# A lecture on K3 surfaces

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A K3 surface over a field $k$ is a complete non-singular variety $S$ of dimension two such that

- (trivial canonical bundle) $\omega_S \simeq \mathcal{O}_S$
- ("simply-connected") $H^1(S, \mathcal{O}_S) = 0$

The name "K3" was given by André Weil in honor of Kummer, Kähler and Kodaira, who made crucial contributions to the field of complex geometry. The story is that the mountain K2 was recently climbed for the first time when K3 surfaces became popular and people said the study of these surfaces is almost as hard as climbing K2 so they called them K3 surfaces.

By definition the canonical bundle $\omega_S = \Omega_S^2$ where the cotangent sheaf $\Omega_S$ of a K3 surface is locally free of rank 2. The natural pairing

$$\Omega_S \times \Omega_S \to \omega_S \simeq \mathcal{O}_S$$

gives an algebraic symplectic structure.

(An equivalent way to say what a K3 surface is that it is a complex surface with a nowhere vanishing holomorphic 2-form, and with no non-trivial holomorphic 1-form. )

I will restrict to $k = \mathbb{C}$ because I want to appeal to Hodge theory, and also view $S$ as a complex manifold (It is known that any complex K3 surface is Kähler, though this is a non-trivial result of Siu). Recall that topologically a smooth complex surface gives us an oriented 4-manifold. The main topological invariants of such a manifold are its fundamental group, Betti numbers $b_i(S) = b_{4-i}(S)$ and its intersection pairing on $H_2(S, \mathbb{Z}) \simeq H^2(S, \mathbb{Z})$ where the latter identification is by Poincaré dualiy. Via this identification, the intersection pairing corresponds to the cup product:

$$H^2(S, \mathbb{Z}) \otimes H^2(S, \mathbb{Z}) \to H^4(S, \mathbb{Z}) \simeq \mathbb{Z}$$

Over $\mathbb{R}$, one can diagonalize this form and we write $b_2^{\pm}(S)$ for the number of positive/negative eigenvalues. Since the pairing is non-degenerate $b_2(S) = b_2^+(S) + b_2^-(S)$.

Recall that the most basic holomorphic invariants of an algebraic surface are

$$q(S) = \dim H^1(S, \mathcal{O}_S)$$

called the *irregularity* and

$$p_g(S) = \dim H^2(S, \mathcal{O}_S) = \dim H^0(S, \omega_S)^\vee$$

called the *geometric genus*, where the latter equality follows from Serre dualiy.

Hodge theory relates these to topological invariants as follows:

$$\begin{aligned} b_1(S) &= 2q(S) \\ b_2(S) &= 2p_g(S) + h^{1,1}(S) \\ b_2^+(S) &= 2p_g(S) + 1 \end{aligned}$$

where $h^{1,1}(S) = \dim H^1(S, \Omega_S)$.

For a K3 surface, by definition we have $q(S) = 0$ hence $b_1(S) = 0$, and $p_g(S) = 1$. On the other hand, recall the *Noether formula* (a special case of Hirzebruch-Riemann-Roch)

$$\chi(S, \mathcal{O}_S) = \frac{c_1^2(S) + c_2(S)}{12}$$

or equivalently

$$12(1 - q + p_g(S)) = c_1^2(S) + c_2(S).$$

For a K3 surface, this implies that the second Chern class of $S$ that coincides with the topological Euler characteristic

$$b_0(S) - b_1(S) + b_2(S) - b_3(S) + b_4(S) = 24$$

Since $b_0(S) = b_4(S) = 1$ and $b_1(S) = b_3(S) = 0$, we get that $b_2(S) = 22$.

For a K3 surface, the Hodge numbers $h^{p,q}(S) := \dim H^q(S, \Omega_S^p)$ are determined as follows: By definition, we have $h^{0,0} = h^{2,0} = h^{0,2} = h^{2,2} = 1$. We have also determined above that $h^{1,1} = 20$, and all other Hodge numbers vanish by Hodge decomposition. So, the Hodge diamond looks as follows:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Next, let us give some examples.

*Complete intersections:* Let $S \subset \mathbb{P}^3$ be a smooth quartic surface. Since $\omega_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(-4)$, by the adjunction formula, we get

$$\omega_S = (\omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(4))_{|S} \simeq \mathcal{O}_S$$

We use the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_S \to 0$$

and the vanishing of cohomology $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) = 0$ to deduce that $H^1(S, \mathcal{O}_S) = 0$.

The most famous quartic surface is given by $x^4+y^4+z^4+w^4 = 0$ called the Fermat quartic.

**Remark 0.1.** *André Weil's original definition of a K3 surface is a compact complex surface which is diffeomorphic to the quartic surface. It is non-trivial statement whose proof requires Seiberg-Witten theory that this definition is equivalent to the more standard definition we gave above.*

Similarly, a smooth complete intersection of type $(d_1, d_2, \ldots, d_n)$ in $\mathbb{P}^{n+2}$ is a K3 surface if and only if $\sum d_i = n+3$. There are only three non-trivial cases: 1) $(\mathbb{P}^3, \mathcal{O}(4))$, 2) $(\mathbb{P}^4, \mathcal{O}(2) \oplus \mathcal{O}(3))$, 3) $(\mathbb{P}^5, \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2))$.

*Exercise*: Here are some other examples. $(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(2,3))$, $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,2,2))$, $(Gr(2,6), \mathcal{O}(1)^{\oplus 6})$. Check that the zero set of a generic section of these vector bundles gives a K3 surface.

Many more examples are obtained if we consider complete intersections in weighted projective spaces. Recall that the weighted projective space associated to weights $(d_0, d_1, \ldots, d_N)$ is given by

$$\mathbb{P}(d_0, d_1, \ldots, d_N) = \mathrm{Proj}(A)$$

where $A = k[x_0, x_1, \ldots, x_N]$ is the graded ring with $|x_i| = d_i$.

It can also be viewed as a toric variety defined by the polytope

$$conv\{(\prod d_i/d_0, 0, \ldots, 0), \ldots, (0, \ldots, \prod d_i/d_N)\}.$$

Let $\mathbf{w} \in A$ be a weighted homogeneous polynomial satisfying:

$$\mathbf{w}(t^{d_0}x_0, t^{d_1}x_1, \ldots, t^{d_N}x_N) = t^h \mathbf{w}(x_0, x_1, \ldots, x_N), \quad t \in \mathbb{G}_m$$

Then, set $S = A/(\mathbf{w})$ and we obtain a hypersurface

$$\mathrm{Proj}(S) \subset \mathbb{P}(d_0, d_1, \ldots, d_N)$$

The canonical bundle of such a hypersurface is given by

$$\omega_S \simeq \mathcal{O}(h - d_0 - d_1 - \ldots - d_N)$$

hence if we arrange the weights correctly we will get K3 surfaces. One of my favourite examples is

$$\mathbf{w}(x,y,z,w) = x^2 + y^3 + z^7 + w^{42}$$

this gives a K3 surface in $\mathbb{P}(21, 14, 6, 1)$.

In the complement of the curve $w = 0$, we get the smooth affine surface

$$x^2 + y^3 + z^7 + 1 = 0$$

which is diffeomorphic to the Milnor fiber of the singulariy $x^2 + y^3 + z^7 = 0$.

The projective surface has three singular points of

$$(0, -1, 1, 0) \text{ of type } A_1$$
$$(1, -1, 0, 0) \text{ of type } A_6$$
$$(1, 0, -1, 0) \text{ of type } A_2$$

Resolving these singularities, we get a minimal model of a smooth K3 surface.

Miles Reid classified and listed all the 95 families of K3 hypersurfaces found in $\mathbb{P}(d_0, d_1, d_2, d_3)$ for some specific $(d_0, d_1, d_2, d_3)$. This was never published but there is a paper by Yonemura and a thesis of Belcastro as useful references.

*Kummer surfaces:* Let $A$ be a two-dimensional complex torus, and $\iota : A \to A$ be the involution $a \to -a$. This has 16 fixed points. $A/\iota$ has only ordinary double point singularities and there are 16 of them. The surface $S \to A/\iota$ given by the minimal resolution of $A/\iota$, obtained by blowing up these 16 points, are K3 surfaces and they are called Kummer surfaces. If $A$ is not algebraic, then we will get a non-algebraic complex K3 surface. This is an important class of K3 surfaces but I will not say more about them in this lecture.

**Picard lattice:** Another important algebraic invariant of a K3 surface $S$ is its Picard group $\mathrm{Pic}(S)$. This is the group of isomorphism classes of invertible sheaves (line bundles) on $S$.

For K3 surfaces $\mathrm{Pic}(S)$ is equivalent to linear equivalence classes of divisors (linear combinations of curves). This coincides with the Néron-Severi lattice $\mathrm{Pic}(S)/\mathrm{Pic}^0(S)$ where $\mathrm{Pic}^0(S)$ is the subgroup of line bundles that are algebraically equivalent to zero. In other words, for a K3 surface, $\mathrm{Pic}^0(S)$ is trivial.

Working over complex numbers, using the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_S \to \mathcal{O}_S^* \to 0$$

we obtain an exact sequence

$$0 \to H^1(S, \mathcal{O}_S^*) \xrightarrow{c_1} H^2(S, \mathbb{Z}) \xrightarrow{\iota_*} H^2(S, \mathcal{O}_S)$$

(This also shows that $H^2(S, \mathbb{Z})$ is torsion free.) Using the canonical isomorphism $\mathrm{Pic}(S) \simeq H^1(S, \mathcal{O}_S^*)$ ([GAGA] identifies analytic and algebraic Pic), we see in fact that there is an embedding of lattices

$$\mathrm{Pic}(S) \hookrightarrow H^2(S, \mathbb{Z})$$

and by the Lefschetz theorem on (1,1)-classes the image of this map can be identified with $H^2(S, \mathbb{Z}) \cap H^{1,1}(S, \mathbb{C})$. Implicit here is that $\mathrm{Pic}(S)$ has a symmetric bilinear form given by summing over local intersection numbers of represensative curves. Under the embedding into $H^2(S, \mathbb{Z})$, this pairing can be computed via cup product on $H^2(S, \mathbb{Z})$.

The abstract isomorphism type of the rank 22 lattice $H^2(S, \mathbb{Z})$ is independent of $S$ (in fact, all K3 surfaces are diffeomorphic).

There is an isomorphism of rank 22 lattices

$$H^2(S, \mathbb{Z}) \simeq (-E_8) \oplus (-E_8) \oplus U \oplus U \oplus U$$

where the negative definite $(-E_8)$ lattice is given by the intersection matrix

$$(-E_8) := \begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & 1 & & & \\ & & 1 & -2 & 0 & & & \\ & & 1 & 0 & -2 & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & 1 \\ & & & & & & 1 & -2 \end{pmatrix}$$

and $U$ is the hyperbolic plane given by

$$U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The isomorphism follows by classification of even, unimodular lattices due to Milnor. The K3 lattice is characterized as the unique unimodular, even lattice of signature $(3, 19)$.

(unimodular means determinant of a matrix representing the symmetric bilinear form is $\pm 1$. even means $(x, x) \in 2\mathbb{Z}$ for all $x$, signature is the number of positive/negative eigenvalues over $\mathbb{R}$).

Picard lattice $\mathrm{Pic}(S)$ is a primitive even sublattice of K3 lattice with signature $(1, \rho - 1)$ where $\rho = \mathrm{rank}(\mathrm{Pic}(S))$ is the *Picard rank*. One can think of this as the classes in $H^2(S, \mathbb{Z})$ represented by algebraic curves.

In general, it is a difficult and interesting problem to compute $\mathrm{Pic}(S)$ for a given $S$. Even determining $\rho(S)$ could be difficult. In characteristic zero, Lefschetz's theorem implies that

$$\rho(S) \le h^{1,1}(S) = 20$$

In fact, every Picard number between 0 and 20 is realized by some complex K3 surface. (Over arbitrary characteristic, one has $\rho(S) \le 22$ by Igusa's theorem.)

**Transcendental lattice:** Another important lattice associated with a surface is the transcendental lattice

$$T(S) := \mathrm{Pic}(S)^{\perp} \subset H^2(S; \mathbb{Z})$$

For a K3 surface, $T(S)$ is also a primitive sublattice of the $K3$ lattice of signature $(2, 20 - \rho)$.

The Picard and transcendental lattices of the Fermat quartic in $\mathbb{P}^3$ defined by $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ are given by

$$\mathrm{Pic}(S) \simeq (-E_8) \oplus (-E_8) \oplus U \oplus (-8) \oplus (-8), \quad T(S) \simeq (8) \oplus (8)$$

A modern proof of this appears in a paper of Schütt, Shioda and van Luijk.

**Hodge theory and Torelli theorem:**

A *Hodge structure of weight* $n \in \mathbb{Z}$ on a free $\mathbb{Z}$-module $V$ is given by a direct sum decomposition of the complex vector space

$$V \otimes \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$.

The most important example of Hodge structures come from the cohomology of smooth projective varieties over $\mathbb{C}$, or more generally, compact Kähler manifolds.

$$H^n(S,\mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}(S)$$

where $H^{p,q}(S)$ can be viewed either as the space of de Rham classes of bidegree $(p,q)$ or the Dolbeault cohomology $H^q(S,\Omega_S^p)$.

For a surface with $H^1(S,\mathbb{Z}) = 0$, the only relevant Hodge structure comes from the second cohomology.

$$H^2(S,\mathbb{Z}) \otimes \mathbb{C} = H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S)$$

For a K3 surface, by definition $H^{2,0}(S) = \mathbb{C} \cdot \Omega$ for a non-vanishing holomorphic 2-form (defined up to a scalar) is of dimension one, $H^{0,2}(S)$ is complex conjugate of $H^{2,0}(S)$ and $H^{1,1}(S)$ is the orthogonal (with respect to intersection pairing) of $H^{2,0}(S) \oplus H^{0,2}(S)$. Hence, the weight two Hodge structure on $H^2(S,\mathbb{Z})$ is determined by the line $\mathbb{C} \cdot \Omega \subset H^2(S,\mathbb{C})$.

The importance of Hodge structures for K3 surfaces is due to the following theorem

**Theorem 0.2.** *(global Torelli theorem) Two complex K3 surfaces $S$ and $S'$ are isomorphic if and only if there exists an isomorphism $H^2(S,\mathbb{Z}) \simeq H^2(S',\mathbb{Z})$ of integral Hodge structures of weight 2 respecting the intersection pairing.*

Note that the Hodge isomorphism is not necessarily induced by isomorphism of varieties. On the other hand, if the Hodge isomorphism sends a Kähler class to a Kähler class, then it is induced by an isomorphisms of varieties.

By our remark above, a Hodge isomorphism is an isometry $f : H^2(S,\mathbb{Z}) \to H^2(S',\mathbb{Z})$ such that $f(H^{2,0}(S)) = H^{2,0}(S')$. Thus, one can interpret the global Torelli theorem by saying that the complex structure of $S$ is determined by the class $[\Omega] \in \mathbb{P}(H^2(S,\mathbb{Z}))$ satisfying the Hodge-Riemann bilinear relations

$$(\Omega,\Omega) = 0, \quad (\Omega,\overline{\Omega}) > 0$$

where $(,)$ refers to the intersection pairing.

To put this into the context of period maps, we first recall the case of curves.

It has been known more than a century that a complex structure on a Riemann surface $C$ of genus $g$ is determined up to isomorphism by the period matrix

$$\Pi = (\int_{\gamma_j} \omega_i)$$

where $(\gamma_1, \ldots, \gamma_{2g})$ is a basis of first homology $H_1(C, \mathbb{Z})$ and $(\omega_1, \ldots, \omega_g)$ is a basis of holomorphic 1-forms, $H^0(C, \Omega^1_C)$. It is possible to choose the bases in such a way that the matrix $\Pi$ has the from $(Z \;\; 1_{g\times g})$ where $Z$ satisfies the *Riemann bilinear relations*, that is, $Z$ is a symmetric complex matrix of size $g$ with positive definite imaginary part. All such matrices are parametrized by a complex domain $\mathfrak{H}_g \subset \mathbb{C}^{g(g+1)/2}$,

$$\mathfrak{H}_g := \{Z \in Mat_g(\mathbb{C}) : Z^t = Z, \mathrm{Im}(Z) > 0\}$$

called the Siegel half plane, which is homogeneous with respect to action of the group $Sp(2g, \mathbb{R})$.

If we choose a different bases with the above property, then we get a different point on $\mathfrak{H}_g$ related to the previous point by the action of $Sp(2g, \mathbb{Z})$. Thus, we get a holomorphic (transcendental) map from the moduli of Riemann surfaces of genus $g$ to the orbit space

$$\mathcal{P} : \mathcal{M}_g \to \mathfrak{H}_g/Sp(2g, \mathbb{Z})$$

called the *period map.*

The fundamental fact is that this map is an isomorphism onto its image which we state as follows:

**Theorem 0.3.** *(Torelli theorem for curves) Two smooth compact complex curves $C$ and $C'$ are isomorphic if and only if there exits an isomorphism $H^1(C, \mathbb{Z}) \simeq H^1(C', \mathbb{Z})$ of integral Hodge structures of weight 1 respecting the intersection pairing.*

Returning back to the case of K3 surfaces, let $\Lambda = (-E_8) \oplus (-E_8) \oplus U \oplus U \oplus U$ be the K3 lattice. We consider the period domain

$$\mathfrak{D} = \{[\Omega] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) : (\Omega, \Omega) = 0, (\Omega, \overline{\Omega}) > 0\}$$

Now given a family of K3 surfaces $S$ over $T$ where $T$ is simply-connected complex-manifold, together with a marking (identification of lattices) $\phi : H^2(S_0, \mathbb{Z}) \xrightarrow{\sim} \Lambda$, we define the *period map*

$$\mathcal{P} : T \to \mathfrak{D}, \quad t \to [\phi(H^{2,0}(S_t))]$$

where $\phi$ gives canonical markings of $H^2(S_t, \mathbb{Z})$ since $T$ is simply-connected.

Local Torelli theorem, due to Kodaira, says that for $T = Def(X_0)$ (the smooth universal deformation space of $S_0$ which is 20-dimensional which can be thought of as a small open disk in $\mathbb{C}^{20}$), the period map is a local isomorphism.

The period map is equivariant with respect to the natural actions of $O(\Lambda)$, though the quotient $O(\Lambda)\backslash\mathcal{D})$ is not Hausdorff. For this reason, one works with polarized moduli spaces of K3 surface (such as considering moduli of pairs $(S, \mathcal{L})$ where $S$ is a K3 surface and $\mathcal{L}$ is an ample line bundle. We will discuss this in the next section).

Local Torelli theorem is valid for a much broader class of varieties but the global Torelli theorem which is equivalent to what we stated above says that

$$\mathcal{P} : \mathfrak{N}/O(\Lambda) \to \mathfrak{D}/O(\Lambda)$$

is a bijection, where $\mathfrak{N}$ is the moduli of marked K3 surfaces (which is a fine moduli space). However, the action of the orthogonal group is not nice. The quotient has no good analytic or algebraic structure, and it is non-Hausdorff.

The group of automorphisms $O(\Lambda)$ of the lattice $\Lambda$ is the group of all bijections $g : \Lambda \to \Lambda$ with $(g(x), g(y)) = (x, y)$ for all $x, y \in \Lambda$). It is a discrete group in $O(\Lambda \otimes \mathbb{R})$.

There are two main sources of elements of $O(\Lambda)$: $Aut(X)$ and $W_\Lambda$ the Weyl group of $\Lambda$. The latter is defined by

$$W_\Lambda = \langle s_\delta : \delta \in \mathrm{Pic(S)}, \langle \delta, \delta \rangle = -2 \rangle$$

where $s_\delta(x) = x + \langle x, \delta \rangle \delta$. It is easy to check that these "reflection maps" are in $O(\Lambda)$.

Global Torelli theorem implies that $Aut(S) \ltimes W_\Lambda$ is of finite index in $O(\Lambda)$.

A nontrivial corollary of the theory of periods of K3 surfaces is the following theorem.

**Theorem 0.4.** *All complex K3 surfaces are diffeomorphic to a nonsingular quartic surface in* $\mathbb{P}^3$.

In passing, let me also mention the derived Torelli theorem due to Orlov.

**Proposition 0.5.** *Two complex projective K3 surfaces $S$ and $S'$ are derived equivalent if and only of there exists a Hodge isometry $\tilde{H}(S, \mathbb{Z}) \simeq \tilde{H}(S', \mathbb{Z})$ where $\tilde{H}$ is defined as*

$$\tilde{H}(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus U$$

*equipped with the Mukai pairing*

$$\langle \alpha, \beta \rangle = (\alpha_2, \beta_2) - (\alpha_0, \beta_4) - (\alpha_4, \beta_0)$$

*It comes with a weight 2 Hodge structure defined by*

$$\tilde{H}^{1,1}(S) := H^{1,1}(S) \oplus (H^0(S) \oplus H^4(S)), \tilde{H}^{2,0}(S) := H^{2,0}(S)$$

**Lattice polarized K3 surfaces:**

As we have mentioned above the moduli of K3 surfaces $O(\Lambda)\backslash\mathfrak{D}$ is not a good space. As usual, better behaved moduli spaces are obtained by choosing extra data. The simplest way is to choose a polarization, that is, for each $d > 0$, we consider K3 surface $S$ equipped with a primitive ample line bundle $L_S$ with $c_1(L_S) = h \in H^2(S, \mathbb{Z})$ primitive element with $h^2 = 2d$. Note that by Riemann-Roch, one has

$$\chi(S, L) = \frac{L^2}{2} + 2$$

One can construct a moduli space of polarized K3 surfaces of degree $2d$. Denote this by $\mathcal{M}_d$. It is known that $\mathcal{M}_d$ is a Deligne-Mumford stack of finite-type and is smooth over $\mathrm{Spec}(\mathbb{Z}[1/(2d)])$. It can be partially compactified by allowing polarized singular K3 surfaces (with only rational double points). Over $\mathbb{C}$, the corresponding coarse moduli space has a description via periods. Namely, given a marked K3 surface with marking $\phi : H^2(S,\mathbb{Z}) \xrightarrow{\sim} \Lambda$ let $l = \phi(h)$, let $l^\perp$ be the orthogonal complement of $\mathbb{Z} \cdot l$ in $\Lambda$. This is isomorphic to

$$l^\perp \simeq (-E_8) \oplus (-E_8) \oplus U \oplus U \oplus \mathbb{Z}\langle -2d \rangle$$

Then the corresponding period space is the 19-dimensional complex variety given by:

$$\mathfrak{D}_d = \{\mathbb{C}[\Omega] \in \mathbb{P}(l^\perp \otimes \mathbb{C}) : (\Omega,\Omega) = 0, (\Omega,\overline{\Omega}) > 0\}$$

There is a corresponding orthogonal group $O(\Lambda_l)$ consisting of elements $g \in O(\Lambda)$ such that $g(l) = l$. A version of the global Torelli theorem states that the quotient

$$\mathfrak{D}_d / O(\Lambda_l)$$

is well-behaved. In particular, it is a quasi-projective variety with only finite quotient singularities, and represents the coarse moduli space of primitively polarized K3 surfaces (possibly with rational double points) of degree $2d$.

By the way $\mathfrak{D}_d$ is a homogeneous space, known as a type IV domain. It has two connected components and the connected component of the identity can be identified with

$$\mathfrak{D}_d^0 = SO(2,19)/SO(2) \times SO(19)$$

The action of $O(\Lambda_l)$ interchanges the connected components.

As a special case, consider the K3 lattice $\Lambda = (-E_8)^{\oplus 2} \oplus U^{\oplus 3}$ and let $l = e_1 + df_1$ where $e_1, f_1$ is the standard basis of $U$. We can then let $\mathcal{D}_l$ to be the corresponding period domain in $\mathbb{P}(l^\perp \otimes \mathbb{C})$. In fact, it follows from lattice theory that by applying a suitable orthogonal transformation of $\Lambda$ any primitive $l \in \Lambda$ with $l^2 = 2d$ is of this form.

A generalization of this notion, due to Nikulin, is to consider K3 surfaces $S$ polarized by an even non-degenerate lattice $M$ of signature $(1, r-1)$. (If $r > 1$, it is called a hyperbolic lattice). It is known that the cone $\{x \in M \otimes \mathbb{R} : x^2 \geq 0\}$ after deleting the zero vecto consists of two connected components. Set $\Delta(M) = \{x \in M : x^2 = -2\}$. Fix a connected component $C^+(M)$ of

$$\{x \in M \otimes \mathbb{R} : x^2 > 0, (x,\delta) \neq 0 \text{ for all } \delta \in \Delta(M)\}$$

The choice of this connected component is not relevant as different choices are permuted by the autoequivalences induced by reflection maps associated the $(-2)$ classes.

An $M$-polarized K3 surfaces is a primitive embedding

$$j : M \hookrightarrow \mathrm{Pic(S)}$$

(where primitive means that the cokernel is a free abelian group) such that $j(C^+(M))$ intersects the closure of the ample cone. The above case of primitively polarized K3 surfaces corresponds to the special case when $M = \mathbb{Z} \cdot l$. The corresponding period domain is

$$\mathfrak{D}_M = \{\mathbb{C}[\Omega] \in \mathbb{P}(M^\perp \otimes \mathbb{C}) : (\Omega, \Omega) = 0, (\Omega, \overline{\Omega}) > 0\}$$

We also have $O(M) = \{g \in O(\Lambda) : g_{|M} = \mathrm{Id}\}$. To define a marked $M$-polarized K3 surface, we in addition introduce an isomorphism of lattices $\phi : H^2(S, \mathbb{Z}) \xrightarrow{\sim} \Lambda$. There is a fine (non-separated) marked moduli stack of $M$-polarized K3 surfaces. The period map gives a map from this moduli stack to the quotient $\mathfrak{D}_M$. To eliminate dependence on the marking, we take the quotient $\mathfrak{D}_M/O(M)$. This turns out to be a quasi-projective variety.

**Mirror symmetry:**

Improv.

## References


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