bundles through a sequence of examples.

1.1 What are Chern classes?

It is usually a difficult problem to classify all non-isomorphic vector bundles over a given fixed base space. The idea of characteristic classes is to provide a topological invariant which allows us to distinguish some classes of vector bundles. Characteristic classes are elements of the cohomology groups of the base space. Chern classes are particular characteristic classes which we associate to complex vector bundles. They are a sequence of functions $c_1, c_2, \ldots$ assigning to each complex vector bundle $E \to B$ a class $c_i(E) \in H^{2i}(B;\mathbb{Z})$ depending only on the isomorphism type of E (and are the unique such sequence satisfying a list of properties which we'll list towards the end). The Chern classes are formally akin to the Stiefel-Whitney classes, which are defined for real vector bundles. The Stiefel-Whitney classes take coefficients in $\mathbb{Z}_2$.

1.2 Vector Bundles

In the course of this lecture we denote by M, a smooth manifold of real dimension n.
A **rank** r **smooth complex vector bundle** is a map $\pi : E \to M$ together with a complex vector space structure on the **fibres** $\pi^{-1}(b)$ for each $m \in M$ such that we have the following local triviality condition: for each $m \in M$ there exists an open neighbourhood U_m of m such that there is a diffeomorphism $d_m : \pi^{-1}(U_m) \to U_m \times \mathbb{C}^r : \pi^{-1}(b) \mapsto \{b\} \times \mathbb{C}^r$ which is a vector space isomorphism. We call d_m the **local trivialisation** of the vector bundle at m. We call E the **total space** and call M the **base space** of the vector bundle. For convenience, we refer to the bundle simply as E with all of the other data above quietly suppressed.

A **smooth section** of the vector bundle E is a smooth right inverse of π. The **trivial bundle** over M is the vector bundle $M \times \mathbb{C}^r$. Any vector bundle E is said to be **trivial** if it is isomorphic to the trivial bundle.

Exercise 1.1. Show how to put the real line bundle structure on the open Möbius band

Exercise 1.2. Show that $L^{\otimes 2}$ is isotopic in $\mathbb{R}^4$ to the trivial embedding of the cylinder.

Fact 1.1. Bundles on contractible manifolds are trivial

Sketch proof: After trivialising at some fibre E_x lying over x, we extend the trivialisation along the homotopy that contracts M to x using a version of Tietze's Extension theorem. We then lift the homotopy to E so that it covers the homotopy on M.

2 Bundles on spheres

2.1 Clutching functions

A convenient way of constructing vector bundles over spheres S^k is by using clutching functions. We start by decomposing S^k as the disjoint union of two disks D^k_+ and D^k_-. Each disc is contractible and so any vector bundle on each piece is trivial by Fact 1.1. We then glue along the boundaries by identifying ∂D^k_+ with ∂D^k_-, whilst applying a map called a **clutching function** (think of engaging gears in machinery) $f : S^{k-1} \to GL_r(\mathbb{C})$ to the vectors in the bundle. We can thicken the intersection slightly to $S^{k-1} \times (-\epsilon, \epsilon)$ to make things smooth.

Fact 2.1. Homotopic clutching functions f, g give isomorphic smooth complex vector bundles E_f, E_g.

Theorem 1. *There is the following bijection between homotopy classes of maps and rank r complex vector bundles over S^k:*

$$[S^{k-1}, GL_r(\mathbb{C})] \leftrightarrow \mathrm{Vect}^r_{\mathbb{C}}(S^k)$$

Proof. See Hatcher, Prop 1.11 □

Remark 2.1. *Hatcher also deals with the slightly more subtle case of real vector bundles over S^k in Proposition 1.14. We have a 'nice' situation in Theorem 1, because $GL_r(\mathbb{C})$ is path-connected (see below for proof). On the other hand $GL_r(\mathbb{R})$ has two path-connected components. In order to draw a similar bijection the notion of an orientation on vector bundles is introduced and also a restriction to path-components of $GL_r(\mathbb{R})$.*

Corollary 1. *Every complex vector bundle over S^1 is trivial.*

Proof. This is equivalent to saying that $GL_r(\mathbb{C})$ is path-connected. By applying elementary row operations, we can diagonalise any matrix $GL_r(\mathbb{C})$. We construct a path to the diagonal matrix by applying an appropriate sequence of row operations with a factor of λ in front of each, then running λ continuously from 0 to 1. Diagonal matrices in $GL_r(\mathbb{C})$ are homeomorphic to r copies of $\mathbb{C} - \{0\}$ which is path-connected. □

Example 2.1. For the line bundle on S^2 we have:
$\pi_1(GL_1(\mathbb{C})) = \pi_1(\mathbb{C}^*) = \pi_1(\mathbb{Z}) \cong \mathbb{Z}$
without wishing to jump the gun, we will soon see that the cohomology class of the generator of this group is the first Chern class of the line bundle.

Remark 2.2. *The bundle corresponding to $n \in \mathbb{Z}$ can be constructed algebraically by using the transition function $z \mapsto z^n$*

Exercise 2.1. Show that a line bundle L is trivial $\iff$ L has a non-vanishing section.
Extension: Show that a rank r vector bundle E is trivial $\iff$ E has r sections $s_1, \ldots, s_r$ such that the vectors $s_1(b), \ldots, s_r(b)$ are linearly independent in each fibre $p^{-1}(b)$

Exercise 2.2. Research and understand the Hopf fibrations.

3 Chern classes as degeneracy loci of generic sections

We specialise slightly and consider $p : E \mapsto M$, a complex vector bundle of rank r, with base space M a compact (orientable) smooth manifold of real dimension n.

A **generic section** is a section of the bundle which intersects the zero-section $\mathcal{O}_M$ transversally. Two sections are said to have **transversal intersection** if at every point of intersection, the tangent spaces of the sections at the point generate the tangent space of the ambient space at the point.

We can define the Chern classes in terms of the zero sets of generic sections, namely the transversal intersections of the generic sections with $\mathcal{O}_M$.

For a generic section s, we write $Z(s)$ for the zero-set of s. Since s is transverse to the zero section, $Z(s)$ is a submanifold of M of real codimension $2r$. Applying Poincarè duality to the fundamental class $[Z(s)] \in H_{n-2r}(M)$, we obtain a cohmology class in $H^{2r}(M)$. This is called the **Euler class**, $e(E)$ of the vector bundle and is also the r^{th} **Chern class**.

$$e(E) = c_r(E) := [Z(s)] \in H_{n-2r}(M) \cong H^{2r}(M; \mathbb{Z})$$

By analogy we extend the definition to give us a sequence of cohomology classes:

$$c_i(E) := PD([Z(s_1 \wedge \ldots \wedge s_{r-i+1})]) \in H^{2i}(M; \mathbb{Z})$$

for $s_1, \ldots, s_{r-i+1}$ generic sections. Here $Z(s_1 \wedge \ldots \wedge s_{r-i+1})$ can be viewed as the set of points $m \in M$ where the sections $s_1(m), \ldots, s_{r-i+1}(m)$ become linearly dependent. Note then that

$$c_1(E) := PD([Z(s_1 \wedge \ldots \wedge s_r)])H^{2r}(M; \mathbb{Z})$$

We define $c_0(E) = 1$ and call the sum $c(E) = 1 + c_1(E) + c_2(E) + \ldots \in H^*(M;Z)$ the **total Chern class** of E. Point (3) in Theorem 2 below demonstrates that such a sum is finite, i.e. a well-defined element of $H^*(M;Z)$

Theorem 2. *There is a unique sequence of functions $c_1, c_2, \ldots$ assigning to each complex vector bundle $E \to B$ a class $c_i(E) \in H^{2i}(B,\mathbb{Z})$ depending only on the isomorphism type of E and satisfying:*

1. $c_i(f^*E) = f^*(c_i(E))$ *for a pullback* f^*E
2. $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$
3. $c_i(E) = 0$ *if* $i > rankE$
4. *For the canonical line bundle* $E \to \mathbb{C}P^\infty$, $c_1(E)$ *generates* $H^2(\mathbb{C}P^\infty;\mathbb{Z})$.

4 Suggested Reading

- A. Hatcher - Vector Bundles and K Theory

  This book is a 'work in progress', but displays welcome familiarity if you have studied Hatcher's 'Algebraic Topology'

- MJ. Atiyah - K Theory

  Richard Thomas' recommendation

5 Additional exercises

1. Why is the class of $[Z(s)]$ independent of choice of generic section?
2. Show that for the line bundle $O(-n)$ the trivialising section 1 on the z-disc on the left glues to the section w^{-n} on the w-disc on the right Interpret in terms of zeros and poles!
3. Suppose that $Z \subset M$ is a codimension-r submanifold with normal bundle N, and assume everything is compact and oriented. Let $[Z] \in H(M)$ denote the fundamental class of Z. Show that its self-intersection $[Z].[Z]$ is (the pushforward from Z of the Poincaré dual of) $c_r(N)$.