Moduli Spaces and Stable Bundles

1 Introduction

A moduli space, loosely speaking, is a 'parameter' space: a space parametrizing certain objects on X, so that the points of the moduli space M correspond to objects on X. However, we need to know about than just the points of M, as the following example illustrates.

Example 1.1. Consider the moduli space of points of $\mathbb{C}^2$ satisfying $y = x^2$ and $y = 0$, that is, $Z(x^2, y) \subset \mathbb{C}^2$. As a set, this is clearly just the origin. However, the ring of functions is $\mathbb{C}[x,y]/(y-x^2,y) \cong \mathbb{C}[x]/(x^2)$, and as a scheme this is Spec $\mathbb{C}[x]/(x^2)$. The scheme is the object which is going to give us information on the non-reduced structure, not the set of closed points.

We will want a moduli space to take into account *deformations*. I define this below, but in this example, if we take $Z(y - x^2, y - t)$, for some t, then we will have two closed points $(x - \sqrt{t}, t), (x + \sqrt{t}, t)$, which will be distinct if $t \neq 0$. We want to be able to see that setting $t = 0$ still gives us more than just a point - it should give us a thickened point, or double point. So we need to consider schemes.

2 Brief aside on schemes

Let us briefly review some definitions.

Definition 2.1. A *presheaf* $\mathcal{O}_X$ of sets (rings/modules) on a topological space X is a set (ring/module) $\mathcal{O}_X(U)$ for every open set $U \subset X$, together with the following data:

1. For every inclusion of open sets $V \subset U$ in X we have a *restriction map* $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ which is a map of sets (rings/modules).
2. For open sets $W \subset V \subset U$ in X, $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$.
3. For all open sets $U \subset X$, the restriction map ρ_{UU} is the identity.

A presheaf is a *sheaf* if we have an additional two axioms:

1. Identity axiom: If $U \subset X$ is an open set, $f, g \in \mathcal{O}_X(U)$, and $\{V_i : i \in I\}$ is an open covering of U, such that for all $i \in I$,
$$\rho_{UV_i}(f) = \rho_{UV_i}(g),$$
then $f = g$.
2. Gluability axiom: If U is an open set in X, and $\{V_i : i \in I\}$ is an open covering of U, and for every V_i we have $f_i \in \mathcal{O}_X(V_i)$ such that for all $i, j \in I$,
$$\rho_{V_i V_i \cap V_j}(f_i) = \rho_{V_j V_i \cap V_j}(f_j),$$
then there exists an $f \in \mathcal{O}_X(U)$ such that $\rho_{UV_i}(f) = f_i$ for all $i \in I$.

A *morphism of sheaves* $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open set $U \subset X$, such that for every inclusion of open sets $V \subset U$ in X the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \downarrow \rho^{\mathcal{F}}_{UV} & & \downarrow \rho^{\mathcal{G}}_{UV} \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V). \end{array}$$

For $x \in X$, the *stalk* at x is the set of equivalence classes $\{[(f,U)] : f \in \mathcal{O}_X(U), x \in U\}$ under the equivalence relation $(f,U) \sim (g,V)$ if there exists an open $x \in W \subset U \cap V$ such that $\rho_{UW}(f) = \rho_{VW}(g)$. We denote the germ $[(f,U)]$ as f_x.

Presheaves can be made into sheaves by sheafification. This can be expressed by a universal property, and hence is unique up to unique isomorphism.

Let $\pi : X \to Y$ be a continuous map of topological spaces, and $\mathcal{O}_X$ a sheaf on X. The *direct image* $\pi_*(\mathcal{O}_X)$ of $\mathcal{O}_X$ is the presheaf on Y given by, for $V \subset Y$ open, $\pi_*\mathcal{O}_X(V) = \mathcal{O}_X(\pi^{-1}(V))$. This is in fact a sheaf. There is also a pullback of a sheaf, $\pi^*\mathcal{O}_Y$.

Definition 2.2. A ringed space is a topological space X together with a sheaf of rings $\mathcal{O}_X$ on X. A morphism of ringed spaces $(\pi, \pi_\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map of topological spaces $\pi : X \to Y$, and a morphism of sheaves on Y, $\pi_\# : \mathcal{O}_Y \to \pi_*\mathcal{O}_X$. Equivalently, we could define $\pi_\#$ to be a morphism of sheaves on X, $\pi^*(\mathcal{O}_Y) \to \mathcal{O}_X$. A *locally ringed space* is a ringed space such that ring of germs at each point is local. A *morphism of locally ringed spaces* is a morphism of ringed spaces, with the additional requirement that it takes the maximal ideal of the germ in X to the maximal ideal of the germ in Y for every $x \in X$. Morphisms of locally ringed spaces induce maps of stalks. That is, if $x \in X, y = \pi(x)$, there is induced morphism of rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, $[(f,U)] \mapsto [(\pi_\#(U)(f), \pi^{-1}(U))]$ where $\pi_\#(U)(f) \in \pi_*\mathcal{O}_X(U) = \mathcal{O}_X(\pi^{-1}(U))$ as needed.

A commutative ring with unity can be made into a locally ringed space using the Spec functor. Let R be a ring. As a topological space, let $\operatorname{Spec} R = \{p : p \text{ is a prime ideal of } R\}$. Define maps

$$Z(-) : \{\text{ideals in } R\} \longrightarrow \{\text{ sets in } \operatorname{Spec} R\}, Z(S) = \{p \in \operatorname{Spec} R | S \subset p\},$$

$$I(-) : \{\text{sets in } \operatorname{Spec} R\} \longrightarrow \{\text{ideals in } R\}, I(K) = \{f \in R | f \in p \text{ for all } p \in K\}.$$

The *Zariski topology* on $\operatorname{Spec} R$ is defined by saying $Z(S)$ is closed for every $S \subset R$. The following lemma lists some well-known facts about these maps.

Lemma 2.3. *Let R be a ring, $J \subset R$ an ideal in R, and $K \subset \operatorname{Spec} R$ a closed set.*

1. $J \subset I(Z(J))$.
2. $K = Z(I(K))$.
3. $I(Z(J)) = \sqrt{J}$, *where* $\sqrt{J} = \{f |$ *there exists* $n \in \mathbb{N}$ *such that* $f^n \in J\}$.

The topology on $\operatorname{Spec} R$ has as an open basis $D(f) = \{p \in \operatorname{Spec} R | f \notin p\}$ for all $f \in R$. We think of elements in R as functions on $\operatorname{Spec} R$, where the value of f at p is the projection of f in R/p. However, because of nilpotents (which are precisely elements in $\bigcap_{p \in \operatorname{Spec} R} p$), functions may not be determined by their values at points. In the particular case of the ring $R = K[X_1, \ldots, X_n]$, where K is an algebraically closed field, functions (elements in R) are determined by their values on the spectrum, and moreover, they are determined by their value at the maximal ideals of R, which are in one to one correspondence with elements in K^n.

The ringed space Spec $R = (X, \mathcal{O}_R)$ is $X = \operatorname{Spec} R$ as a topological space, together with the sheaf on the base of distinguished open set (sets of the form $D(f), f \in R$), where $\mathcal{O}_R(D(f))$ is the localization of R at the set of all elements $g \in R : D(f) \subset D(g)$. This in fact defines a sheaf on a base. An affine scheme is a ringed space which is isomorphic to $(\operatorname{Spec} R, \mathcal{O}_R)$ for some ring R. A scheme is a ringed space $(X, \mathcal{O}_X)$ which can be covered by open sets such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. If $\phi : R \to S$ is a morphism of commutative rings, then it induces a morphisms of affine sheaves $\operatorname{Spec} S \to \operatorname{Spec} R$. We want morphisms of schemes to locally look like the morphisms that arise in this way. One can define morphisms of schemes like this, but equivalently, morphisms of locally ringed spaces coincide with them, which gives an alternative definition.

3 Back to Moduli Spaces

One begins by finding as many discrete invariants as possible, and then considering the moduli problem for fixed invariants: for example, nonsingular projective curves of a fixed genus. Moreover, we want the moduli space to be natural, in that the geometric structure of the moduli space should reflect the geometry of the problem.

Moduli spaces are the solution to a moduli problem. To pose a moduli problem you need the following ingredients:

- The set of objects A you want to classify, perhaps up to an equivalence (for example, isomorphism).
- A definition of a *family* of these objects with a base space S, which satisfies
  - If $S = \{s\}$, then a family over S is just an object in A.
  - Families pull back. In other words, if $\phi : S_1 \to S_2$ is a morphism, and X is a family over S_2, then we need a family over S_2, which we denote $\phi^*(X)$. Pulling back must be functorial, and respect equivalence.

Notice that this means if X is a family over S, and $s \in S$, then the pullback of X over the inclusion $\{s\} \in S$ is an object in A, which we denote X_s.

Example 3.1. Right notions of families:

- If we want to consider vector bundles over X, the correct notion of a family over a base S, where S is a scheme, is a bundle E on $X \times_k S$ such that the pullback bundle $X_s := E|_{X \times \{s\}}$ is stable over X for all $s \in S$. A *deformation* of a vector bundle E over X is a family of vector bundles over X with base S such that the fibre X_{s_0} is E for some $s_0 \in S$.
- To generalize the above, for sheaves over X, a family is a sheaf $\mathcal{F}$ on $X \times S$ which is *flat* over S - it sort of varies smoothly over S. Flatness implies that the Hilbert polynomial is constant over the fibres X_s, $s \in S$ (the fibres over each s have the same topology), for X reduced.
- A family of complex projective varieties over a base S, also a complex variety, is a proper surjective morphism $\pi : T \to S$ such that π is flat with reduced fibres, and has maximal rank. A *deformation* of a complex projective variety M is a family $\pi : T \to S$ together with an isomorphism $\pi^{-1}(s_0) \cong M$ for some $s_0 \in S$.
- For the moduli problem of subschemes of X, a family is a subscheme of $X \times S$, which is flat over S.

As mentioned, we need to know more about M than just the points - we need the whole scheme structure. The notion of a family is what allows us to put an algebraic structure on the moduli space. This is because it gives rise to a natural functor Schemes $\to$ Sets,

$$S \mapsto \{\text{families with base } S\},$$

called the moduli functor.

Exercise 3.2. Let $\mathcal{M}$ be a scheme, and consider the "functor of points" of $\overline{\mathcal{M}}$, Schemes $\to$ Sets, that takes $S \mapsto \mathrm{Hom}(S, \mathcal{M})$. Show that $\mathcal{M}$ is determined by its functor of points.

Remark 3.3. If $\mathcal{M}, S$ are schemes, then the S-valued points of M is the set $\overline{\mathcal{M}}(S)$. If $M = \mathrm{Spec}(k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ for a field k, and $S = \mathrm{Spec}(R)$ for a k-algebra R, then the S-valued points of M correspond to ring morphisms $k[x_1, \dots, x_n] \to R$ whose kernel contains $(f_1, \dots, f_r)$. That is, it is a choice $x_i \mapsto r_i$ such that $f_1(r_1, \dots, r_n) = \cdots = f_r(r_1, \dots, r_n) = 0$. So S points of $\mathcal{M}$ are just solutions to $f_1 = \cdots = f_r = 0$ in R.

Applying the functor of points of a scheme $\mathcal{M}$ to just a point $\{s\}$ tells us just about the points of $\mathcal{M}$.

There is also a relative version of the functor of points. If $\mathcal{M}$ is a scheme over S (that is, $\mathcal{M}$ and S are schemes and we have a morphism $\mathcal{M} \to S$; morphisms between two schemes over S must make the usual diagram commute), then $\overline{\mathcal{M}}_S(T) = \mathrm{Hom}_S(T, \mathcal{M}) \subset \overline{\mathcal{M}}(T)$. If it is clear, you can suppress the S. Another thing that will come up is the notion of a *subfunctor*. Let A be a category, then $F : A \to \mathrm{Set}$ is a subfunctor of $G : A \to \mathrm{Set}$ if for all objects $B \in A$, $F(B) \subset F(A)$, and if $f : B_1 \to B_2$, then $F(f)$ is the restriction of $G(f)$ from $G(B_i)$ to $F(B_i)$.

Definition 3.4. Suppose we are given a moduli problem. A *(fine) moduli space* is a scheme $\mathcal{M}$ such that $\overline{\mathcal{M}}$ is the functor given by the moduli problem. That is, $\mathcal{M}$ *represents* $S \mapsto \{\text{families with base } S\}$: $\mathrm{Mor}(S, M) \cong \{\text{families with base } S\}$.

There is a unique family U with base M corresponding to the identity map $1_{\mathcal{M}} \in \mathrm{Mor}(\mathcal{M}, \mathcal{M})$, and this is called the *universal family*, because if F is a family over base X, then there is a unique morphism $\phi : S \to M$ corresponding to F in $\mathrm{Mor}(S, \mathcal{M})$, and $\phi^*(U) \cong F$.

The above definition shows that the notion of family over a base S, where S is more than just a point, is what gives us the non-reduced information about $\mathcal{M}$, because it corresponds to considering $\overline{\mathcal{M}}$ for more that just points.

Consider vector bundles over X. Then a moduli space $\mathcal{M}$ has points vector bundles over X. The universal bundle U on $\mathcal{M} \times X$ should have fibre E over $\{E\} \times X$.

$$\mathrm{Mor}(S, \mathcal{M}) \leftrightarrow \{\text{bundles on } X \times S\}$$

$$f : S \to \mathcal{M} \mapsto (f \times \mathrm{Id})^*(U)$$

$$\{s \in S \mapsto E|_{X \times \{s\}}\} \leftarrow E.$$

Universal families, or fine moduli spaces, rarely exist. Thus, there is a weaker notion of a *coarse moduli space*, where instead of a natural isomorphism from the moduli functor to $\overline{\mathcal{M}}$, there is just a natural transformation between them, which is universal among such natural transformations.

Exercise 3.5. Show that $\mathbb{C}^*$ automorphisms of stable bundles mean the moduli functor is not representable (for example, do this for line bundles on a curve of genus g).

This exercise is related to the *jump phenomenon*, when, given a family F over a connected base S for some moduli problem, and $s_0 \in S$, for all $s, t \in S, s, t \neq s_0$, $F_s = F_t$, but $F_{s_0} \neq F_s$.

Exercise 3.6. Show that there exists a family of vector bundles on $\mathbb{CP}^n$ with base $\mathbb{C}$, where for $s \neq 0$, $X_s \cong \mathcal{O} \oplus \mathcal{O}$, and $X_0 \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$.

Jump phenomena prevent any moduli space from being Hausdorff. Similarly to in GIT, one solution to this is to throw away the 'bad' spaces. In fact, the solution to a moduli problem often comes down to forming an orbit space.

4 Moduli Space of Bundles

Let $(X, \mathcal{O}_X(1))$ be a projective variety (so that $\mathcal{O}_X(1)$ is an ample line bundle). Let ξ be a coherent sheaf on X. Define $\xi(m) = \xi \otimes \mathcal{O}_X(m)$. The *Hilbert function* of ξ is $P_\xi(m) = \dim H^0(X, \xi(m)) = \dim\Gamma(X, \xi(m))$ (this depends on the line bundle we have fixed). Alternatively, define this as $P_\xi(m) = \chi(\xi(m))$. The idea is that for a sufficiently large m, all non-zero cohomology of $\xi(m)$ will vanish (because $\mathcal{O}_X(1)$ is ample). Define the *Hilbert polynomial* to be a polynomial in m which agrees with $P_\xi(m)$ for $m >> 0$ (this exists). Thanks to Riemann-Roch, the Hilbert polynomial depends only on the Chern classes of ξ and of $\mathcal{O}_X(1)$.

If $\xi = E$ is a vector bundle, then in fact,

$$P_E(m) = a_0 m^r + a_1 m^{r-1} + \cdots,$$

where $a_0 = \mathrm{rank} E \int_X \frac{\omega^n}{n!}$, $a_1 = \int_X (c_1(E) + \mathrm{rank}E \frac{c_1(X)}{2}).\frac{\omega^{n-1}}{(n-1)!}$. We can consider the monic version of this polynomial.

Fix a coherent sheaf $\mathcal{F}$ on $(X, \mathcal{O}_X(1))$. A quotient of $\mathcal{F}$ is an exact sequence

$$\mathcal{F} \to \mathcal{Q} \to 0.$$

A *family* of quotients with base S is a quotient of $\pi^*\mathcal{F}$, where $\pi : X \times S \to X$ is the projection, which is flat over S and $\mathcal{Q}|_{X\times\{s\}}$ has the same Hilbert polynomial P, for all $s \in S$.

We put an equivalence relation on quotients by identifying quotients with the same kernel.

Exercise 4.1. Check that when $S = \{s\}$ this is just a quotient.

The *Quot functor* takes a scheme S to the set of all families of quotients $\mathcal{Q}$ of $\mathcal{F}$ with base S. We can also fix a polynomial P and restrict to quotients with that Hilbert polynomial.

Example 4.2. If $\mathcal{F} = \mathcal{O}$, then a quotient is just the structure sheaf of some closed subscheme $Z \subset X$, by considering the support of $\mathcal{Q}$.

To see this, take an affine open set U: $\mathcal{O}_X(U) \cong \mathrm{Spec}(A)$. A quotient is a surjective map $A \to \mathcal{Q}(U)$, so $\mathcal{Q}(U) \cong A/I$, for some ideal I. Then $Z \cap U = \mathrm{Spec} A/I = Z(I) \subset X$. Let $\mathcal{I}$ be the ideal sheaf of Z, i.e., on an open set U, $\mathcal{I}(U) = \ker(\mathcal{F}(U) \to \mathcal{Q})$. So in this case, the Quot functor takes a scheme S to the set of all families of subschemes of X with base S. This is called the *Hilbert functor*. If it is representable, it is called the *Hilbert scheme*.

If we consider Example 1.1, the Hilbert scheme is Spec $\mathbb{C}[t]/(t^2)$.

Exercise 4.3. Set the Hilbert polynomial to be $P(n) \equiv 1$. Show that $\mathrm{Quot}(\mathcal{O}_X, P) \cong X$ (as a scheme/functor of points - not just as a set).

Example 4.4. To show that the Quot functor is representable, we will need the Grassmannian functor. Let k be a field, and V a finite dimensional vector space over k. Let r be an integer such that $0 \leq r \leq \dim(V)$. Then $\mathrm{Grass}(V, r)$ is a functor from the opposite category of Sch/k to Sets, such that

$$\mathrm{Grass}(V, r)(X) = \{\text{quotients } \mathcal{O}_X \otimes_k V \to F \to 0 \text{ that are locally free and of constant rank } r\}.$$

I will sketch how to show that this functor is representable. First, fix a $W \subset V$ of dimension r. Consider the subset of $\mathrm{Grass}(V, r)$ such that the restriction of $\mathcal{O}_X \otimes_k V \to F$ to $\mathcal{O}_X \otimes W$ gives an isomorphism $\mathcal{O}_X \otimes W \to F$. Let G_W be the subfunctor defined by these subsets.

G_W is representable. To see this, note that there is an isomorphism $\mathcal{O}_X \otimes W \leftarrow F$. Then we get a composition $\mathcal{O}_X \otimes_k V \to F \to \mathcal{O}_X \otimes_k W$, which is the right inverse of the inclusion $\mathcal{O}_X \otimes_k W \to \mathcal{O}_X \otimes_k V$. So G_W corresponds to morphisms $V \to W$ which splits the inclusion. If $K \subset W$ such that $V = K \oplus W$, then G_W corresponds to morphisms $K \to W$. So G_W is the functor $\mathbf{Hom}(K, W)$, defined by

$$\mathbf{Hom}(K, W)(X) = \mathrm{Hom}(\mathcal{O}_X \otimes_k K, \mathcal{O}_X \otimes_k W).$$

$G_W = \mathbf{Hom}(K, W)$ is representable by $\overline{G}_W = \mathrm{Spec}(\mathrm{Sym}(\mathrm{Hom}(K, W)^*))$. To see this, recall that the universal property of the symmetric algebra is the following: given a k-vector space V and a k-module B, then there is a natural isomorphism $\mathrm{Hom}_{k-\mathrm{v.s.}}(V, B) = \mathrm{Hom}_{k-\mathrm{alg}}(\mathrm{Sym}V, B)$. Thus,

$$\mathrm{Hom}_{\mathrm{sch/k}}(X, \mathrm{Spec}(\mathrm{Sym}(\mathrm{Hom}(K, W)^*))) \cong \mathrm{Hom}_{k-\mathrm{alg}}(\mathrm{Sym}(\mathrm{Hom}(K, W)^*)), \mathcal{O}_X(X))$$

$$\cong \mathrm{Hom}_{k-\mathrm{v.s.}}(\mathrm{Hom}(K, W)^*, \mathcal{O}_X(X)) \cong \mathcal{O}_X(X) \otimes \mathrm{Hom}(K, W).$$

The idea now is to show that the G_W are open (we know that they are affine) subfunctors, that agree on intersections, and so they can be glued together to get $\mathrm{Grass}(V, r)$. Fix X and W, and consider a quotient $\phi : \mathcal{O}_X \otimes_k V \to F$. Consider open subschemes Y of X such that the restriction of ϕ to Y lies in the image of $G_W(Y)$, and choose a maximal such Y, call it Y_W. Then this is the definition of G_W being an *open sub-functor*. As W varies, the Y_W form an open covering of X, so the G_W are an affine open covering of G. After checking the cocycle condition, we can see that we can glue the $\overline{G}_W$ together, and this scheme represents G.

Finally, you can use the Plucker embedding to show that $\mathrm{Grass}(V, r)$ is projective (and indeed a smooth irreducible variety).

Theorem 4.5. *Let $\mathcal{F}$ be a coherent sheaf over a projective variety $(X, \mathcal{O}(1))$. Then Quot$(\mathcal{F}, P)$ is representable.*

Outline: First, let's fix a quotient $0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{Q} \to 0$. As $\mathcal{O}(1)$ is ample, there is an $m >> 0$ such that for all $i > 0$, $H^i(\mathcal{K}(m)) = 0$. Moreover, for m sufficiently large, $\mathcal{K}(m)$ is globally generated. This means that there is an exact sequence

$$H^0(\mathcal{K}(m)) \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{K}(m) \to 0.$$

Sections of $\mathcal{K}(m)$ generate $\mathcal{K}(m)$ inside $\mathcal{F}(m)$, and taking the quotient defines $\mathcal{Q}(m)$. So the quotient is determined by the exact sequence of vector spaces

$$0 \to H^0(\mathcal{K}(m)) \to H^0(\mathcal{F}(m)) \to H^0(\mathcal{Q}(m)) \to 0.$$

But this exact sequence is a point of $\mathrm{Gr}(H^0(\mathcal{F}(m)), P(m))$ (recall that by definition $P(m) = \dim H^0(\mathcal{F}(m))$. We assume that you can choose m universally.

We then need to generalize and do the above over a base S, in order to get a subfunctor of the Grassmannian. The Grassmannian is representable, so this gives a subscheme of the Grassmannian, called Quot. For the details of the proof, see Huybrechts-Lehn. $\square$

Now we can give the outline of Simpson's construction of the moduli space of vector bundles over $(X, \mathcal{O}(1))$ with given Hilbert polynomial P.

1. Fix a bundle ξ on X with Hilbert polynomial P.
2. Find $N > 0$ such that all higher cohomology of $\xi(N)$ vanishes, and $\xi(N)$ is generated by its global sections. Again, this means that
$$H^0(\xi(N)) \otimes \mathcal{O}(-N) \to \xi \to 0.$$
3. Fix an identification $H^0(\xi(N)) \cong \mathbb{C}^{P(N)}$.
4. Under this identification, ξ is a a quotient sheaf of $\mathcal{O}(-N)^{\oplus P(N)}$, so $\xi \in \mathrm{Quot}(\mathcal{O}(-N)^{\oplus P(N)}, P)$.
5. Divide by the choice of identification in 3). That is, divide by the action of $SL(P(N))$.
6. Find a stability condition to do GIT for this quotient, and let Quot' denote the semi-stable points.
7. Go back to step 2, and show that N can be chosen universally for all bundles satisfying the stability condition.
8. Check that Quot'$//SL(N)$ is a moduli space of stable bundles/sheaves.

In step 4, we identified ξ as an element of $\mathrm{Quot}(H^0(\xi(N)) \otimes H^0(\mathcal{O}(-N)), P)$, divided by the action.

In a previous exercise, we showed that the stability condition for a subspace $A \subset V \otimes W$, under the $SL(V)$ action on $\mathrm{Gr}(V \otimes W, r)$ was for all proper subspaces $S \subset V$,
$$\frac{\dim A \cap (S \otimes W)}{\dim S} < \frac{\dim A}{\dim V}.$$
So as an element of $\mathrm{Quot}(H^0(\xi(N)) \otimes H^0(\mathcal{O}(-N)), P$, ξ is stable by checking proper subspaces S of $H^0(\xi(N))$. By unwinding $\xi(N)$ and evaluating sections, S gives rise to a subsheaf of ξ. ξ is stable if and only if for all proper subsheaves $\mathcal{F} \subset \xi$, and $n >> 0$,
$$\frac{P_{\mathcal{F}}(n)}{\mathrm{rank}\mathcal{F}} < \frac{P_\xi(n)}{\mathrm{rank}\xi}.$$
This is called Gieseker stability. If we consider the leading coefficient as $n \to \infty$, then we get *slope stability*,
$$\frac{\int_X c_1(\mathcal{F})\omega^{n-1}}{\mathrm{rank}\mathcal{F}} < \frac{\int_X c_1(\xi)\omega^{n-1}}{\mathrm{rank}\xi}.$$
Slope stability implies Gieseker stability, and on a curve, these are equivalent.

Stability is a generic condition.

Exercise 4.6. Show that both Gieseker stability and slope stability ($u(E)$ is the slope of E) satisfy the see-saw property. That is, for all exact sequences
$$0 \to A \to \xi \to B \to 0,$$
such that $A \neq 0$, $u(A) < u(E) \Leftrightarrow u(\xi) < u(B) \Leftrightarrow u(A) < u(B)$, and similarly when we consider Gieseker stability instead of slope stability.

Exercise 4.7. Show that a stable vector bundle is simple.

Stability rules out the possibility of the 'jump phenomenon'.

Lemma 4.8. *If $\xi, \mathcal{F}$ are families of stable sheaves parametrized by $\mathbb{C}$, and for all $t \in \mathbb{C}^*$, $\xi_t \cong \mathcal{F}_t$, then $\xi_0 \cong \mathcal{F}_0$.*

Proof. By definition, $\xi, \mathcal{F}$ are sheaves on $X \times \mathbb{C}$, flat over $\mathbb{C}$ with the same Hilbert polynomial on each fibre, and stable on each fibre.

Let $\pi : X \times \mathbb{C} \to \mathbb{C}$ be the projection. $\mathcal{H}om(\xi, \mathcal{F})$ is a sheaf on $X \times \mathbb{C}$ (defined by $\mathcal{H}om(\xi, \mathcal{F})(U) = \mathrm{Hom}(\xi|_U, \mathcal{F}|_U)$), so we can construct the direct image sheaf under π, $\pi_* \mathcal{H}om(\xi, \mathcal{F})$. By definition, for an open $U \subset \mathbb{C}$, $\pi_* \mathcal{H}om(\xi, \mathcal{F})(U) = \mathrm{Hom}(\xi|_{X \times U}, \mathcal{F}|_{X \times U})$. The fiber over $t \in \mathbb{C}$ is $\mathrm{Hom}(\xi_t, \mathcal{F}_t)$, as these are coherent sheaves.

Because $\xi_t, \mathcal{F}_t$ are simple by the previous exercise, the fiber over $t \in \mathbb{C} - \{0\}$ is $\mathrm{Hom}(\xi_t, \mathcal{F}_t) \cong \mathbb{C}$.

Exercise 4.9. Show that $\pi_* \mathcal{H}om(\xi, \mathcal{F})$ is torsion free over $\mathbb{C}$. Recall that a sheaf $\mathcal{G}$ over a scheme $(X, \mathcal{O}_X)$ is torsion-free if for all $p \in X$, $\mathcal{G}_p$ is a torsion-free $\mathcal{O}_{X,p}$- module.

If $\xi, \mathcal{F}$ are families of vector bundles, then $\pi_* \mathcal{H}om(\xi, \mathcal{F})$ is a vector bundle, so as it is a line bundle over $\mathbb{C} - \{0\}$, it is a line bundle over $\mathbb{C}$. For sheaves, we need the previous exercise to show that it is a a line bundle over $\mathbb{C}$.

Pick a nonzero element of the fiber over 0. By definition, this is a non-zero morphism $\phi : \xi_0 \to \mathcal{F}_0$. So we can form two exact sequences:

$$0 \to \ker\phi \to \xi_0 \to \mathrm{im}\phi \to 0,$$

$$0 \to \mathrm{im}\phi \to \mathcal{F}_0 \to \mathrm{coker}\phi \to 0.$$

Using the see-saw property of μ and the stability of $\xi_0, \mathcal{F}_0$, if $\mathrm{im}\phi \neq \mathcal{F}_0$, we have that $\mu(\mathrm{im}\phi) > \mu(\xi_0)$, and $\mu(\mathrm{im}(\phi)) < \mu(\mathcal{F}_0)$. By definition of μ, it depends on the Hilbert polynomial of a sheaf, and so because ξ and $\mathcal{F}$ are families of fixed Hilbert polynomials, $\mu(\mathcal{F}_0) = \mu(\xi_0)$. So we have a contradiction, unless $\mathrm{im}\phi = \mathcal{F}_0$ and $\ker(\phi) = 0$. So ϕ is an isomorphism. $\square$

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