POINCARÉ DUALITY

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Contents

1. Introduction—Intersections

Recommended reading: Bott and Tu, the first fifty pages or so.

1.1. **The flat model.** Two (affine) subspaces in $\mathbb{R}^n$ of complementary dimension k and $n-k$. If in 'general position' they will meet in a point. If this is not the case (for example line contained in or parallel to a plane in $\mathbb{R}^3$) then a generic perturbations (of the coefficients defining the subspaces) will cause them to meet in a point.

1.2. **Transverse intersection.** Let M be a smooth manifold. Let Y and Z be submanifolds. Suppose that $p \in Y \cap Z$. We say that Y and Z intersect transversally at p if

$$T_pM = T_pY + T_pZ$$

i.e. the two tangent bundles of Y and Z at p together span T_pM. Examples in 3 dimensions.

Further, Y and Z intersect transversally if they intersect transversally at every point p in $Y \cap Z$. If Y and Z intersect transversally, their intersection is smooth and the codimension of $Y \cap Z$ is the sum of the codimensions of Y and Z. In particular, if the Y and Z are of complementary dimension, then their transverse intersection is a set of points and if M is

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compact (or more generally, at least one of Y and Z is compact) then this is a finite set of points.

Now suppose that M is orientable and fix an orientation. If now Y and Z are oriented closed submanifolds then we define

$$Y \cdot Z = \sum_{p \in Y \cap Z} \pm 1$$

where we take the $+$ sign if the orientation of $T_pY \oplus T_pZ$ agrees with the orientation of T_pM and the $-$ sign otherwise.

With these signs, this turns out to be a *topological invariant*: the intersection pairing of Y and Z depends only upon the homology classes defined by the submanifolds Y and Z.

Remark 1.1. Without orientations, one can always define the mod-2 intersection number of homology classes.

A first statement of PD is that, extended to homology, the intersection product

$$H_k(M, \mathbb{Z}) \otimes H_{n-k}(M, \mathbb{Z}) \longrightarrow \mathbb{Z}, \ \ (Y, Z) \mapsto Y \cdot Z \tag{1.1}$$

is perfect: every $\mathbb{Z}$-linear map $H_{n-k}(M, \mathbb{Z}) \to \mathbb{Z}$ arises from intersection pairing with some $Y \in H_k(M, \mathbb{Z})$ and on the other hand $Z \mapsto Y \cdot Z$ vanishes identically on H_{n-k} iff Y is in the torsion part of H_k.

As a first consequence, defining $b_k(M)$ to be the rank of the free part of $H_k(M, \mathbb{Z})$, equivalently the dimension of the real vector space $H_k(M, \mathbb{R})$, we have $b_k = b_{n-k}$ for any orientable compact n-manifold without boundary.

Further consequences:

Suppose that compact orientable M has even dimension $n = 2m$. On the middle-dimensional homology $H_m(M, \mathbb{R})$ the intersection form is a bilinear form which is symmetric if m is even and skew if m is odd. Since the pairing is non-degenerate, $H_m(M)$ must be a symplectic vector space and hence even-dimensional if m is odd. If m is even (so n is divisible by 4) then we have a quadratic form on $H_m(M, \mathbb{R})$ which turns out to be an important topological invariant of our manifold M. In particular it has a *signature* (number of positive eigenvalues minus number of negative eigenvalues), which is a more refined invariant than just the number $b_m(M)$.

In dimension 4, there is a classification (Hasse–Minkowski) of the quadratic forms over the integers that can occur. There are two families:

Odd forms $\mathrm{diag}(\pm 1, \ldots, \pm 1)$

Even forms $rE_8 + sH$, where $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and

$$E_8 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$$

It is still an open question which of the even forms occur as intersection forms of smooth 4-manifolds. (It is known that all occur as intersection-forms of compact oriented simply connected topological 4-manifolds.) Cf. breakthrough work of Freedman and Donaldson (seperately) in the early 1980s.

Remark 1.2. Self-intersections: If Y is an oriented m-dimensional submanifold of M^{2m}, and we want to compute $Y \cdot Y$, then we have to work a bit, because Y certainly does not intersect itself transversely!

This number is computed by taking a transverse perturbation (somehow) Y' of Y in its homology class, then counting intersection points (with sign).

2. de Rham comohology

2.1. **Definitions.** De Rham complex, d wedge product. Compact supports, two cohomology theories. Functorial properties. Extension by zero for compact supports.

2.2. **Poincaré Lemmas.** These state that:

$$H^*(\mathbb{R}^n) = \mathbb{R} \text{ in degree } 0, \text{ and vanishes otherwise}$$

for 'ordinary' cohomology and

$$H^*_c(\mathbb{R}^n) = \mathbb{R} \text{ in degree } n, \text{ and vanishes otherwise.}$$

Write down representatives.

Proofs Induction on the dimension. Can be done by hand if $n = 1$. Suppose known for n, and try to prove for $n + 1$. For convenience write $x_{n+1} = y$. If ω is of positive degree, write

$$\omega = \alpha + \mathrm{d}y \wedge \beta \tag{2.1}$$

where α and β are forms with no $\mathrm{d}y$ term. This decomposition is unique, so in particular $\omega = 0$ iff $\alpha = 0$ and $\beta = 0$. Write d' for differentiation in the n x variables, so

$$\mathrm{d} = \mathrm{d}' + \mathrm{d}y \frac{\partial}{\partial y}. \tag{2.2}$$

Then we have

$$\mathrm{d}\omega = \mathrm{d}'\alpha + \mathrm{d}y \wedge (\partial_y \alpha - \mathrm{d}'\beta) \tag{2.3}$$

and there is no $\mathrm{d}y$ in $\mathrm{d}'\alpha$ or in $\partial_y\alpha - \mathrm{d}'\beta$. By the above remark,

$$\mathrm{d}\omega = 0 \Leftrightarrow \mathrm{d}'\alpha = 0 \text{ and } \partial_y\alpha - \mathrm{d}'\beta = 0.$$

coefficients of the forms will (generally) be functions of y.

To show that ω is exact, consider a form u with no $\mathrm{d}y$ term (but whose coefficients may depend on y). Then the equation

$$\mathrm{d}u = \omega \Leftrightarrow \partial_y u = \beta, \;\; \mathrm{d}'u = \alpha.$$

Define u so as to satisfy the first equation $\partial_y u = \beta$. This can be done simply by integrating from the origin, say. Then

$$\omega - \mathrm{d}u = \alpha - \mathrm{d}'u$$

and there is no $\mathrm{d}y$ on the RHS. The RHS is d-closed, and this implies (because no $\mathrm{d}y$) that the coefficients of the RHS are independent of y, so that the RHS is a form of positive degree in the n variables $(x_1, \ldots, x_n)$. By induction, this is exact, and it follows that ω is exact on $\mathbb{R}^n$.

Poincaré Lemma for compact supports.

We shall use induction again, let us do $n = 1$ in a bit more detail.

For $n = 1$, $H^0_c(\mathbb{R}) = 0$ because a constant function with compact support must be 0. If $\omega = f(x)\,\mathrm{d}x$ is a compactly supported 1-form, consider

$$u(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t.$$

Then $\mathrm{d}u = \omega$, but u only has compact support if

$$\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t = 0.$$

Thus integration gives a map

$$I : \Omega^1_c(\mathbb{R}) \to \mathbb{R},$$

clearly surjective and $\ker(I) = \mathrm{d}\Omega^0_c(\mathbb{R})$. This shows $H^1_c(\mathbb{R}) = \mathbb{R}$.

An interesting slant on the preceding proof is obtained by making it a bit more formal. For $p \in \mathbb{R}$, define

$$L : \Omega^k(\mathbb{R}^{n+1}) \to \Omega^{k-1}(\mathbb{R}^{n+1}), \;\; L\omega = \int_p^y \beta(x,t)\,\mathrm{d}t, \tag{2.4}$$

Lemma 2.1. *Let L be above and let $j : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be the inclusion*

$$x \mapsto (x, p).$$

We have the homotopy identity:

$$dL + Ld = \mathrm{Id} - j^* \tag{2.5}$$

Proof. Exercise using the definitions of d and the fundamental theorem of calculus. □

It follows directly that $H^k(\mathbb{R}^{n+1}) = H^k(\mathbb{R}^n)$ for any n.

For compact supports, take $p = -\infty$. Then

$$L : \Omega_c^k(\mathbb{R}^{n+1}) \to \Omega_{\text{prop}}^{k-1}(\mathbb{R}^{n+1}) \tag{2.6}$$

where the subscript means proper support with respect to the projection onto the first n variables (support contained in some $K \times \mathbb{R}$, where $K \subset \mathbb{R}^n$ is compact). In place of (2.5) we now have

$$\mathrm{d}L + L\mathrm{d} = \mathrm{Id} \tag{2.7}$$

which might make the cohomology look completely trivial, but this is not the case because $L\omega$ does not in general have compact support. To mend this problem, define a map

$$I : \Omega_c^k(\mathbb{R}^{n+1}) \to \Omega_c^k(\mathbb{R}^n) \tag{2.8}$$

by integrating out the last variable

$$I[\omega] = I[\alpha + \mathrm{d}y \wedge \beta] = \int_{-\infty}^{\infty} \beta(x,t)\,\mathrm{d}t \tag{2.9}$$

This is a (signed) cochain map:

Lemma 2.2. *We have*

$$dI = I d. \tag{2.10}$$

Proof. Exercise, an easier one than the proof of the previous Lemma! □

Note that $I\omega = 0$ iff $L\omega$ has compact support—cf. the 1-dimensional case.

Observe (note the similarity to what we did in one dimension) that

$$I\omega = 0 \Leftrightarrow L\omega \in \Omega_c^{k-1}(\mathbb{R}^{n+1}) \tag{2.11}$$

To exploit all this there is one more ingredient: pick a 1-form ρ on $\mathbb{R}$ with compact support and total integral equal to 1. Let

$$K : \Omega_c^{k-1}(\mathbb{R}^n) \to \Omega_c^k(\mathbb{R}^{n+1}) \tag{2.12}$$

be defined by wedge-product with ρ

$$K\eta = \rho \wedge \eta. \tag{2.13}$$

(More precisely ρ here is the pull-back of our form by projection onto the last variable $(x,y) \mapsto y$. This is an abuse of notation.)

Lemma 2.3. *We have*

$$dK + K d = 0, \quad IK\eta = \eta. \tag{2.14}$$

Thus K is a right-inverse to I but it is not a left-inverse. However there is a homotopy formula

$$\mathrm{Id} - KI = \mathrm{d}L' + L'\mathrm{d}, \quad \text{where } L' = L(1 - KI). \tag{2.15}$$

This follows by applying the chain homotopy formula to $(1 - KI)\omega$ and commuting d through K and I. At the level of cohomology, then K and I are inverses, and it follows that

$$I : H_c^k(\mathbb{R}^{n+1}) \longrightarrow H_c^{k-1}(\mathbb{R}^n) \tag{2.16}$$

is invertible. Induction now gives the compactly supported Poincaré Lemma.

2.3. **Stokes Theorem: the duality between homology and cohomology.** If Y is an oriented k-submanifold of M, then we have a linear map

$$\Omega_c^k(M) \to \mathbb{R}, \alpha \longmapsto \int_Y \alpha. \tag{2.17}$$

This descends to $H_c^k(M)$ because Stokes Theorem says that

$$\int_Y \mathrm{d}\eta = 0 \text{ for all } \eta \in \Omega_c^{k-1}(M). \tag{2.18}$$

If Y is a compact oriented k-submanifold, then we get a linear map $H^k(M) \to \mathbb{R}$ by the same token. The point is that at least one of Y and α should have compact support in M. Most of the time we are interested in compact manifolds anyway, in which case compact support is automatic.

The definition extends to k-cycles. These usally have compactness built in, so are formal linear combinations of smoothly embedded k-simplices in M. The value $\int_Y \alpha$ only depends on the homology class of Y, for if $Y = \partial Z$, then the other part of Stokes gives

$$\int_Y \alpha = \int_{\partial Z} \alpha = \int_Z \mathrm{d}\alpha = 0.$$

Theorem 2.4. *de Rham cohomology is dual to (smooth) singular homology (with real coefficients) in the sense that the integration pairing*

$$H_k(M, \mathbb{R}) \otimes H^k(M, \mathbb{R}), \quad (Y, \alpha) \mapsto \int_Y \alpha \tag{2.19}$$

is perfect.

This is not Poincaré duality! In the sense that this is always true, even if M is not a manifold, in which case $H^k(M, \mathbb{R})$ is defined in terms of singular cochains (linear functions on chains).

2.4. **Finite-dimensionality.** Mayer–Vietoris as a tool for computation. Good covers. $H^*(M)$ is finite-dimensional if M admits a finite good cover (always the case if M is compact). This can be proved by induction on cardinality of good cover using the Poincaré Lemmas and the Mayer–Vietoris sequences.

2.5. **Poincaré Duality in de Rham theory.** As before, let M be a compact oriented n-manifold. Then we have a map (cup product)

$$\alpha \cup \beta = \int_M \alpha \wedge \beta \tag{2.20}$$

at the level of forms which descends to a bilinear pairing in cohomology

$$H^k(M, \mathbb{R}) \otimes H^{n-k}(M, \mathbb{R}) \to \mathbb{R}. \tag{2.21}$$

PD states that this is a non-degenerate pairing, identifying each space with the dual of the other. There are various generalizations, for example if M is non-compact but admits a finite good cover, then the cohomology groups are still finite-dimensional, the cup product is defined as a pairing

$$H^k(M,\mathbb{R}) \otimes H_c^{n-k}(M,\mathbb{R}) \to \mathbb{R} \tag{2.22}$$

and is non-degenerate.

2.6. **Sketch Proof.** In the last exercise of the sheet you are invited to try to complete the details of the following sketch. Let M be a smooth oriented manifold with a finite good cover. (If M is compact, such covers always exist.)

Theorem 2.5. *Let M be as above. Then wedge product descends to define a perfect pairing $H_c^k(M) \otimes H^{n-k}(M) \to \mathbb{R}$.*

Use induction on the cardinality, N, of the cover. If $N = 1$, the statement is just the Poincaré Lemma. Suppose it is known for all manifolds with a cover of cardinality N, and suppose that M has a good cover of cardinality $N+1$. Write $M = U \cup V$, where

$$U = U_1 \cup \cdots \cup U_N, \quad V = U_{N+1}.$$

Write down the Mayer–Vietoris sequences for H^* and H_* for $M = U \cup V$. Beware: the maps for the two theories go in opposite directions. This is because if U is an open subset of M, then the inclusion induces a map $\Omega_c^k(U) \to \Omega_c^k(M)$ given by 'extension by zero'.

This yields a diagram of the following kind in which the rows are exact. (Check you

$$\begin{array}{ccccccccc}
\longrightarrow & H^{k-1}(U\cap V) & \longrightarrow & H^k(M) & \longrightarrow & H^k(U)\oplus H^k(V) & \longrightarrow & H^k(U\cap V) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H_c^{n-k+1}(U\cap V)^* & \longrightarrow & H_c^{n-k}(M)^* & \longrightarrow & H_c^{n-k}(U)^*\oplus H_c^{n-k}(V)^* & \longrightarrow & H_c^{n-k}(U\cap V)^* & \longrightarrow
\end{array}$$

understand exactly what is going on here.) Now use the five lemma to complete the proof by induction. The inductive assumption is supposed to give that most of the vertical arrows are isomorphisms, as needed for an application of the five lemma.

3. The Poincaré dual of a submanifold

Let M be a compact oriented smooth n-manifold. Since $H_k(M,\mathbb{R})$ and $H^{n-k}(M,\mathbb{R})$ are both dual to $H^k(M,\mathbb{R})$ we have an isomorphism

$$H_k(M) \simeq H^{n-k}(M) \tag{3.1}$$

This means that there is a correspondence between closed k-cycles on M and closed $(n-k)$-forms on M, $Y \mapsto \eta_Y$, with the property

$$\int_Y \alpha = \int_M \alpha \wedge \eta_Y \tag{3.2}$$

for all $\alpha \in H^k(M)$. The form η_Y is often referred to as the 'Poincaré dual' of Y.

The 'explicit' construction of the Poincaré dual η_Y of Y is interesting but we only have space for an example or two. Note that its degree is the codimension of Y. It turns out that one can find representatives of η_Y which are supported in arbitrarily small tubular neighbourhoods of Y, and which look, transversely, like a bump $(n-k)$-form in the normal variables.

Remark 3.1. Currents.

Intersection form is PD to cup product

Suppose that Y and Z are (closed) oriented submanifolds of complementary dimensions k and $n-k$ which intersect transversally. By the above we have dual classes η_Y and η_Z of degrees respectively $(n-k)$ and k. Let p be an intersection point and choose coordinates $x = (x', x'')$ centred at p and defined in a small open subset U such that

$$Y \cap U = \{x'' = 0\} = \{x_{k+1} = \cdots = x_n = 0\} \text{ and } Z \cap U = \{x' = 0\} = \{x_1 = \cdots = x_k = 0\}. \tag{3.3}$$

It turns out that it's not far off the truth to say that in U,

$$\eta_Y = \rho(x'')\mathrm{d}x'', \quad \eta_Z = \rho(x')\mathrm{d}x'$$

where ρ stands for a bump function in the given number of variables, of total mass 1. We may suppose that the support of each is arbitrarily close to 0 and then it is clear that

$$\int_U \eta_Y \wedge \eta_Z = \pm 1$$

Thus $\int_M \eta_Y \wedge \eta_Z$ is a sum with signs over the points of $Y \cap Z$. This is a sketch of the principle that the intersection of transverse submanifolds in $H_*(M)$ goes over to wedge product under the isomorphism (3.6).

3.1. **Self-intersection of the diagonal and the Euler characteristic.** Let M be a smooth compact oriented manifold of dimension n.

Theorem 3.2. *Hopf index Theorem. Let V be a generic vector field on M with isolated simple zeros. Then the number of zeros, counted with sign, is equal to the Euler characteristic,*

$$\chi(M) = \sum (-1)^k \dim H^k(M, \mathbb{R}). \tag{3.4}$$

Proof. We do not prove this in full, in particular it would be too much of a diversion to give the full definition of how to count zeros of vector fields.

Let Δ be the diagonal of $M \times M$. We shall prove

$$\Delta \cdot \Delta = \chi(M). \tag{3.5}$$

This is well on the way to proving the theorem. To sketch why, note that a tubular neighbourhood U of Δ in $M \times M$ can be identified diffeomorphically with TM, the tangent bundle of M, and so $\Delta \cdot \Delta$ is the same as the self-intersection of the zero-section of TM. We may perturb the zero section simply by replacing it by a section of TM, which is the same as a vector field. The intersections are the zeros of the vector field. There is a way of counting zeros of vector fields which gives the self-intersection of the zero-section in TM. So if you believe all that, it remains to prove (3.5).

Since $H_k(M)$ and $H^{n-k}(M)$ are dual to $H^k(M)$ we have an isomorphism

$$H_k(M) \simeq H^{n-k}(M) \tag{3.6}$$

If the class in H_k is represented by a submanifold Y, then this is the class of η_Y.

Let η be the Poincare dual of the diagonal Δ in $M \times M$ so that for every class $\theta \in H^n(M \times M)$,

$$\int_\Delta \theta = \int \theta \wedge \eta. \tag{3.7}$$

By the Künneth formula for cohomology, $H^*(M \times M) = H^*(M) \otimes H^*(M)$. Thus we can choose a basis in the following way: let (e_j) and (f_j) be dual bases for wedge product (possible by PD), that is $\int e_j \wedge f_k = \delta_{jk}$. Now let α_j be the pull-back by the first projection of e_j and let β_k be the pull-back by the second projection of f_k. Then $\alpha_j \wedge \beta_k$ is a basis of $H^*(M \times M)$. Thus

$$\eta = \sum c_{rs} \pi_1^* e_r \wedge \pi_2^* f_s \tag{3.8}$$

for some real coefficients c_{rs}.

Claim:

$$\eta = \sum (-1)^{\deg e_r} \pi_1^* e_r \wedge \pi_2^* f_r$$

The claim is proved as follows. Fix e_q of degree k and f_p of degree m. Then

$$\int_\Delta \pi_1^* f_p \wedge \pi_2^* e_q = \int_M f_p \wedge e_q = (-1)^{mk} \delta_{pq} = (-1)^{k(n-k)} \delta_{pq}. \tag{3.9}$$

By the defining property of η, we also have

$$\begin{aligned}
\int_\Delta \pi_1^* f_p \wedge \pi_2^* e_q &= \int_{M\times M} \pi_1^* f_p \wedge \pi_2^* e_q \wedge \eta && (3.10)\\
&= \sum_{r,s} c_{rs} \int_{M\times M} \pi_1^* f_p \wedge \pi_2^* e_q \wedge \pi_1^* e_r \wedge \pi_2^* f_s && (3.11)\\
&= \sum_{r,s} c_{rs} (-1)^{(m+k)(n-m)} \int \pi_1^*(e_r \wedge f_p) \wedge \pi_2^*(e_q \wedge f_s) && (3.12)\\
&= \sum_{r,s} c_{rs} (-1)^{(m+k)(n-m)} \delta_{rp} \delta_{qs} && (3.13)
\end{aligned}$$

Equating, and using that $m + k = n$ when $p = q$,

$$c_{pq} = (-1)^k \text{ where } \deg e_p = k. \tag{3.14}$$

To complete the proof, we compute

$$\Delta \cdot \Delta = \int_\Delta \eta = \sum c_{pq} \int e_p \wedge f_q = \sum_p (-1)^{\deg e_p}. \tag{3.15}$$

Because the e_p form a basis of $H^*(M)$, this sum just counts up the elements in the basis, with a plus sign if the basis element is in the even cohomology and a minus sign if in the odd cohomology. The proof is complete. □