Mixed Hodge Structures

1 Introduction

In these notes we give an informal introduction to the theory of mixed Hodge structures. Guided by the example of a nodal curve, we illustrate how naturally mixed Hodge structures arise and, then, we describe how the cohomology of a smooth variety admits such a rich structure. Finally, we state the Invariant cycles theorem as an example where the theory of mixed Hodge structures is used.

2 Linear Algebra and Hodge Theory

Consider a finite dimensional real vector space V, endowed with an almost complex structure, i.e. an endomorphism $J : V \to V$ such that $J^2 = -id$. Note that such a V has even dimension and that its complexification admits a natural decomposition. Indeed, let

$$V_{\mathbb{C}} \coloneqq V \otimes_{\mathbb{R}} \mathbb{C},$$

and consider the $\mathbb{C}$-linear extension of J to $V_{\mathbb{C}}$. It has eigenvalues $\pm i$ and, according to its decomposition in eigenspaces, we have

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1},$$

where $V^{1,0}$ denotes the eigenspace of i and $V^{0,1}$ the one of $-i$.

endowed with an almost complex structure, i.e. an endomorphism $J : V \to V$ such that $J^2 = -id$. First, note that V has even dimension. Moreover, we can decompose its complexification in the obvious way. Indeed, let

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and consider the $\mathbb{C}$-linear extension of J to $V_{\mathbb{C}}$. It has eigenvalues $\pm i$ and, according to its decomposition in eigenspaces, we have

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1},$$

where $V^{1,0}$ denotes the eigenspace of i and $V^{0,1}$ the one of $-i$. Finally, complex conjugation on $V_{\mathbb{C}}$ induces an isomorphism between $V^{1,0}$ and $V^{0,1}$. This is an example of a pure (real) Hodge structure of weight 1. Besides, Hodge structures arise in many more complicated contexts. For instance, let X be an m-dimensional compact oriented Riemannian manifold and consider the sheaf of n-forms on X by $\mathcal{A}_X^n$, with the exterior derivative $d : \mathcal{A}_X^n \to \mathcal{A}_X^{n+1}$. Let us define the Laplacian operator

$$\Delta_d \coloneqq dd^* + d^*d,$$

where $d^* = (-1)^{m(n-1)-1} * d* : \mathcal{A}_X^n \to \mathcal{A}_X^{n-1}$ and $*$ is the Hodge operator. Now, consider the set of harmonic forms

$$\mathcal{H}^n(X) = \{\alpha \in \mathcal{A}_X^n \text{ such that } \Delta_d \alpha = 0\}.$$

Then, there is a natural map

$$\mathcal{H}^n(X) \longrightarrow H^n(X, \mathbb{R}),$$

sending any harmonic form to its cohomology class. Thanks to the work of Hodge, Kodaira et al, this map turns out to be an isomorphism of vector spaces, hence

$$\mathcal{H}^n(X) \simeq H^n(X, \mathbb{R}).$$

Take a complex manifold X and consider the sheaf of complex n-forms $(\mathcal{A}_X^n, d)$, which can be decomposed into the direct sum

$$\mathcal{A}_X^n = \bigoplus_{i+j=n} \mathcal{A}_X^{i,j},$$

where $\mathcal{A}_X^{i,j}$ denotes the sheaf of (i,j)-forms and $d = \partial + \bar{\partial}$, where $\partial : \mathcal{A}_X^{i,j} \to \mathcal{A}_X^{i+1,j}$ and $\bar{\partial} : \mathcal{A}_X^{i,j} \to \mathcal{A}_X^{i,j+1}$ denote the Dolbeault operators. We may try to see if $H^1(X)$ is even dimensional, but, unfortunately,

$$[J, d] \neq 0,$$

which means that J does not act on closed forms. The picture becomes much nicer when we restrict to Kähler manifolds. Indeed, it turns out that

$$[\Delta_d, J] = 0,$$

so J acts on harmonic forms. Moreover, we have

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}.$$

This leads to a decomposition of $\mathcal{H}^n(X)$ into the direct sum of the spaces of (p,q)-harmonic forms $H^{p,q}(X)$,

$$\mathcal{H}^n(X) = \bigoplus_{p+q=n} H^{p,q}(X),$$

where $\overline{H^{p,q}(X)} = H^{q,p}(X)$. When we restrict our attention to a compact Kähler manifold, the isomorphism

$$H^n(X, \mathbb{C}) \simeq \mathcal{H}^n(X)$$

induces the (Hodge) decomposition

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X),$$

where J acts by $(i)^p(-i)^q$ on each $H^{p,q}(X)$. Moreover, we get a decreasing (Hodge) filtration, which is an equivalent data to the decomposition above:

$$F^0 H^n(X, \mathbb{C}) = H^n(X, \mathbb{C}) \supseteq F^1 H^n(X, \mathbb{C}) \supseteq \cdots \supseteq F^p H^n(X, \mathbb{C}) \supseteq \cdots,$$

where

$$F^pH^n(X,\mathbb{C}) = \bigoplus_{\substack{i+j=n \\ i\geq p}} H^{i,j}(X).$$

Note that the two data are equivalent since, given a filtration $F^\bullet$ on $H^n(X,\mathbb{C})$, we recover the decomposition simply defining

$$H^{p,q} = F^p \cap \overline{F^q}.$$

Definition 2.1. An integral Hodge structure pure of weight $k \in \mathbb{Z}$ is a finitely generated free abelian group $V_\mathbb{Z}$, with a decomposition of $V_\mathbb{C} = V_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$,

$$V_\mathbb{C} = \bigoplus_{p+q=k} V^{p,q},$$

such that

$$\overline{V^{p,q}} = V^{q,p}.$$

Equivalently, as discussed above, we can replace the decomposition with a finite decreasing filtration of $V_\mathbb{C}$, $F^\bullet V_\mathbb{C}$ such that

$$F^pV_\mathbb{C} \cap \overline{F^qV_\mathbb{C}} = 0 \text{ and } F^pV_\mathbb{C} \oplus \overline{F^qV_\mathbb{C}} = V_\mathbb{C},$$

whenever $p+q = k+1$. In the case of compact Kähler manifolds,

$$V_\mathbb{Z} = H^k(X,\mathbb{Z})/\text{torsion}$$

is a pure Hodge structure of weight k.

Definition 2.2. Given $(V_\mathbb{Z}, V^{p,q})$ and $(W_\mathbb{Z}, W^{p,q})$ two pure $\mathbb{Z}$-Hodge structures of weight k and $k+2r$, then a morphism of Hodge structures of bidegree (r,r) is a group homomorphism $\phi : V_\mathbb{Z} \to W_\mathbb{Z}$ such that

$$\phi(V^{p,q}) \subseteq W^{p+r,q+r}, \text{ or } \phi(F^pV_\mathbb{C}) \subseteq F^{p+r}W_\mathbb{C}.$$

We restrict our attention to morphisms of bidegree $(0,0)$ introducing the Tate twist.

Definition 2.3. Define the Tate-Hodge structure $\mathbb{Z}(1)$ to be the pure Hodge structure of weight -2

$$(2\pi i\mathbb{Z}, H^{-1,-1}), \text{ where } H^{-1,-1} = \mathbb{Z}(1) \otimes \mathbb{C}.$$

Moreover, given a pure Hodge structure $(V, V^{p,q})$ of weight k and an integer c, define the Tate twist to be the pure Hodge structure $(V(-c), V(-c)^{p,q})$ of weight $k+2c$, defined by

$$V(-c) = V \text{ and } (V(-c))^{p,q} = V^{p-c,q-c}.$$

We refer to morphism of pure Hodge structures V, W of weight k and $k+2r$ as a morphism of Hodge structures of bidegree $(0,0)$ between $V(-r)$ and W.

The first example of morphism between pure Hodge structure comes again from Kähler manifolds. Given $f : X \to Y$ holomorphic map between compact Kähler manifolds, we get

$$f^* : H^n(Y,\mathbb{Z}) \to H^n(X,\mathbb{Z}),$$

preserving the weight! So f^* is a morphism of Hodge structure. What about the Gysin morphism f_*? Note that f_* does not preserve the weight, since

$$f_* : H^n(X, \mathbb{Z}) \to H^{n+2r}(Y, \mathbb{Z}), \text{ where } r = \dim_{\mathbb{C}}(Y) - \dim_{\mathbb{C}}(X),$$

but induces a morphism of Hodge structures between $H^n(X, \mathbb{Z})(-r) \to H^{n+2r}(Y, \mathbb{Z})$.
Exercise. Prove that a morphism of Hodge structures $\phi : V_{\mathbb{Z}} \to W_{\mathbb{Z}}$ (of bidegree $(0, 0)$) is strict for the Hodge filtration, i.e. for all p

$$\operatorname{im} \phi \cap F^p W_{\mathbb{C}} = \phi(F^p V_{\mathbb{C}}).$$

3 Mixed Hodge Structures

Hodge theory tells us that the cohomology of a compact Kähler manifold is a pure Hodge structure. What if we consider an algebraic variety (over $\mathbb{C}$) in general? The question whether its cohomology should have a pure Hodge structure has negative answer. Consider a nodal elliptic curve C. It turns out that $H^1(C) \simeq \mathbb{Z}\langle dy \rangle$, hence it does not admit a complex structure and, in particular, is not a pure Hodge structure of weight 1. If we resolve the singularity, we get $\bar{C}$ with exceptional divisor E such that $C = \bar{C}/E$ and we have a long exact sequence of relative homology groups

$$0 \longleftarrow \tilde{H}_0(E) \longleftarrow H_1(\bar{C}, E) \longleftarrow H_1(\bar{C}) \longleftarrow H_1(E) = 0 \longleftarrow \cdots,$$

and the relative exact sequence in cohomology

$$0 \longrightarrow \tilde{H}^0(E) \longrightarrow H^1(\bar{C}, E) \longrightarrow H^1(\bar{C}) \longrightarrow 0.$$

Since $H^1(C) \simeq H^1(\bar{C}, E)$, we can see $H^1(C)$ as the extension of $H^0(E)$ and $H^1(\bar{C})$, which are, respectively, pure Hodge structures of weight 0 and 1. More precisely, we have an increasing filtration

$$\operatorname{im} H^0(E) = W^0 \subset W^1 = H^1(C),$$

such that the graded piece

$$\mathrm{Gr}_1^{\mathrm{W}} := H^1(C)/\tilde{H}^0(E) \simeq H^1(\bar{C})$$

is a pure Hodge structure of weight 1.

Definition 3.1. A mixed Hodge structure $(H_{\mathbb{Z}}, W^\bullet, F^\bullet)$ consists of a free abelian group $H_{\mathbb{Z}}$ together with an increasing filtration of $H_{\mathbb{Q}}$

$$W^0 \subseteq W^1 \subseteq W^2 \subseteq \cdots,$$

and a dicreasing filtration of $H_{\mathbb{C}}$

$$H_{\mathbb{C}} = F^0 \supset F^1 \supset F^2 \supset \cdots,$$

such that $F^\bullet$ defines a pure Hodge structure of weight k on the graded piece

$$\mathrm{Gr}_{\mathrm{k}}^{\mathrm{W}} H_{\mathbb{C}} = W^k/W^{k-1}.$$

A morphism of mixed Hodge structures is a $\mathbb{Z}$-linear map which is compatible with the two filtrations of filtered vector spaces.

Proposition 3.2. Any morphism $f : (V_\mathbb{Z}, W, F) \to (V'_\mathbb{Z}, W', F')$ of mixed Hodge structures is strict, i.e. every element of F'^p which is in the image of f comes from F^p and similarly the same holds for the weight filtration.

In view of the (singular) case above, we would like to have a (canonical and functorial) result independent of resolution and compactification, which allows us to associate a mixed Hodge structure to any algebraic variety over $\mathbb{C}$. Indeed, we have the following.

Theorem 3.3. *Quasi-projective varieties have canonical mixed Hodge structure on their cohomology.*

Moreover, we get that the filtrations are independent of the choice of resolution, and so, by Hironaka's theorem, the result is extended to any algebraic variety over $\mathbb{C}$.

4 The open smooth case

Let U be a smooth variety and $X \supseteq U$ a smooth compactification, such that $X - U = D$ is a (smooth) normal crossing divisor. Consider the long exact sequence of relative cohomology groups

$$\cdots \longrightarrow H^i(X,U) \longrightarrow H^i(X) \longrightarrow H^i(U) \longrightarrow H^{i+1}(X,U) \longrightarrow \cdots$$

By excision and by Thom isomorphism, we have

$$H^i(X,U) \simeq H^i(\mathcal{N}_D, \mathcal{N}_D - D) \simeq H^{i-2}(D),$$

where $\mathcal{N}_D$ is the normal bundle of D in X. Thus, we have that $H^i(X,U)$ can be seen as a pure Hodge structure but of the wrong weight, since we would like

$$H^{i-2}(D) \longrightarrow H^i(X)$$

to be a morphism of pure Hodge structures (of bidegree $(0,0)$) but they have weights $i-2$ and i. Thus, we take the Tate twist so that our long exact sequence looks like

$$\cdots \longrightarrow H^{i-2}(D)(-1) \longrightarrow H^i(X) \longrightarrow H^i(U) \longrightarrow H^{i-1}(D)(-1) \longrightarrow \cdots .$$

$H^i(U)$ admits a mixed Hodge structure with induced weight filtration

$$W^k(H^i(U)) = \begin{cases} 0 & k < i \\ \operatorname{im}(H^i(X) \to H^i(U)) & i = k \\ H^i(U) & k \geq i+1. \end{cases}$$

Where does the decreasing filtration arise? In the classical case, i.e. for a complex manifold X, we consider the complex of sheaves on X of holomorphic $\bullet$-forms $\Omega^\bullet_X$ and its naive filtration

$$F^p\Omega^\bullet_X \coloneqq \Omega^{\geq p}_X = 0 \to \cdots 0 \to \Omega^p_X \to \Omega^{p+1}_X \to \cdots .$$

It is easy to check that induces a decreasing filtration on the hypercohomology of $\Omega_X^\bullet$, which is defined as

$$F^p\mathbb{H}^n(X,\Omega_X^\bullet) \coloneqq \operatorname{im}(\mathbb{H}^n(X,F^p\Omega_X^\bullet) \to \mathbb{H}^n(X,\Omega_X^\bullet)).$$

Since $\Omega_X^\bullet$ is a resolution of the locally constant sheaf of stalk $\mathbb{C}$ over X, we have

$$\mathbb{H}^n(X,\Omega_X^\bullet) \simeq H^n(X,\mathbb{C}),$$

which induces a decreasing filtration on $H^n(X,\mathbb{C})$. In the case of compact Kähler manifolds, this is what we are looking for and, indeed, the filtration leads us to the proof of (a weaker version of) the Hodge decomposition. Unfortunately, if we try to use the same construction for U, in general we get no extra information. For instance, for U affine,we would get

$$F^nH^n(U,\mathbb{C}) = H^n(U,\mathbb{C}).$$

We need to replace our complex of sheaves $\Omega_X^\bullet$ with something clever. It turns out that if we allow logarithmic singularities along D, we are able to induce a (non-trivial) decreasing filtration on $H^i(U,\mathbb{C})$ from the naive one on the new complex. More precisely, let $\Omega_X^\bullet(\log D)$ be the subcomplex of $j_*\Omega_U^\bullet$, where $j : U \hookrightarrow X$ is the inclusion of U in its compactification such that

- a meromorphic differential k-form α on an open $V \subset X$, holomorphic on $V \cap U$ is an element of $\Omega_X^k(\log D)|_V$ if α and $d\alpha$ admit poles of order at most 1 in $V \cap D$.

We can give an explicit local description of the elements of $\Omega_X^k(\log D)$: Let $z_1,\dots,z_n$ be local coordinates on an open set V of X, in which $V \cap D$ is defined by the equation

$$z_1\cdots z_r = 0, \text{ for } r \le n.$$

Then, the elements of the form

$$\frac{dz_{i_1}}{z_{i_1}} \wedge \frac{dz_{i_2}}{z_{i_2}} \wedge \cdots \frac{dz_{i_l}}{z_{i_l}} \wedge dz_{j_1} \wedge \cdots dz_{j_m},$$

for $l+m=k$, $i_\bullet \le r$, $j_\bullet > r$, form a basis of $\Omega_X^k(\log D)|_V$.

Theorem 4.1.

$$H^k(U,\mathbb{C}) = \mathbb{H}^k(X,\Omega_X^\bullet(\log D)).$$

Hence, we can consider the filtration $F^pH^k(U,\mathbb{C})$ induced by the naive

$$F^p\Omega_X^\bullet(\log D) \coloneqq \Omega_X^{\ge p}(\log D).$$

Such a filtration induces a pure Hodge structure of weight i on

$$\operatorname{Gr}_i^W H^k(U) = \begin{cases} 0 & i<k \\ \operatorname{im}(H^k(X) \to H^k(U)) \simeq H^k(X)/H^k(X,U) & i=k \\ \ker(H^{k+1}(X,U) \to H^k(X)) \simeq H^k(U)/H^k(X) & i=k+1 \\ 0 & i>k+1. \end{cases}$$

Remark 4.2. In the case of X singular, we take a smooth resolution $\hat{X}$ with exceptional set $E = \cup E_i$, where E_i are simple normal crossing divisors. We iterate the procedure briefly explained in the previous section in a similar manner. Moreover, if $f : \bar{X} \to X$ is proper and surjective, then

$$W^{k-1}H^k(X) = \ker f^*.$$

5 Applications

Let $X \to C$ be a surjective projective map, with X a complex smooth projective variety and C curve, and denote $C - \{\text{critical values}\}$ by C^*. Then, $H^k(X_t)^{\pi_1(C^*)}$ contains the image of

$$H^k(X - (\text{critical fibers})) \to H^k(X_t),$$

where X_t is a smooth fiber. What can we say about the converse? One of the amazing applications of the theory of mixed Hodge structures leads to a proof that any invariant cycle lifts to $H^k(X)$. Indeed, in Thèorie de Hodge II, Deligne proved the following result.

Theorem 5.1 (Deligne's global invariant cycles theorem)**.** *The subspace of monodromy invariants is*

$$H^k(X_t)^{\pi_1(C^*)} = \text{im}(H^k(X) \to H^k(X_t)).$$

Two of the main ingredients of the proof are the degeneration on the first page of the Leray spectral sequence for f and the properties of the mixed Hodge structures (in particular, strictness of maps between them). This is a clear example where the existence of this rich structure on the cohomology of X allows us to prove something of topological nature.